Multivariate Igusa Theory: Decay Rates of Exponential Sums

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1 Introduction

For $f = (f_1, \ldots, f_r)$ an r-tuple of restricted power series over $\mathbb{Q}_p$ in the variables $x = (x_1, \ldots, x_n)$ and for $y \in \mathbb{Q}_p^r$, we consider the exponential integral

$$E_f(y) = \int_{\mathbb{Z}_p^n} \psi\big(y \cdot f(x)\big)|dx|, \tag{1.1}$$

where ψ is a nontrivial additive character on $\mathbb{Q}_p$, $|dx|$ denotes the normalized Haar measure on $\mathbb{Q}_p^n$, and $y \cdot f(x) = \sum_j y_j f_j(x)$.

With $|y| = \max(|y_i|)_i$ and $\ll$ the Vinogradov symbol, we obtain the following general upper bounds.

Theorem 1.1. If $f(\mathbb{Z}_p^n)$ has nonempty interior in $\mathbb{Q}_p^r$, there exists a real number $\alpha < 0$ such that

$$E_f(y) \ll \min\big\{|y|^\alpha, 1\big\}. \tag{1.2}$$
$\square$

In his book [9], Igusa proves Theorem 1.1 in the case when $r = 1$ with $f = f_1$ a non-constant homogeneous polynomial, and he formulates the problem of generalizing this to the case of $r > 1$. In this case Igusa is also able to give an explicit $\alpha < 0$ in terms of the numerical data of an embedded resolution of f. By a very fine analysis of embedded

resolutions of f, Lichtin [11] is able to prove Theorem 1.1 in the case of a dominant map of $r = 2$ polynomials, where he also gives an explicit $\alpha < 0$ in terms of the geometry of f. At present, these proofs seem to be difficult to be generalized to the case of $r > 2$ polynomials. In the case of one nonconstant polynomial $f = f_1$, Theorem 1.1 can be proven by elementary methods, see, for instance, the work of Čubarikov [2] and Loxton [12]. In the last section we show how the results of [2] can be used to derive Theorem 1.1, when $f_1, \ldots, f_r$ are polynomials, in a very short way (even with explicit upper bounds and weaker suppositions). We indicate there why the situation for analytic maps is more difficult.

1.1

In this paper we present a new technique to study exponential integrals of a general nature, namely, by studying rather general p-adic integrals by means of p-adic cell decomposition and the theory of subanalytic sets. Examples of such general exponential integrals are given below. These techniques are also used in other contexts, for example, by Denef [3] to prove the rationality of the Serre-Poincaré series associated to the p-adic points on a variety.

1.2

For readers not familiar with p-adic integration, we indicate how $E_f(y)$ can be understood as an exponential sum. In the case when the f_i are restricted power series over $\mathbb{Z}_p$, $\psi(x) = \exp(2\pi i(x \bmod \mathbb{Z}_p))$ (abbreviated by $\exp(2\pi i x)$), and

$$y = \left(\frac{u_1}{p^m}, \ldots, \frac{u_r}{p^m} \right), \tag{1.3}$$

with u_i integers satisfying $(u_1, \ldots, u_r, p) = 1$, $m \geq 0$, we can write

$$E_f(y) = \frac{1}{p^{mn}} \sum_{x \in (\mathbb{Z}_p/p^m)^n} \exp\left(2\pi i \frac{\sum_{j=1}^{r} u_j f_j(x)}{p^m} \right). \tag{1.4}$$

Note that for general $y' \in \mathbb{Q}_p^r$ there can always be found a tuple y of the form (1.3) such that $E_f(y') = E_f(y)$. Theorem 1.1 then says that $|E_f(y)|$ can be bounded by $cp^{m\alpha}$ for some $c > 0$ and $\alpha < 0$, uniform in y.

1.3

We use the notion of subanalytic sets as in [8] and the recent notion of subanalytic constructible functions as in [1] (see below for the definitions).

Let $G : \mathbb{Q}_p^r \to \mathbb{Q}$ be an integrable subanalytic constructible function and let $G^*(y) := \int_{\mathbb{Q}_p^r} G(x)\psi(x \cdot y)|dx|$ be its Fourier transform. We obtain the following general upper bounds.

Theorem 1.2. There exists a real number $\alpha < 0$ such that $G^*(y) \ll \min\{|y|^\alpha, 1\}$. □

1.4

We indicate how Theorem 1.1 follows from Theorem 1.2. It is well known that, whenever $f(\mathbb{Z}_p^n)$ has nonempty interior in $\mathbb{Z}_p^r$, E_f is the Fourier transform of an integrable function $F_f : \mathbb{Q}_p^r \to \mathbb{Q}$ (see [9] or [15]). We prove that we can take F_f to be a subanalytic constructible function (see Theorem 3.1 below). Theorem 1.1 then follows immediately from Theorem 1.2.

In fact, a similar reasoning leads to the following much more general result.

Theorem 1.3. If $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^r$ is a subanalytic map and $\phi : \mathbb{Q}_p^n \to \mathbb{Q}$ is an integrable subanalytic constructible function such that the support of ϕ is contained in f^{-1} (Regular values of f) $\cup A$, with A a set of measure zero, then there exists a real number $\alpha < 0$ such that $E_{\phi,f}(y) \ll \min\{|y|^\alpha, 1\}$, with

$$E_{\phi,f}(y) := \int_{\mathbb{Q}_p^n} \phi(x)\psi\big(y \cdot f(x)\big)|dx|. \tag{1.5}$$
□

We end Section 5 with an open question about what happens if f is analytic but no longer subanalytic. All results of the paper also hold for finite field extensions of $\mathbb{Q}_p$.

Possible applications, or possible subjects for future research, lie in the search for candidate exponents α of Theorem 1.1 using the numerical data of a (parameterized) resolution of singularities of the family $\sum_{i=1}^r u_i f_i$ with parameters u_i (if such resolution exists); as noted before, candidate exponents can be found in this way when $r = 1$, see [6]. Also, one can try to establish, under similar conditions as in [9], an analytic analogue of the Poisson summation formula considered by Igusa [9].

1.5   Notation and terminology

We fix a p-adic field K (i.e., $[K : \mathbb{Q}_p]$ is finite) and write R for the valuation ring of K, π_0 for a uniformizer of R, and q for the cardinality of the residue field. For $x \in K$, $\nu(x) \in \mathbb{Z} \cup \{\infty\}$ denotes the p-adic valuation of x and $|x| = q^{-\nu(x)}$ the p-adic norm. We write P_n for the collection of nth powers in $K^{\times} = K \setminus \{0\}$, $n > 0$, and $\lambda P_n = \{\lambda x \mid x \in P_n\}$ for $\lambda \in K$. Let ψ be a nontrivial additive character on K. We write $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ for $x, y \in K^n$, $n > 0$.

The Vinogradov symbol $\ll$ has its usual meaning, namely that for complex-valued functions f and g with g taking nonnegative real values, $f \ll g$ means $|f| \leq cg$ for some constant c.

A restricted analytic function $R^n \to K$ is an analytic function given by a single restricted power series over K in n variables (by definition, this is a power series over K which converges on R^n). We extend each restricted analytic function $R^n \to K$ to a function $K^n \to K$ by putting it zero outside R^n. A key notion is the following.

Definition 1.4. A subset of K^n is called (globally) *subanalytic* if it can be obtained in finitely many steps by taking finite unions, intersections, complements, and linear projections of zero loci of polynomials and of zero loci of restricted analytic functions in K^{n+e}, $e \geq 0$. A function $f : X \subset K^m \to K^n$ is called subanalytic if its graph is a subanalytic set.

We recall a basic result on subanalytic sets.

Proposition 1.5 (see [8, Proposition 3.29]). Let $X \subset K^n$ be a subanalytic set and $f : X \to K$ a subanalytic function. Then there exists a finite partition of X into p-adic submanifolds A_j of K^n such that the restriction of f to each A_j is analytic and such that each A_j is subanalytic.   □

We refer to [1, 5, 7, 8, 13] for the theory of subanalytic sets.

2   Cell decomposition and p-adic integration

Cell decomposition is well suited to describe piecewise several kinds of p-adic maps, for example, polynomial maps, restricted analytic maps, subanalytic constructible functions, and so on. It allows one to partition the domain of such functions into p-adic manifolds of a simple form, called cells, and to obtain on each of these cells a nice description of the way the function depends on a specific special variable (for an example of such an application, see Lemma 2.5). By induction one gets a nice description of the function with respect to the other variables.

Cells are defined by induction on the number of variables.

Definition 2.1. A cell $A \subset K$ is a (nonempty) set of the form

$$\{t \in K \mid |\alpha|\square_1|t - c|\square_2|\beta|, \ t - c \in \lambda P_n\}, \tag{2.1}$$

with constants $n > 0, \lambda, c \in K, \alpha, \beta \in K^\times$, and $\square_i$ either $<$ or no condition. A cell $A \subset K^{m+1}$, $m \geq 0$, is a set of the form

$$\{(x, t) \in K^{m+1} \mid x \in D, \ \big|\alpha(x)\big|\square_1\big|t - c(x)\big|\square_2\big|\beta(x)\big|, \ t - c(x) \in \lambda P_n\}, \tag{2.2}$$

with $(x, t) = (x_1, \ldots, x_m, t), n > 0, \lambda \in K, D = \pi_m(A)$ a cell, where π_m is the projection $K^{m+1} \to K^m$, subanalytic functions $\alpha, \beta : K^m \to K^\times$ and $c : K^m \to K$, and $\square_i$ either $<$ or no condition, such that the functions α, β, and c are analytic on D. We call c the center of the cell A and λP_n the coset of A.

Note that either a cell is the graph of an analytic function defined on D (namely if $\lambda = 0$) or, for each $x \in D$, the fiber $A_x = \{t \mid (x, t) \in A\}$ is nonempty open (if $\lambda \neq 0$).

Theorem 2.2 (p-adic cell decomposition, [1]). Let $X \subset K^{m+1}$ be a subanalytic set and $f_j : X \to K$ subanalytic functions for $j = 1, \ldots, r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that

$$\big|f_j(x, t)\big| = \big|\delta_{ij}(x)\big| \cdot \left|\left(t - c_i(x)\right)^{a_{ij}} \lambda_i^{-a_{ij}}\right|^{1/n_i}, \quad \text{for each } (x, t) \in A_i, \tag{2.3}$$

with $(x, t) = (x_1, \ldots, x_m, t)$, integers a_{ij}, and $\delta_{ij} : K^m \to K$ subanalytic functions, analytic on $\pi_m(A_i), j = 1, \ldots, r$. If $\lambda_i = 0$, the convention that $a_{ij} = 0$ is used. □

Theorem 2.2 is a generalization of cell decomposition for polynomial maps by Denef [3, 4]. Recently, in [1, 7], cell decomposition has been used to study parametrized integrals as follows.

Definition 2.3. For each subanalytic set X, let $\mathcal{C}(X)$ be the $\mathbb{Q}$-algebra generated by the functions $|h|$ and $\nu(h)$ for all subanalytic functions $h : X \to K^\times$. Call $G \in \mathcal{C}(X)$ a *subanalytic constructible function* on X.

To any function G in $\mathcal{C}(K^{m+n}), m, n \geq 0$, we associate a function $I_m(G) : K^m \to \mathbb{Q}$ by putting

$$I_m(G)(x) = \int_{K^n} G(x, y)|dy| \tag{2.4}$$

if the function $y \mapsto G(x, y)$ is absolutely integrable for all $x \in K^m$, and by putting $I_m(G)(x) = 0$ otherwise.

Theorem 2.4 (basic theorem on p-adic analytic integrals [1])**.** For any function $G \in \mathcal{C}(K^{m+n})$, the function $I_m(G)$ is in $\mathcal{C}(K^m)$.   □

Lemma 2.5. Let $X \subset K^{m+1}$ be a subanalytic set and let G_j be functions in $\mathcal{C}(X)$ in the variables $(x_1, \ldots, x_m, t)$ for $j = 1, \ldots, r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $G_j|_{A_i}$ is a finite sum of functions of the form

$$\left| \left(t - c_i(x) \right)^a \lambda_i^{-a} \right|^{1/n_i} \nu \left(t - c_i(x) \right)^s h(x), \tag{2.5}$$

where $h : K^m \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers. Also, for any function $G \in \mathcal{C}(K^n)$, there exists a closed subanalytic set $A \subset K^n$ of measure zero such that G is locally constant on $K^n \setminus A$.   □

Proof. The description is immediate from Theorem 2.2 and the definitions. The statement about $G \in \mathcal{C}(K^n)$ follows from Proposition 1.5 and the definitions.   ■

The following corollary is immediate.

Corollary 2.6. Let G be in $\mathcal{C}(K)$. Suppose that if $|y|$ tends to ∞ then $G(y)$ converges to zero. Then there exists a real number $\alpha < 0$ such that $G(y) \ll |y|^\alpha$.   □

We prove the following addendum to Theorem 2.4.

Proposition 2.7. Let G in $\mathcal{C}(K^{r+n})$ be such that $G(x, \cdot) : K^n \to \mathbb{Q}$ is integrable for almost all $x \in K^r$. Then there exists a function $F \in \mathcal{C}(K^r)$ such that for all $x \in K^r \setminus B$, with B a subanalytic set of measure zero,

$$F(x) = \int_{K^n} G(x, y)|dy|. \tag{2.6}$$
                    □

Proof. By induction and by Fubini's theorem, it is enough to treat the case $n = 1$.

By Lemma 2.5, we can partition K^{r+1} into cells A with center c and coset λP_m such that $G|_A$ is a finite sum of functions of the form

$$H(x, y) = \left| \left(y - c(x) \right)^a \lambda^{-a} \right|^{1/m} \nu \left(y - c(x) \right)^s h(x), \tag{2.7}$$

where $h : K^r \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers.

Claim 1. Possibly after refining the partition, we can assure that for each A either the projection $A' := \pi_r(A) \subset K^r$ has zero measure or we can write $G|_A$ as a sum of terms H of the form (2.7) such that the function $H(x, \cdot)$ is integrable over $A_x := \{y \mid (x, y) \in A\}$ for all $x \in A'$. $\qquad\square$

First we prove the claim. By partitioning further, we may suppose that either $\nu(y - c)$ is constant on A or it takes infinitely many values on A, and in the case when $\nu(y - c)$ is constant on A, we may assume that $a = s = 0$. Regroup the terms with the same exponents (a, s) by summing up the respective functions h.

By the description (2.7) of H and by the definition of cells, the fact that the function

$$H(x, \cdot) : A_x \longrightarrow \mathbb{Q} : y \longmapsto H(x, y) \tag{2.8}$$

is integrable over A_x only depends on the exponents (a, s), on whether $h(x)$ is zero or not, and on the particular form of the cell A_x. Also, if terms $H_1, \ldots, H_k$ have different exponents (a_i, s_i), then they have a different asymptotical behavior for y going to $c(x)$ with x fixed, and hence, if their sum is integrable over A_x, then each H_i is integrable over A_x.

Suppose now that A has nonempty interior. Let H be a term with exponents (a, s) and function h as in (2.7). Then, either $h(x)$ is almost everywhere zero or there exists by [Lemma 2.5](#) a nonempty open $U \subset A'$ such that $h(x)$ is constant and nonzero on U. If there exists such nonempty U, then, by the above discussion, the term $H(x, \cdot)$ is integrable over A_x for each $x \in U$ and hence for each $x \in A'$. If $h(x)$ is almost everywhere zero, then we can, by partitioning A' further using [Lemma 2.5](#), reduce to the case when A' has zero measure or $h(x)$ is identically zero on A', in which case we can skip the term H. This proves the claim.

Suppose that the statements of the claim are fulfilled for our partition of K^{r+1} into cells. Let $\mathcal{P}$ be the set of cells A such that $\pi_r(A)$ has measure zero. Put $B := \cup_{A \in \mathcal{P}} \pi_r(A)$ and $C := \cup_{A \in \mathcal{P}} A$. Let G' be the constructible function $G(1 - \chi_C)$, where χ_C is the characteristic function of C. Then, B has measure zero in K^r, G' satisfies

$$\int_K G(x, y)|dy| = \int_K G'(x, y)|dy| \tag{2.9}$$

for all $x \in K^r \setminus B$, and $G'(x, \cdot)$ is integrable for all $x \in K^r$. Putting $F := I_r(G')$, an application of [Theorem 2.4](#) ends the proof. $\qquad\blacksquare$

3   Exponential sums as Fourier transforms

We fix a nontrivial additive character ψ on K. For $\phi \in \mathcal{C}(K^n)$ an integrable function, for $f : K^n \to K^r$ a subanalytic function, and for $y \in K^r$, we consider the exponential integral

$$E_{\phi,f}(y) = \int_{K^n} \phi(x)\psi\big(y \cdot f(x)\big)|dx|. \tag{3.1}$$

We call $z \in K^r$ a *regular value* of f if $f^{-1}(z)$ is nonempty, if f is C^1 on a neighborhood of $f^{-1}(z)$, and if the rank of the Jacobian matrix of f is maximal at each point $x \in f^{-1}(z)$. We denote the set of regular values of f by Reg_f and the support of ϕ by $\mathrm{Supp}\,\phi$.

Theorem 3.1. Let $f : K^n \to K^r$ be a subanalytic function and let $\phi \in \mathcal{C}(K^n)$ be an integrable function satisfying $\mathrm{Supp}\,\phi \subset f^{-1}(\mathrm{Reg}_f) \cup A$ with A a set of measure zero. Then there exists an integrable function $F_{\phi,f}$ in $\mathcal{C}(K^r)$ such that for any bounded continuous function $G : K^r \to \mathbb{C}$,

$$\int_{K^r} F_{\phi,f}(z)G(z)|dz| = \int_{K^n} \phi(x)G\big(f(x)\big)|dx|, \tag{3.2}$$

and hence, the following Fourier transformation formula holds:

$$E_{\phi,f}(y) = \int_{z \in K^r} F_{\phi,f}(z)\psi(z \cdot y)|dz|. \tag{3.3}$$
$$\square$$

Theorem 3.1 is a generalization of [7, Corollary 1.8.2] by Denef, which treats the case when the f_i are polynomials and ϕ is a Schwartz-Bruhat function. Igusa has given an analogon of Theorem 3.1 in the case of $r = 1$ polynomial (cf. the asymptotic expansions of [9]), and Lichtin [10] has given one in the case of $r = 2$ polynomials, both in the case when ϕ is a Schwartz-Bruhat function. Igusa and Lichtin also relate the asymptotic expansions to the numerical data of an embedded resolution of f, the counterpart (however not easily computable) of which would be here to apply cell decomposition to get explicit asymptotic expansions for given f and ϕ.

Note that $F_{\phi,f}$ is determined, up to a set of measure zero, by the universal property stated in the theorem. The function $F_{\phi_{\mathrm{triv}},f}$, with ϕ_{triv} the characteristic function of R^n and f a dominant polynomial mapping, is called the local singular series of f and plays an important role in number theory, for example, in the circle method.

Proof of Theorem 3.1. Clearly $f^{-1}(\mathrm{Reg}_f)$ is subanalytic. Without loss of generality, we may assume that for all $x \in f^{-1}(\mathrm{Reg}_f)$ one has

$$J(x) := \det\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j=1,\dots,r} \neq 0. \tag{3.4}$$

By the inverse function theorem, Proposition 1.5, [8, Theorem 3.2] on the existence of bounds, and the subanalytic selection theorem [8, Theorem 3.6], we may also suppose that

$$T : f^{-1}(\mathrm{Reg}_f) \longrightarrow K^n : x \longmapsto y = (f(x), x_{r+1}, \dots, x_n) \tag{3.5}$$

is injective and a C^1 bijection onto its image with C^1 inverse. Applying the change-of-variables formula, we obtain

$$\begin{aligned}
\int_{K^n} \phi(x)G(f(x))|dx| &= \int_{f^{-1}(\mathrm{Reg}_f)} \phi(x)G(f(x))|dx| \\
&= \int_{T(f^{-1}(\mathrm{Reg}_f))} \phi \circ T^{-1}(y)G(y_1,\dots,y_r)|J \circ T^{-1}(y)|^{-1}|dy|.
\end{aligned} \tag{3.6}$$

By Fubini's theorem and Proposition 2.7, there exists a function $F_{\varphi,f}$ in $\mathcal{C}(K^r)$ with the property that

$$F_{\varphi,f}(y_1,\dots,y_r) = \int_{K^{n-r}} \phi \circ T^{-1}(y)|J \circ T^{-1}(y)|^{-1}|dy_{r+1} \wedge \cdots \wedge dy_n|, \tag{3.7}$$

for almost all $(y_1,\dots,y_r) \in K^r$, where we have extended the integrand by zero to K^{n-r}. This function clearly satisfies the requirements of the theorem. ∎

4 Estimates for Fourier transforms

For an integrable function G in $\mathcal{C}(K^r)$, we write G^* for its Fourier transform

$$G^* : K^r \longrightarrow \mathbb{C} : y \longmapsto \int_{K^r} G(x)\psi(x \cdot y)|dx|. \tag{4.1}$$

The following is a generalization of Theorem 1.2.

Theorem 4.1. For each integrable $G \in \mathcal{C}(K^r)$, there exists a real number $\alpha < 0$ such that $G^*(y) \ll \min\{|y|^{\alpha}, 1\}$. □

Proof. For simplicity, we suppose that $\psi(R) = 1$ and $\psi(x) \neq 1$ for $x \notin R$ (any other additive character is of the form $x \mapsto \psi(ax)$ with $a \in K$). It is clear that $G^*(y) \ll 1$ since

$$\left|G^*(y)\right| \leq \int_{K^r} \left|G(x)\right| |dx| < \infty. \tag{4.2}$$

Hence, it is enough to prove, for $i = 1, \ldots, r$, that

$$G^*(y) \ll \left|y_i\right|^{\alpha_i} \tag{4.3}$$

for some $\alpha_i < 0$. We prove that $G^*(y) \ll |y_r|^{\alpha_r}$ for some $\alpha_r < 0$. Write $x = (\widehat{x}, x_r)$ with $\widehat{x} = (x_1, \ldots, x_{r-1})$. By Lemma 2.5, we can partition K^r into cells A with center c and coset λP_m such that $G|_A$ is a finite sum of functions of the form

$$H(x) = \left|\left(x_r - c(\widehat{x})\right)^a \lambda^{-a}\right|^{1/m} v\left(x_r - c(\widehat{x})\right)^s h(\widehat{x}), \tag{4.4}$$

where $h: K^{r-1} \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers.

Claim 2. Possibly after refining the partition, we can assure that for each A either the projection $A' := \pi_{r-1}(A) \subset K^r$ has zero measure or we can write $G|_A$ as a sum of terms H of the form (4.4) such that H is integrable over A and $H(\widehat{x}, \cdot)$ is integrable over $A_{\widehat{x}} := \{x_r \mid (\widehat{x}, x_r) \in A\}$ for all $\widehat{x} \in A'$. Moreover, doing so, we can assure that each such term H does not change its sign on A. □

As this claim and its proof are similar to Claim 1 we will give only an indication of its proof.

Partitioning further, we may suppose that $v(x_r - c(\widehat{x}))$ does not change its sign on A, and that it takes either only one value on A or infinitely many values. If $v(x_r - c(\widehat{x}))$ only takes one value on A, we may suppose that the exponents a and s as in (4.4) are zero. Now apply Lemma 2.5 to each h and to the norms of all the subanalytic functions appearing in the description of the cells A in a similar way (in particular, make similar assumptions as above). Do this inductively for each variable. This way, the claim is reduced to a summation problem over (Presburger set of) integers, which is easily solved (cf. the proof of Claim 1). This proves the claim.

Fix a cell A and a term H as in the claim. The cell A has, by definition, the following form:

$$A = \{x \mid \widehat{x} \in A', \; v(\alpha(\widehat{x}))\square_1 v(x_r - c(\widehat{x}))\square_2 v(\beta(\widehat{x})), \; x_r - c(\widehat{x}) \in \lambda P_m\}, \tag{4.5}$$

where $A' = \pi_{r-1}(A)$ is a cell, $\square_i$ is $<$ or no condition, and $\alpha, \beta : K^{r-1} \to K^\times$ and $c : K^{r-1} \to K$ are subanalytic functions. We focus on a cell A with nonempty interior, in particular, $\lambda \neq 0$ and A' has nonempty interior. For $\widehat{x} \in A'$ and $y \in K^r$, we denote by $I(\widehat{x}, y)$ the value

$$I(\widehat{x}, y) = \int_{x_r \in A_{\widehat{x}}} H(x)\psi(x \cdot y)|dx_r|. \tag{4.6}$$

Let $\chi_{\lambda P_m} : K \to \mathbb{Q}$ be the characteristic function of λP_m and write $\widehat{y} = (y_1, \ldots, y_{r-1})$. We easily find that $I(\widehat{x}, y)$ equals

$$\psi(\widehat{x} \cdot \widehat{y} + cy_r)h(\widehat{x})|\lambda|^{-a/m} \sum_{\substack{(4.8)}} q^{-ja/m}j^s \int_{v(x_r - c) = j} \chi_{\lambda P_m}(x_r - c)\psi((x_r - c)y_r)|dx_r|, \tag{4.7}$$

where $c = c(\widehat{x})$ and the summation is over

$$\{j \mid v(\alpha(\widehat{x}))\square_1 j\square_2 v(\beta(\widehat{x}))\}. \tag{4.8}$$

By Hensel's lemma, there exists an integer e such that all units u with $u \equiv 1 \bmod \pi_0^e$ are mth powers (here, π_0 is such that $v(\pi_0) = 1$). Hence

$$\int_{v(u) = j} \chi_{\lambda P_m}(u)\psi(uy_r)|du| \tag{4.9}$$

is zero whenever $j + v(y_r) + e < 0$ (since, in this case, one essentially sums a nontrivial character over a finite group). In consequence, the only terms contributing to the sum (4.7) are those for which $-v(y_r) - e \leq j$.

We thus have

$$\left| \int_{x \in A} H(x) \psi(x \cdot y) |dx| \right| = \left| \int_{\widehat{x} \in A'} I(\widehat{x}, y) |d\widehat{x}| \right| \tag{4.10}$$

$$\leq \int_{B_{y_r}} |H(x)||dx| \tag{4.11}$$

with $B_{y_r} = \{x \in K^r \mid x \in A, -\nu(y_r) - e \leq \nu(x_r - c(\widehat{x}))\}$.

The integrability of H over A, the fact that H does not change its sign on A, and Theorem 2.4 imply that the integral (4.11), considered as a function in the variable y_r, is in $\mathcal{C}(K)$.

Next we prove that (4.11) goes to zero when $|y_r|$ goes to infinity. First suppose that A is contained in a compact set. Since $B_{y_r} \subset A$, the measure of B_{y_r}, and hence also (4.11), goes to zero when $|y_r|$ tends to infinity. In the case when A is not contained in a compact set, let A_b be the intersection of A with $(\pi_0^b R)^r$, for $b < 0$. Clearly each A_b is contained in a compact set. Also, for each $\varepsilon > 0$, there exists a b_0 such that for each $b < b_0$ and for each y_r, one has $\int_{B_{y_r} \backslash A_b} |H(x)||dx| < \varepsilon$, by the integrability of H over A. By the previous discussion, $\int_{B_{y_r} \cap A_b} |H(x)||dx|$, and hence also (4.11), goes to zero when $|y_r|$ goes to ∞.

An application of Corollary 2.6 now finishes the proof. ∎

Remark 4.2. The fact that $|G^*|$ in Theorem 4.1 goes to zero when $|y|$ goes to infinity also follows directly from the lemma of Riemann-Lebesgue in general Fourier analysis (cf. the section on Fourier transforms in [14]). However, to know this is not enough to apply Corollary 2.6 as is done to finish the proof of Theorem 4.1 since, in general, $|G^*|$ is not subanalytic constructible.

5   Decay rates of exponential sums

We use the notation of Section 3 for $E_{\phi, f}$. Combining Theorem 4.1 with the Fourier transformation formula of Theorem 3.1, we obtain the following generalization of Theorem 1.3.

Theorem 5.1. If $f : K^n \to K^r$ is a subanalytic map and $\phi \in \mathcal{C}(K^n)$ is integrable and satisfies $\mathrm{Supp}\, \phi \subset f^{-1}(\mathrm{Reg}_f) \cup A$ with A a set of measure zero, then there exists a real number $\alpha < 0$ such that

$$E_{\phi, f}(y) \ll \min\{|y|^\alpha, 1\}. \tag{5.1}$$

□

Combining this theorem with the fact that the set of singular points of a dominant polynomial mapping $K^n \to K^r$ (or a dominant restricted analytic mapping $R^n \to K^r$) has measure zero, we obtain the following corollary.

Corollary 5.2. If $f : K^n \to K^r$ is a dominant polynomial mapping and if $\phi \in \mathcal{C}(K^n)$ is integrable, then there exists $\alpha < 0$ such that (5.1) holds. The same conclusion holds for $E_{\phi,f}$ with $f : R^n \to K^r$ a restricted analytic map, extended by zero to a map $K^n \to K^r$, such that $f(R^n)$ has nonempty interior in K^r. $\hfill\square$

5.1

We end this section with an open question. Let $f = (f_1, \ldots, f_r) : K^n \to K^r$ be an analytic map given by r power series $f_1, \ldots, f_r \in K[[x]]$ which converge on the whole of K^n. Suppose that $\phi \in \mathcal{C}(K^n)$ is integrable and that $f(K^n)$ contains a nonempty open set. The question is whether there exists an $\alpha < 0$ such that

$$\int_{K^n} \phi(x)\psi\big(y \cdot f(x)\big)|dx| \ll \min\{|y|^\alpha, 1\}. \tag{5.2}$$

6   Polynomial mappings

In this section we use elementary methods to deduce explicit upper bounds for polynomial exponential sums. Theorem 6.1 below is of a nature different than our main Theorem 5.1 (and its proof is much more easy), in the sense that it uses the degree of the polynomial mapping as exponent in the upper bound. Such bound based on the degree would give a trivial bound when naively adapted to the analytic case. Similar problems occur when the explicit bounds of Loxton [12] are naively adapted to the analytic case. Since we use a result of [2] formulated there for polynomials over $\mathbb{Z}$, we will work over $\mathbb{Q}_p$.

For g a polynomial in $\mathbb{Q}_p[x]$ with $x = (x_1, \ldots, x_n)$, let $d_j(g)$ be the degree of g with respect to the variable x_j for $j = 1, \ldots, n$, and let $e(g)$ be the minimum of the p-adic orders of the coefficients of $g(x) - g(0)$. For $f = (f_1, \ldots, f_r)$ a tuple of polynomials in $\mathbb{Q}_p[x]$, let $d(f)$ be $\max_{ij}(d_j(f_i))$.

A function $\phi : \mathbb{Q}_p^n \to \mathbb{Q}$ is a Schwartz-Bruhat function if it is locally constant and has compact support. In this section we consider

$$E_{\phi,f}(y) = \int_{\mathbb{Q}_p^n} \phi(x)\psi\big(y \cdot f(x)\big)|dx|, \tag{6.1}$$

with $f = (f_1, \ldots, f_r)$ a tuple of polynomials in $\mathbb{Q}_p[x]$, $\phi : \mathbb{Q}_p^n \to \mathbb{Q}$ a Schwartz-Bruhat function, ψ a nontrivial additive character on $\mathbb{Q}_p$, and $y \in \mathbb{Q}_p^r$.

By elementary methods we easily deduce the following from the work of Čubarikov [2].

Theorem 6.1. Suppose that $f_1, \ldots, f_r$ are polynomials in $x = (x_1, \ldots, x_n)$ over $\mathbb{Q}_p$ which satisfy that $\sum_i a_i f_i + a_0 = 0$ implies $a_i = 0$ for $a_i \in \mathbb{Q}_p$ and $i = 0, \ldots, n$. Let $\phi : \mathbb{Q}_p^n \to \mathbb{Q}$ be a Schwartz-Bruhat function. Then, for any $\varepsilon > 0$,

$$E_{\phi,f}(y) \ll \min\left\{|y|^{\varepsilon - 1/d(f)}, 1\right\}. \tag{6.2}$$

Moreover, for y with $v(y) < 0$,

$$E_{\phi,f}(y) \ll \left(-v(y)\right)^{n-1} |y|^{-1/d(f)}. \tag{6.3}$$
$\square$

Proof. For simplicity, we may assume that $\psi(\mathbb{Z}_p) = 1$ and $\psi(x) \neq 1$ for $x \notin \mathbb{Z}_p$, and that at least one coefficient of $f_1(x) - f_1(0)$ has p-adic order 0. Since ϕ is a finite linear combination of characteristic functions of compact balls, we may moreover assume that ϕ is ϕ_{triv}, that is, the characteristic function of $\mathbb{Z}_p^n$. Čubarikov in [2, Lemma 3] proves that for any polynomial $g \in \mathbb{Z}[x]$ with $e(g) = 0$, $d(g) \leq d$ for some $d \in \mathbb{N}$, and each $z \in \mathbb{Q}_p$ with $v(z) < 0$, one has

$$\left|E_{\phi_{\text{triv}},g}(z)\right| < c(d,n)\left(-v(z)\right)^{n-1}|z|^{-1/d} \tag{6.4}$$

with $c(d,n)$ a constant depending only on d and n.

Rewrite $E_{\phi,f}(y)$ as

$$E'(z, u_1, \ldots, u_r) = \int_{\mathbb{Z}_p^n} \psi\left(z(u \cdot f(x))\right)|dx|, \tag{6.5}$$

with $z \in \mathbb{Q}_p$, $u \in \mathbb{Z}_p^r$ with $|u| = 1$, and $y = (zu_1, \ldots, zu_r)$. For any such u, the number $d(u \cdot f)$ cannot exceed $d(f)$. By the compactness of $\{u \in \mathbb{Z}_p^r \mid |u| = 1\}$, also the number $e(u \cdot f)$ is bounded uniformly in u, say, by N, since otherwise $\sum_i a_i f_i + a_0 = 0$ for some nontrivial $a_i \in \mathbb{Q}_p$.

One easily deduces from the mentioned result of [2] that for $v(z) < -N$,

$$\left|E'(z, u_1, \ldots, u_r)\right| < c\left(d(f),n\right)p^{N/d(f)}\left(-v(z)\right)^{n-1}|z|^{-1/d(f)}, \tag{6.6}$$

with $c(d(f),n)$ as above. The theorem follows. ∎

Remark 6.2. Note that from the proof of Theorem 6.1 and [2, Lemma 3], one can construct a (nonoptimal) constant c, depending only on ψ, ϕ, and f, such that for each y with $v(y) < 0$,

$$\left| E_{\phi,f}(y) \right| < c\left(-v(y) \right)^{n-1} |y|^{-1/d(f)}. \tag{6.7}$$

We leave the determination of the optimal c to a future work.

Acknowledgments

References

[1]   R. Cluckers, *Analytic p-adic cell decomposition and integrals*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1489–1499.

[2]   V. N. Čubarikov, *Multiple rational trigonometric sums and multiple integrals*, Mat. Zametki **20** (1976), no. 1, 61–68 (Russian), translated in Math. Notes **20** (1976), 589–593.

[3]   J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. **77** (1984), no. 1, 1–23.

[4]   ———, *p-adic semi-algebraic sets and cell decomposition*, J. reine angew. Math. **369** (1986), 154–166.

[5]   ———, *Multiplicity of the poles of the Poincaré series of a p-adic subanalytic set*, Séminaire de Théorie des Nombres (Talence, 1987/1988), University Bordeaux I, Talence, 1988, Exp. No. 43, pp. 1–8.

[6]   ———, *Report on Igusa's local zeta function*, Astérisque **201–203** (1991), 359–386, Séminaire Bourbaki, Vol. 1990/91 Exp. No. 741.

[7]   ———, *Arithmetic and geometric applications of quantifier elimination for valued fields*, Model Theory, Algebra, and Geometry, Math. Sci. Res. Inst. Publ., vol. 39, Cambridge University Press, Cambridge, 2000, pp. 173–198.

[8]   J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Ann. of Math. (2) **128** (1988), no. 1, 79–138.

[9]   J.-I. Igusa, *Lectures on Forms of Higher Degree. Notes by S. Raghavan*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 59, Springer-Verlag, New York, 1978.

[10]   B. Lichtin, *On a question of Igusa: towards a theory of several variable asymptotic expansions. I*, Compositio Math. **120** (2000), no. 1, 25–82.

[11]   ———, *On a question of Igusa: towards a theory of several variable asymptotic expansions. II*, to appear in Compositio Math.

[12]   J. H. Loxton, *Estimates for complete multiple exponential sums*, Acta Arith. **92** (2000), no. 3, 277–290.

[13]   L. van den Dries, D. Haskell, and D. Macpherson, *One-dimensional p-adic subanalytic sets*, J. London Math. Soc. (2) **59** (1999), no. 1, 1–20.

[14]   A. Weil, *L'Intègration dans les Groupes Topologiques et Ses Applications*, Actual. Sci. Ind., no. 869, Hermann et Cie., Paris, 1940.

[15]   ———, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87 (French).

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