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## GROTHENDIECK RINGS OF $\mathbb{Z}$ -VALUED FIELDS

RAF CLUCKERS<sup>†</sup> AND DEIRDRE HASKELL<sup>††</sup>

**Abstract.** We prove the triviality of the Grothendieck ring of a  $\mathbb{Z}$ -valued field  $K$  under slight conditions on the logical language and on  $K$ . We construct a definable bijection from the plane  $K^2$  to itself minus a point. When we specialize to local fields with finite residue field, we construct a definable bijection from the valuation ring to itself minus a point.

At the Edinburgh meeting on the model theory of valued fields in May 1999, Luc B elair posed the question of whether there is a definable bijection between the set of  $p$ -adic integers and the set of  $p$ -adic integers with one point removed. At the same meeting, Jan Denef asked what is the Grothendieck ring of the  $p$ -adic numbers, as did Jan Kraj ı cek independently in [K]. A general introduction to Grothendieck rings of logical structures was recently given in [KS] and in [DL2, par. 3.7]. Calculations of non-trivial Grothendieck rings and related topics such as motivic integration can be found in [DL] and [DL2]. The logical notion of the Grothendieck ring of a structure is analogous to that of the Grothendieck ring in the context of algebraic  $K$ -theory and has analogous elementary properties (see [S]). Here we recall the definition.

**DEFINITION 1.** Let  $\mathcal{M}$  be a structure and  $Def(\mathcal{M})$  the set of definable subsets of  $M^n$  for every positive integer  $n$ . For any  $X, Y \in Def(\mathcal{M})$ , write  $X \cong Y$  if and only if there is a definable bijection (an isomorphism) from  $X$  to  $Y$ . Let  $F$  be the free abelian group whose generators are isomorphism classes  $[X]$  with  $X \in Def(\mathcal{M})$  (so  $[X] = [Y]$  if and only if  $X \cong Y$ ) and let  $E$  be the subgroup generated by all expressions  $[X] + [Y] - [X \cup Y] - [X \cap Y]$  with  $X, Y \in Def(\mathcal{M})$ . Then the Grothendieck group of  $\mathcal{M}$  is the quotient group  $F/E$ . Write  $[X]$  for the image of  $X \in Def(\mathcal{M})$  in  $F/E$ . The Grothendieck group has a natural structure as a ring with multiplication induced by  $[X] \cdot [Y] = [X \times Y]$  for  $X, Y \in Def(\mathcal{M})$ . We call this ring the Grothendieck ring  $K_0(\mathcal{M})$  of  $\mathcal{M}$ .

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It is easy to see that the above questions are related: the Grothendieck ring is trivial if and only if there is a definable bijection between  $M^k$  and itself minus a point for some  $k$ , which happens if and only if the Grothendieck group is trivial. Moreover, if we find such a  $k$  then we have for any  $X \in \text{Def}(\mathcal{M})$  a definable bijection from the disjoint union of  $M^k \times X$  and  $X$  to  $M^k \times X$ ; if there is a definable injection from  $M^k$  into  $X$  we find a definable bijection from  $X$  to itself minus a point.

In this paper we answer the questions posed by Bélair and Denef. Furthermore, we prove the triviality of the Grothendieck ring of any  $\mathbb{Z}$ -valued field which satisfies some slight conditions and give in this general setting an explicit bijection from the plane to itself minus a point. For the fields  $\mathbb{Q}_p$  and  $\mathbb{F}_q((t))$  we explicitly construct a definable bijection from the valuation ring to itself minus a point.

Dave Marker independently produced a definable bijection from  $\mathbb{Z}_p$  to  $\mathbb{Z}_p \setminus \{0\}$ , after it was noticed by Lou van den Dries that its existence followed from unpublished notes of the second author. The first author has proved further that there is a definable bijection between any two definable sets in the  $p$ -adics if and only if they have the same dimension. This will appear in a later paper. We thank the referee for encouraging us to present these results in greater generality than had been our original intention.

Fix a  $\mathbb{Z}$ -valued field  $K$ , that is, a field with a valuation  $v: K^\times \rightarrow \mathbb{Z}$  to an ordered group  $\mathbb{Z}$  which is elementarily equivalent to the integers in the Presburger language. Let  $R = \{x \in K \mid v(x) \geq 0\}$  be the valuation ring,  $R^* = R \setminus \{0\}$  and  $\bar{K} = R/m$  the residue field, with  $m$  the maximal ideal of  $R$  and natural projection  $R \rightarrow \bar{K}: x \rightarrow \bar{x}$ . An angular component map is a homomorphism  $ac: K^\times \rightarrow \bar{K}^\times$  such that  $ac(x) = \bar{x}$  if  $v(x) = 0$ . We extend  $ac$  to a map  $ac: K \rightarrow \bar{K}$  by putting  $ac(0) = 0$  (for the existence of angular component maps, see [P] and [B]).

**DEFINITION 2.** Let  $\mathcal{L}$  be an extension of the language of rings with  $K$  as a model. We say that the structure  $(K, \mathcal{L})$  satisfies condition  $(*)$  if we can choose an angular component map  $ac$  and an  $\mathcal{L}$ -definable element  $\pi \in R$  with  $v(\pi) = 1$  and  $ac(\pi) = 1$  such that the sets  $R$  and  $R^{(1)} = \{x \in R \mid ac(x) = 1\}$  are  $\mathcal{L}$ -definable.

Notice that if condition  $(*)$  is satisfied, the set

$$\{(x, y) \in K^2 \mid v(x) \leq v(y)\}$$

is  $\mathcal{L}$ -definable by the formula  $\exists z \in R (zx = y)$ . A bijection  $X \rightarrow Y$  with  $X, Y \in \text{Def}(K, \mathcal{L})$  with  $\mathcal{L}$ -definable graph will be called an isomorphism.

Let  $X \subset K^m$  and  $Y \subset K^n$  be definable sets,  $m \geq n$ . Let  $X' = \{0\} \times X$  and  $Y' = \{1\}^{m-n+1} \times Y$ . Then we define the disjoint union  $X \sqcup Y$  of  $X$  and  $Y$  up to isomorphism to be  $X' \cup Y'$ . We say that a set  $W$  is isomorphic to  $X \sqcup Y$  if  $W$  is isomorphic to  $X' \cup Y'$  and then obviously  $[W] = [X] + [Y]$ . If

$(K, \mathcal{L})$  satisfies condition  $(*)$  then we can find  $W \subset R^m$  with  $W \cong X \sqcup Y$  as follows. The map  $i: K \rightarrow R$  which sends  $x$  to  $\pi x$  if  $v(x) \geq 0$  and to  $1 + 1/x$  if  $v(x) < 0$  is a definable injection. For  $m = n = 1$ , put  $X'' = \pi.i(X)$  and  $Y'' = 1 + \pi.i(Y)$ . Then  $X'' \cong X$ ,  $Y'' \cong Y$  and  $X'' \cap Y'' = \emptyset$ , so  $W = X'' \cup Y''$  is isomorphic to  $X \sqcup Y$ . For  $m > 1$ , use the same method in each coordinate.

**PROPOSITION 1.** *Let  $K$  be a  $\mathbb{Z}$ -valued field, which is a model for the language  $\mathcal{L}$ . If the structure  $(K, \mathcal{L})$  satisfies condition  $(*)$ , then the following holds:*

- (i) *The disjoint union of  $R$  and  $R^{(1)}$  is isomorphic to  $R^{(1)}$  and thus  $[R] = 0$ .*
- (ii) *The disjoint union of two copies of  $R^{*2}$  is isomorphic to  $R^{*2}$  itself, and hence  $[R^{*2}] = 0$ .*

**PROOF.**

(i) The map

$$\{0\} \times R \cup \{1\} \times R^{(1)} \rightarrow R^{(1)}: \begin{cases} (0, x) \mapsto 1 + \pi x, \\ (1, x) \mapsto \pi x, \end{cases}$$

is easily seen to be an isomorphism as required. This yields in the Grothendieck ring  $[R] + [R^{(1)}] = [R^{(1)}]$ , so  $[R] = 0$ .

(ii) Define the sets

$$\begin{aligned} X_1 &= \{ (x, y) \in R^{*2} \mid v(x) \leq v(y) \}, \\ X_2 &= \{ (x, y) \in R^{*2} \mid v(x) > v(y) \}, \end{aligned}$$

then  $X_1, X_2$  form a partition of  $R^{*2}$ . The isomorphisms

$$\begin{aligned} \{0\} \times R^{*2} &\rightarrow X_1: (0, x, y) \mapsto (x, xy), \\ \{1\} \times R^{*2} &\rightarrow X_2: (1, x, y) \mapsto (\pi xy, y), \end{aligned}$$

imply that  $R^{*2} \sqcup R^{*2}$  is isomorphic to  $X_1 \cup X_2 = R^{*2}$ . It follows that  $2[R^{*2}] = [R^{*2}]$ , so  $[R^{*2}] = 0$ . Notice that the proof of (ii) does not use the full power of  $(*)$ , only that  $R$  is definable. ◻

**THEOREM 1.** *Let  $K$  be a  $\mathbb{Z}$ -valued field, which is a model for the language  $\mathcal{L}$ . If the structure  $(K, \mathcal{L})$  satisfies condition  $(*)$ , then the Grothendieck ring  $K_0(K)$  is trivial and there exists an isomorphism from  $R^2 \setminus \{(0, 0)\}$  to  $R^2$ .*

**PROOF.** Since  $0 = [R] = [R^*] + [\{0\}]$  we have  $[R^*] = -1$ . Together with  $0 = [R^{*2}] = [R^*]^2$  this yields  $1 = 0$ , so  $K_0(K)$  is trivial.

Define the isomorphisms  $\psi: R^2 \rightarrow \pi^3 R^2: (x, y) \mapsto (\pi^3 x, \pi^3 y)$  and

$$\varphi_i: R^2 \rightarrow (\pi^i + \pi^3 R) \times (\pi^i + \pi^3 R): (x, y) \mapsto (\pi^i + \pi^3 x, \pi^i + \pi^3 y)$$

for  $i = 1, 2$ .

Since clearly  $\psi(R^* \times R^*) \cup \varphi_1(R^* \times R^*)$  is isomorphic to  $R^{*2} \sqcup R^{*2}$ , we can find by Proposition 1 (ii) an isomorphism

$$f_1: \varphi_1(R^* \times R^*) \rightarrow \psi(R^* \times R^*) \cup \varphi_1(R^* \times R^*).$$

Define  $f_2$  by

$$f_2: \psi(R \times R^*) \cup \varphi_2(R^{(1)} \times R^*) \rightarrow \varphi_2(R^{(1)} \times R^*):$$

$$\begin{cases} \psi(x, y) \mapsto \varphi_2(1 + \pi x, y), \\ \varphi_2(x, y) \mapsto \varphi_2(\pi x, y). \end{cases}$$

Analogously, we can modify the function given in the proof of Proposition 1 (i) to get an isomorphism

$$f_3: \varphi_2(\{0\} \times R^{(1)}) \rightarrow \varphi_2(\{0\} \times R^{(1)}) \cup \psi(\{0\} \times R).$$

Finally,  $g: R^2 \setminus \{(0, 0)\} \rightarrow R^2$ :

$$x \mapsto \begin{cases} f_1(x) & \text{if } x \in \varphi_1(R^* \times R^*), \\ f_2(x) & \text{if } x \in \psi(R \times R^*) \cup \varphi_2(R^{(1)} \times R^*), \\ f_3(x) & \text{if } x \in \varphi_2(\{0\} \times R^{(1)}), \\ x & \text{else,} \end{cases}$$

is the required isomorphism. \(\dashv\)

We give some examples for the conditions of Theorem 1 to be satisfied. Let  $\mathcal{L}_{ac}$  be the language of rings with an extra constant symbol to denote  $\pi$  and a relation symbol to denote the set  $R^{(1)}$ . Let  $\mathcal{L}_{ac,R}$  be the language  $\mathcal{L}_{ac}$  with an extra relation symbol to denote  $R$ .

- Let  $K$  be a valued field with valuation to the integers  $\mathbb{Z}$ . Then we can define an angular component as follows. Choose  $\pi \in K$  with  $v(\pi) = 1$  and put  $ac(x) = \overline{\pi^{-v(x)}x}$  for  $x \neq 0$ . Then clearly  $ac(\pi) = 1$  and  $(K, \mathcal{L}_{ac,R})$  satisfies condition (\*).

- Let  $K$  be a Henselian field with valuation to the integers  $\mathbb{Z}$ . Then  $R$  is already definable in the language of rings: if  $\text{char}(\bar{K}) \neq 2$  we have  $R = \{x \in K \mid \exists y \in K, y^2 = 1 + \pi x^2\}$  and if  $\text{char}(\bar{K}) = 2$  then we use the formula  $\exists y \in K, y^3 = 1 + \pi x^3$  to define  $R$ . This implies that  $(K, \mathcal{L}_{ac})$  satisfies condition (\*).

- For definability of the valuation ring in fields of rational functions within the language of rings, see [D] and [KR].

Now we specialize our attention to local fields with finite residue field.

**THEOREM 2.** *Let  $K = \mathbb{F}_q((t))$  be the formal Laurent series over the finite field  $\mathbb{F}_q$  and  $\mathcal{L}_t$  the language of rings with a constant symbol to denote  $t$ . Then  $K_0(K)$  is trivial and we have an isomorphism  $R \rightarrow R^*$ .*

**PROOF.** We first show that  $K$  satisfies condition (\*). Since  $K$  is a Henselian field,  $R$  is definable as shown above. For each  $x \in \mathbb{F}_q$  we have  $x^{q-1} = 1$ , so

we can define  $R^{(1)}$  as

$$R^{(1)} = \left\{ x \in R \mid \exists y \in R^*, \bigvee_{n=0}^{q-2} t^n y^{q-1} = x \right\},$$

again by Hensel’s lemma.

By Theorem 1 we have an isomorphism  $f : R^2 \rightarrow R^2 \setminus \{(0, 0)\}$ . For a Laurent series  $H(t) \in K$  we have  $H(t)^p = H(t^p)$ . Consequently, the map

$$g : K^2 \rightarrow K : (x, y) \mapsto x^p + ty^p$$

is an injection from the plane into the line. We obtain the isomorphism

$$R \rightarrow R^* : x \mapsto \begin{cases} g \circ f \circ g^{-1}(x) & \text{if } x \in g(R^2), \\ x & \text{else.} \end{cases} \quad \dashv$$

Now let  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers and  $K$  a fixed finite field extension of  $\mathbb{Q}_p$ . Choose an element  $\pi$  with  $v(\pi) = 1$ , then  $ac(x) = \pi^{-v(x)}x \pmod{(\pi)}$  defines an angular component for  $x \neq 0$ . We work with  $\mathcal{L}_\pi$ , the language of rings with an extra constant symbol to denote  $\pi$ . For a definable set  $X \subset K$  and  $k \in \mathbb{N}_0$  we write

$$X^{(k)} = \{ x \in X \mid v(\pi^{-v(x)}x - 1) \geq k \},$$

which corresponds with our previous definition of  $R^{(1)}$ . The set  $R$  and each  $X^{(k)}$  is definable by the same argument as in the proof of Theorem 2, so  $(K, \mathcal{L}_\pi)$  satisfies condition (\*). We put  $P_n = \{ x \in K^\times \mid \exists y \in K, y^n = x \}$  and  $\bar{P}_n = P_n \cap R$ . Recall that  $P_n$  is a subgroup of finite index in  $K^\times$  for each  $n$ .

For convenience, we recall the following easy corollary of Hensel’s Lemma.

**COROLLARY 1.** *Let  $n > 1$  be a natural number. For each  $k > v(n)$ , and  $k' = k + v(n)$  the function*

$$K^{(k)} \rightarrow P_n^{(k')} : x \mapsto x^n$$

*is an isomorphism.*

In the next proposition we exhibit some isomorphisms between definable sets.

**PROPOSITION 2.** *Let  $K$  be a finite field extension of the  $p$ -adic numbers and  $\mathcal{L}_\pi$  the language of rings with an extra constant symbol to denote  $\pi$ . Then we have*

- (i) *for each  $k > 0$ , the union of two disjoint copies of  $R^{(k)}$  is isomorphic to  $R^{(k)}$ ;*
- (ii) *the union of two disjoint copies of  $R^*$  is isomorphic to  $R^*$ .*

PROOF.

(i) **Case 1:**  $p \neq 2$ . The map  $R^{(k)} \rightarrow \bar{P}_2^{(k)} : x \mapsto x^2$  is an isomorphism for each  $k > 0$  by Corollary 1. By Hensel's Lemma,  $R^{(k)} = \bar{P}_2^{(k)} \cup \pi \bar{P}_2^{(k)}$  is a partition. Hence the function

$$\{0\} \times R^{(k)} \cup \{1\} \times R^{(k)} \rightarrow R^{(k)} : \begin{cases} (0, x) \mapsto x^2, \\ (1, x) \mapsto \pi x^2, \end{cases}$$

is an isomorphism.

**Case 2:**  $p = 2$ . The map  $R^{(k)} \rightarrow \bar{P}_3^{(k)} : x \mapsto x^3$  is an isomorphism by Corollary 1, and by Hensel's Lemma  $R^{(k)} = \bar{P}_3^{(k)} \cup \pi \bar{P}_3^{(k)} \cup \pi^2 \bar{P}_3^{(k)}$  is a partition. Explicitly, we see that cubing and multiplying by 1,  $\pi$  or  $\pi^2$  is an isomorphism from three disjoint copies of  $R^{(k)}$  to  $R^{(k)}$ . First suppose that  $k > v(2)$  and put  $k' = k + v(2)$ , then  $R^{(k)} \rightarrow \bar{P}_2^{(k')} : x \mapsto x^2$  is an isomorphism by Corollary 1. By Hensel's lemma, we have a partition  $R^{(k)} = \bigcup_{i=1}^{2^l} \alpha_i \bar{P}_2^{(k')}$  for some  $l \in \mathbb{N}_0$ . Thus we can say there are isomorphisms from  $R^{(k)}$  to  $2^l$  disjoint copies of  $R^{(k)}$  and to three disjoint copies of  $R^{(k)}$ . Some arithmetic on the number of disjoint copies yields the required isomorphism for  $k > v(2)$ .

If  $k \leq v(2)$  then  $R^{(k)}$  admits a finite partition into parts of the form  $\alpha R^{(v(2)+1)}$ , with  $v(\alpha) = 0$ , and hence that the required isomorphism exists follows from property (i) for  $R^{(v(2)+1)}$ .

(ii) Since  $R^*$  admits a finite partition with parts of the form  $\alpha R^{(1)}$  with  $v(\alpha) = 0$ , this follows from (i). □

Now we give the solution of the problems raised by J. Denef and L. Bélaïr.

**THEOREM 3.** *Let  $K$  be a finite field extension of  $\mathbb{Q}_p$  and  $\mathcal{L}_\pi$  the language of rings with an extra constant symbol to denote  $\pi$ . Then  $K_0(K) = 0$  and we have an isomorphism from  $R$  to itself minus a point.*

PROOF. The triviality of the Grothendieck ring follows from Theorem 1.

We write the isomorphism explicitly in the case  $p \neq 2$ . First let

$$W = 1 + \pi^2 R^* \cup \pi^2 R \cup \pi + \pi^2 R^{(1)}.$$

As in the proof of Proposition 2, we can write

$$R^* = \bigcup_{i=1}^l \alpha_i R^{(1)} = \bigcup_{i=1}^l (\alpha_i \bar{P}_2^{(1)} \cup \pi \alpha_i \bar{P}_2^{(1)})$$

as a partition for some  $l \in \mathbb{N}_0$ . Thus the function

$$f_1 : \pi^2 R^* \cup 1 + \pi^2 R^* \rightarrow 1 + \pi^2 R^* : \begin{cases} \pi^2 \alpha_i x \mapsto 1 + \pi^2 (\alpha_i x^2), \\ 1 + \pi^2 \alpha_i x \mapsto 1 + \pi^2 (\pi \alpha_i x^2), \end{cases}$$

where  $x \in R^{(1)}$ , is a well-defined isomorphism. Modify the function given in the proof of Proposition 1 (i), to get

$$f_2: \pi^2 R \cup \pi + \pi^2 R^{(1)} \rightarrow \pi + \pi^2 R^{(1)}: \begin{cases} \pi^2 x \mapsto \pi + \pi^2(1 + \pi x), \\ \pi + \pi^2 x \mapsto \pi + \pi^2(\pi x). \end{cases}$$

Then the function

$$f: W \rightarrow W \setminus \{0\}: x \mapsto \begin{cases} f_1^{-1}(x) & \text{if } x \in 1 + \pi^2 R^*, \\ f_2(x) & \text{if } x \in \pi^2 R \cup \pi + \pi^2 R^{(1)}, \end{cases}$$

is an isomorphism. Finally,

$$g: R \rightarrow R^*: x \mapsto \begin{cases} f(x) & \text{if } x \in W \\ x & \text{if } x \notin W \end{cases}$$

is an isomorphism.

In the case  $p = 2$ , we know from Proposition 2 (i) that there is a function which plays the role of  $f_1$ . The rest is as above.  $\dashv$

REMARK.

- The construction of the bijection  $\mathbb{Z}_p \rightarrow \mathbb{Z}_p \setminus \{0\}$  also works for the field  $\mathbb{F}_q((t))$  if  $2 \nmid q$ . If  $2 \mid q$  the proof of Proposition 2 (i) collapses since the index of the squares in  $\mathbb{F}_q((t))^\times$  is infinite.

- The triviality of the Grothendieck ring of a structure  $\mathcal{M}$  implies that every Euler characteristic on the definable sets is trivial. An Euler characteristic is a map  $\chi: \text{Def}(\mathcal{M}) \rightarrow R_\chi$  with  $R_\chi$  a ring, such that  $\chi(X) = \chi(Y)$  if  $X \cong Y$ ,  $\chi(X \cup Y) = \chi(X) + \chi(Y)$  if  $X \cap Y = \emptyset$  and  $\chi(X \times Y) = \chi(X)\chi(Y)$ . In general an Euler characteristic on  $\text{Def}(\mathcal{M})$  factorizes through  $\text{Def}(\mathcal{M}) \rightarrow K_0(\mathcal{M}): X \mapsto [X]$ .

## REFERENCES

- [1] L. BÉLAIR, *Types dans les corps valués munis d'applications coefficients*, *Illinois Journal of Mathematics*, vol. 43 (1999), no. 2, pp. 410–425.
- [2] J. DENEFF, *The diophantine problem for polynomial rings and fields of rational functions*, *Transactions of the American Mathematical Society*, vol. 242 (1978), pp. 391–399.
- [3] J. DENEFF and F. LOESER, *Definable sets, motives and  $p$ -adic integrals*, *Journal of the American Mathematical Society*, to appear, 45 pages.
- [4] ———, *Germes of arcs on singular algebraic varieties and motivic integration*, *Inventiones Mathematicae*, vol. 135 (1999), pp. 201–232.
- [5] K. H. KIM and F. W. ROUSH, *Diophantine unsolvability over  $p$ -adic function fields*, *Journal of Algebra*, vol. 176 (1995), no. 1, pp. 83–110.
- [6] J. KRAJÍČEK, *Uniform families of polynomial equations over a finite field and structures admitting an Euler characteristic of definable sets*, *Proc. LMS*, vol. 81 (2000), pp. 257–284.
- [7] J. KRAJÍČEK and T. SCANLON, *Combinatorics with definable sets: Euler characteristics and Grothendieck rings*, this BULLETIN, vol. 6 (2000), pp. 311–330.



[8] J. PAS, *On the angular component map modulo  $p$* , *The Journal of Symbolic Logic*, vol. 55 (1990), pp. 1125–1129.

[9] J. R. SILVESTER, *Introduction to algebraic  $K$ -theory*, Mathematics Series, Chapman and Hall, 1981.

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