Sato–Tate theorem for families and low-lying zeros of automorphic *L*-functions

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Abstract We consider certain families of automorphic representations over number fields arising from the principle of functoriality of Langlands. Let *G* be a reductive group over a number field *F* which admits discrete series representations at infinity. Let ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$ be the associated *L*-group and $r: {}^L G \to \mathrm{GL}(d, \mathbb{C})$ a continuous homomorphism which is irreducible and does not factor through $\mathrm{Gal}(\bar{F}/F)$. The families under consideration consist of discrete automorphic representations of $G(\mathbb{A}_F)$ of given weight and level and we let either the weight or the level grow to infinity. We establish a quantitative Plancherel and a quantitative Sato–Tate equidistribution theorem for the Satake parameters of these families. This generalizes earlier results in the subject, notably of Sarnak (Prog Math 70:321–331, 1987) and Serre (J Am Math Soc 10(1):75–102, 1997). As an application we study the distribution of the low-lying zeros of the associated family of *L*-functions $L(s, \pi, r)$, assuming from the principle of functoriality that these *L*-functions are automorphic. We find that the distribution of the 1-level densities coincides with the distribution of

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the 1-level densities of eigenvalues of one of the unitary, symplectic and orthogonal ensembles, in accordance with the Katz–Sarnak heuristics. We provide a criterion based on the Frobenius–Schur indicator to determine this symmetry type. If r is not isomorphic to its dual $r^\vee$ then the symmetry type is unitary. Otherwise there is a bilinear form on $\mathbb{C}^d$ which realizes the isomorphism between r and $r^\vee$. If the bilinear form is symmetric (resp. alternating) then r is real (resp. quaternionic) and the symmetry type is symplectic (resp. orthogonal).

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Contents

1 Introduction

The non-trivial zeros of automorphic L-functions are of central significance in modern number theory. Problems on individual zeros, such as the Riemann hypothesis (GRH), are elusive. There is however a theory of the statistical distribution of zeros in families. The subject has a long and rich history. A unifying modern viewpoint is that of a comparison with a suitably chosen model of random matrices: the Katz–Sarnak heuristics. There are both theoretical and numerical evidences for this comparison. Comprehensive results in the function field case [59] have suggested an analogous picture in the number field case as explained in [60]. In a large number of cases, and with high accuracy, the distribution of zeros of automorphic L-functions coincide with the distribution of eigenvalues of random matrices. See [37,85] for numerical investigations and conjectures and see [40,49,50,53,68,82,84] and the references therein for theoretical results.

The concept of families is central to modern investigations in number theory. We want to study in the present paper certain families of automorphic representations over number fields in a very general context. The families under consideration are obtained from the discrete spectrum by imposing constraints on the local components at archimedean and non-archimedean places and by applying Langlands global functoriality principle.

Our main result is a Sato–Tate equidistribution theorem for these families (Theorem 1.3). As an application of this main result we can give some evidence towards the Katz–Sarnak heuristics [60] in general and establish a criterion for the random matrix model attached to families, i.e. for the symmetry type.

1.1 Sato–Tate theorem for families

The original Sato–Tate conjecture is about an elliptic curve E, assumed to be defined over $\mathbb{Q}$ for simplicity. The number of points in $E(\mathbb{F}_p)$ for almost all primes p (with good reduction) gives rise to an angle θ_p between $-\pi$ and π. The conjecture, proved in [7], asserts that if E does not admit complex multiplication then $\{\theta_p\}$ are equidistributed according to the measure $\frac{2}{\pi} \sin^2 \theta d\theta$. In the context of motives a generalization of the Sato–Tate conjecture was formulated by Serre [96].

To speak of the automorphic version of the Sato–Tate conjecture, let G be a connected split reductive group over $\mathbb{Q}$ with trivial center and π an automorphic representation of $G(\mathbb{A})$. Here G is assumed to be split for simplicity (however we stress that our results are valid without even assuming that G is quasi-split; see Sect. 5 below for details). The triviality of center is not serious as it essentially amounts to fixing central character. Let T be a maximal split torus of G. Denote by $\widehat{T}$ its dual torus and Ω the Weyl group. As $\pi = \otimes'_v \pi_v$ is unramified at almost all places p, the Satake isomorphism identifies π_p with a point on $\widehat{T}/\Omega$. The automorphic Sato–Tate conjecture should be a prediction about the equidistribution of π_p on $\widehat{T}/\Omega$ with respect to a natural measure (supported on a compact subset of $\widehat{T}/\Omega$). It seems nontrivial to specify this measure in general. The authors do not know how to do it without invoking the (conjectural) global L-parameter for π. The automorphic Sato–Tate conjecture is known in the limited cases of (the restriction of scalars of) GL$_1$ and GL$_2$ [6,7]. In an ideal world the conjecture should be closely related to Langlands functoriality.

In this paper we consider the Sato–Tate conjecture for a *family* of automorphic representations, which is easier to state and prove but still very illuminating. Our working definition of a family $\{\mathcal{F}_k\}_{k \geqslant 1}$ is that each $\mathcal{F}_k$ consists of all automorphic representations π of $G(\mathbb{A})$ of level N_k with π_∞ cohomological of weight ξ_k, where $N_k \in \mathbb{Z}_{\geqslant 1}$ and ξ_k is an irreducible algebraic representation of G, such that either

(1) (level aspect) ξ_k is fixed, and $N_k \to \infty$ as $k \to \infty$ or
(2) (weight aspect) N_k is fixed, and $m(\xi_k) \to \infty$ as $k \to \infty$,

where $m(\xi_k) \in \mathbb{R}_{\geqslant 0}$ should be thought of as the minimal distance of the highest weight of ξ_k to root hyperplanes. (See Sect. 6.4 below for the precise definition.) Note that each $\mathcal{F}_k$ has finite cardinality and $|\mathcal{F}_k| \to \infty$ as $k \to \infty$. (For a technical reason $\mathcal{F}_k$ is actually allowed to be a multi-set. Namely the same representation can appear multiple times, for instance more than its automorphic multiplicity.) In principle we could let ξ_k and N_k vary simultaneously but decided not to do so in the current paper in favor of transparency of arguments. For instance families of type (i) and (ii) require somewhat different ingredients of proof in establishing the Sato–Tate theorem for families, and the argument would be easier to understand if we separate them. It should be possible to treat the mixed case (where both N_k and ξ_k vary) by combining techniques in the two cases (i) and (ii).

Let $\widehat{T}_c$ be the maximal compact subtorus of the complex torus $\widehat{T}$. The quotient $\widehat{T}_c/\Omega$ is equipped with a measure $\widehat{\mu}^{\mathrm{ST}}$, to be called the Sato–Tate measure, coming from the Haar measure on a maximal compact subgroup of $\widehat{G}$ (of which $\widehat{T}_c$ is a maximal torus). The following is a rough version of our result on the Sato–Tate conjecture for a family.

Theorem 1.1 *Suppose that $G(\mathbb{R})$ has discrete series representations. Let $\{\mathcal{F}_k\}_{k \geqslant 1}$ be a family in the level aspect (resp. weight aspect) as above. Let $\{p_k\}$ be a strictly increasing sequence of primes such that N_k (resp. ξ_k) grows faster than any polynomial in p_k in the sense that $\dfrac{\log p_k}{\log N_k} \to 0$ (resp. $\dfrac{\log p_k}{\log m(\xi_k)} \to 0$) as $k \to \infty$. Assume that the members of $\mathcal{F}_k$ are unramified at p_k for every k. Then the Satake parameters $\{\pi_{p_k} : \pi \in \mathcal{F}_k\}_{k \geqslant 1}$ are equidistributed with respect to $\widehat{\mu}^{\mathrm{ST}}$.*

To put things in perspective, we observe that there are three kinds of statistics about the Satake parameters of $\{\pi_{p_k} : \pi \in \mathcal{F}_k\}_{k \geqslant 1}$ depending on how the arguments vary.

  (i) Sato–Tate: $\mathcal{F}_k$ is fixed (and a singleton) and $p_k \to \infty$.
 (ii) Sato–Tate for a family: $|\mathcal{F}_k| \to \infty$ and $p_k \to \infty$.
(iii) Plancherel: $|\mathcal{F}_k| \to \infty$ and p_k is a fixed prime.

The Sato–Tate conjecture in its original form is about equidistribution in case (i) whereas our Theorem 1.1 is concerned with case (ii). The last item is marked as Plancherel since the Satake parameters are expected to be equidistributed with respect to the Plancherel measure (again supported on $\widehat{T}_c/\Omega$) in case (iii). This has been shown to be true under the assumption that $G(\mathbb{R})$ admits discrete series in [99]. We derive Theorem 1.1 from an error estimate (depending on k) on the difference between the Plancherel distribution at p and the actual

distribution of the Satake parameters at p_k in $\mathcal{F}_k$. This estimate (see Theorem 1.3 below) refines the main result of [99] and is far more difficult to prove in that several nontrivial bounds in harmonic analysis on reductive groups need to be justified.

1.2 Families of L-functions

An application of Theorem 1.1 is to families of L-functions. We are able to verify to some extent the heuristics of Katz and Sarnak [60] and determine the symmetry type, see Sect. 1.3 below. In this subsection we define the relevant families of L-functions and record some of their properties.

Let $r\colon {}^{L}G \to \mathrm{GL}(d, \mathbb{C})$ be a continuous L-homomorphism. We assume the Langlands functoriality principle: for all $\pi \in \mathcal{F}_k$ there exists an isobaric automorphic representation $\Pi = r_*\pi$ of $\mathrm{GL}(d, \mathbb{A})$ which is the functorial lift of the automorphic representation π of $G(\mathbb{A})$, see Sect. 4.3 for a review of the concept of isobaric representations and Sect. 10 for the precise statement of the hypothesis. This hypothesis is only used in Theorem 1.5, Sects. 11 and 12. By the strong multiplicity one theorem Π is uniquely determined by all but finitely many of its local factors $\Pi_v = r_*\pi_v$.

To an automorphic representation Π on $\mathrm{GL}(d, \mathbb{A})$ we associate its principal L-function $L(s, \Pi)$. By definition $L(s, \pi, r) = L(s, \Pi)$. By the theory of Rankin–Selberg integrals or by the integral representations of Godement–Jacquet, $L(s, \Pi)$ has good analytic properties: analytic continuation, functional equation, growth in vertical strips. In particular we know the existence and some properties of its non-trivial zeros, such as the Weyl's law (Sect. 4.4).

We denote by $\mathfrak{F}_k = r_*\mathcal{F}_k$ the set of all such $\Pi = r_*\pi$ for $\pi \in \mathcal{F}_k$. Since the strong multiplicity one theorem implies that Π is uniquely determined by its L-function $L(s, \Pi)$. We simply refer to $\mathfrak{F} = r_*\mathcal{F}$ as a family of L-functions.

In general there are many ways to construct interesting families of L-functions. In a recent manuscript [87], Sarnak attempts to sort out these constructions into a comprehensive framework and proposes[1] a working definition (see also [67]). The families of L-functions under consideration in the present paper fit well into that framework. Indeed they are harmonic families in the sense that their construction involves inputs from local and global harmonic analysis. Other types of families include geometric families constructed as Hasse–Weil L-functions of arithmetic varieties and Galois families associated to families of Galois representations.

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[1] Sarnak and the authors gave a more refined and updated framework in [89] while our paper was under review.

1.3 Criterion for the symmetry type

Katz and Sarnak [60] predict that one can associate a symmetry type to a family of L-functions. By definition the symmetry type is the random matrix model which is conjectured to govern the distribution of the zeros. There is a long and rich history for the introduction of this concept.

Hilbert and Pólya suggested that there might be a spectral interpretation of the zeros of the Riemann zeta function. Nowadays strong evidence for the spectral nature of the zeros of L-functions comes from the function field case: zeros are eigenvalues of the Frobenius acting on cohomology. This is exemplified by the equidistribution theorem of Deligne and the results of Katz and Sarnak [59] on the distribution of the low-lying eigenvalues in geometric families.

In the number field case the first major result towards a spectral interpretation is the pair correlation of high zeros of the Riemann zeta function by Montgomery. Developments then include Odlyzko's extensive numerical study and the determination of the n-level correlation by Hejhal and Rudnick and Sarnak [86]. The number field analogue of the Frobenius eigenvalue statistics of [59] concerns the statistics of low-lying zeros.

More precisely [60] predicts that the low-lying zeros of families of L-functions are distributed according to a determinantal point process associated to a random matrix ensemble. This will be explained in more details in Sects. 1.5 and 1.6 below. We shall distinguish between the three determinantal point processes associated to the unitary, symplectic and orthogonal ensembles.[2] Accordingly the symmetry type associated to a family $\mathfrak{F}$ is defined to be unitary, symplectic or orthogonal (see Sect. 1.6 for typical results).

Before entering into the details of this theory in Sect. 1.5 below, we state here our criterion for the symmetry type of the harmonic families $r_*\mathcal{F}$ defined above. We recall in Sect. 6.8 the definition of the Frobenius–Schur indicator $s(r) \in \{-1, 0, 1\}$ associated to an irreducible representation. We shall prove that the symmetry type is determined by $s(r)$. This is summarized in the following which may be viewed as a refinement of the Katz–Sarnak heuristics.

Criterion 1.2 *Let $r : {}^L G \to \mathrm{GL}(d, \mathbb{C})$ be a continuous L-homomorphism which is irreducible and non-trivial when restricted to $\widehat{G}$. Consider the family $r_*\mathcal{F}$ of automorphic L-functions of degree d as above.*

(i) *If r is not isomorphic to its dual $r^\vee$ then $s(r) = 0$ and the symmetry type is unitary.*

(ii) *Otherwise there is a bilinear form on $\mathbb{C}^d$ which realizes the isomorphism between r and $r^\vee$. By Schur lemma it is unique up to scalar and is either symmetric or alternating. If it is symmetric then r is real, $s(r) = 1$ and*

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[2] In this paper we do not distinguish in the orthogonal ensemble between the O, SO(*odd*) and SO(*even*) symmetries. We will return to this question in a subsequent work.

*the symmetry type is symplectic. If it is alternating then r is quaternionic,
$s(r) = -1$ and the symmetry type is orthogonal.*

We note that the conditions that r be irreducible and non-trivial when restricted
to $\widehat{G}$ are optimal. If r were trivial when restricted to $\widehat{G}$ then $L(s, \pi, r)$ would be
constant and equal to a single Artin L-function and the low-lying zeros would
correspond to the eventual vanishing of this Artin L-function at the central
point (which is a different problem). Also the universality exhibited in our
criterion may be compared with the GUE universality of the high zeros of [86].

If r were reducible then the L-functions would factor as a product
$L(s, \pi, r_1)L(s, \pi, r_2)$. Suppose that both r_1 and r_2 are irreducible and non-
trivial when restricted to $\widehat{G}$. If $r_1 = r_2$ then clearly the distribution of zeros
will be as before but with multiplicity two. If $r_1 \not\simeq r_2$ then we expect that the
zeros will follow the distribution of the *independent* superposition of the two
random matrix ensembles attached to r_1 and r_2. In other words the zeros of
$L(s, \pi, r_1)$ are uncorrelated to the zeros of $L(s, \pi, r_2)$, and one could verify
this using the methods of this paper to some extent. In particular we expect no
repulsion between the respective sequences of zeros.

It would be interesting to study families of automorphic representations over
a function field $k = \mathbb{F}_q(X)$ of a curve X. To our knowledge the Katz–Sarnak
heuristics for such families are not treated in the literature, except in the case
of $G = GL(1)$ where harmonic families coincide with the geometric families
treated by Katz–Sarnak (e.g. Dirichlet L-series with quadratic character are
the geometric families of hyperelliptic curves in [59, §10]). Over function
fields our criterion has the following interpretation. We consider families of
automorphic representations π of $G(\mathbb{A}_k)$; for simplicity we suppose that each
automorphic representations π of $G(\mathbb{A}_k)$ in the family $\mathcal{F}$ is attached to an
irreducible ℓ-adic representation $\rho\colon \mathrm{Gal}(k^{sep}/k) \to {}^{L}G$. Then $r_*\pi$ is attached
to the Galois representation $r \circ \rho$, and corresponds to a constructible ℓ-adic
sheaf F of dimension d on the curve X. The zeros of the L-function $L(s, \pi, r)$
are the eigenvalues of Frobenius on the first cohomology, more precisely the
numerator of the L-function $L(s, \pi, r)$ is

$$\det(1 - q^{-s}\mathrm{Fr}|H^1(X, F)).$$

If $s(r) = -1$ [resp. $s(r) = 1$] then there is an alternating (resp. symmet-
ric) pairing on the sheaf F. The natural pairing on $H^1(X, F)$ induced by the
cup product is symmetric (resp. alternating) and invariant by the action of
Frobenius. Thus the zeros of $L(s, \pi, r)$ are the eigenvalues of an orthogonal
(resp. symplectic) matrix. This is in agreement with the assertion (ii) of our
Criterion 1.2. We also note the related situation [58].

Known analogies between L-functions and their symmetries over number
fields and function fields are discussed in [60, §4]. Overall we would like

propose Criterion 1.2 and its analogue for geometric families as an answer to the question mark in the entry 6-A of Table 2 in [60].

1.4 Automorphic Plancherel density theorem with error bounds

We explain a more precise version of the theorem and method of proof for the Sato–Tate theorem for families (Sect. 1.1). The key is to bound the error terms when we approximate the distribution of local components of automorphic representations in a family with the Plancherel measure.

For simplicity of exposition let us assume that G is a split reductive group over $\mathbb{Q}$ with trivial center as in Sect. 1.1. A crucial hypothesis is that $G(\mathbb{R})$ admits an $\mathbb{R}$-anisotropic maximal torus [in which case $G(\mathbb{R})$ admits discrete series representations]. Let $\mathcal{A}_{\mathrm{disc}}(G)$ denote the set of isomorphism classes of discrete automorphic representations of $G(\mathbb{A})$. We say that $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ has level N and weight ξ if π has a nonzero fixed vector under the adelic version of the full level N congruence subgroup $K(N) \subset G(\mathbb{A}^\infty)$ and if $\pi_\infty \otimes \xi$ has nonzero Lie algebra cohomology. In this subsection we make a further simplifying hypothesis that ξ has regular highest weight, in which case π_∞ as above must be a discrete series representation. (In the main body of this paper, the latter assumption on ξ is necessary only for the results in Sects. 9.6–9.8, where more general test functions are considered)

Define $\mathcal{F} = \mathcal{F}(N, \xi)$ to be the finite multi-set consisting of $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ of level N and weight ξ, where each such π appears in $\mathcal{F}$ with multiplicity

$$a_{\mathcal{F}}(\pi) := \dim(\pi^\infty)^{K(N)} \in \mathbb{Z}_{\geqslant 0}.$$

This quantity naturally occurs as the dimension of the π-isotypical subspace in the cohomology of the locally symmetric space for G of level N with coefficient defined by ξ. The main motivation for allowing π to appear $a_{\mathcal{F}}(\pi)$ times is to enable us to compute the counting measure below with the trace formula.

Let p be a prime number. Write $G(\mathbb{Q}_p)^\wedge$ for the unitary dual of irreducible smooth representations of $G(\mathbb{Q}_p)$. The unramified (resp. unramified and tempered) part of $G(\mathbb{Q}_p)^\wedge$ is denoted $G(\mathbb{Q}_p)^{\wedge,\mathrm{ur}}$ [resp. $G(\mathbb{Q}_p)^{\wedge,\mathrm{ur,temp}}$]. There is a canonical isomorphism

$$G(\mathbb{Q}_p)^{\wedge,\mathrm{ur,temp}} \simeq \widehat{T}_c/\Omega. \tag{1.1}$$

The unramified Hecke algebra of $G(\mathbb{Q}_p)$ will be denoted $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$. There is a map from $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ to the space of continuous functions on $\widehat{T}_c/\Omega$:

$$\phi \mapsto \widehat{\phi} \text{ determined by } \widehat{\phi}(\pi) = \mathrm{tr}\,\pi(\phi), \quad \forall \pi \in G(\mathbb{Q}_p)^{\wedge,\mathrm{ur,temp}}.$$

There are two natural measures supported on $\widehat{T}_c/\Omega$. The Plancherel measure $\widehat{\mu}_p^{\mathrm{pl,ur}}$, dependent on p, is defined on $G(\mathbb{Q}_p)^{\wedge,\mathrm{ur}}$ and naturally arises in local

harmonic analysis. The Sato–Tate measure $\widehat{\mu}^{\mathrm{ST}}$ on $\widehat{T}_c/\Omega$ is independent of p and may be extended to $G(\mathbb{Q}_p)^{\wedge,\mathrm{ur}}$ by zero. Both $\widehat{\mu}_p^{\mathrm{pl},\mathrm{ur}}$ and $\widehat{\mu}^{\mathrm{ST}}$ assign volume 1 to $\widehat{T}_c/\Omega$. There is yet another measure $\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}$ on $G(\mathbb{Q}_p)^{\wedge,\mathrm{ur}}$, which is the averaged counting measure for the p-components of members of $\mathcal{F}$. Namely

$$\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}} := \frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{F}} \delta_{\pi_p} \tag{1.2}$$

where δ_{π_p} denotes the Dirac delta measure supported at π_p. [Each $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ contributes $a_{\mathcal{F}}(\pi)$ times to the above sum.] Our primary goal is to bound the difference between $\widehat{\mu}_p^{\mathrm{pl},\mathrm{ur}}$ and $\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}$. [Note that our definition of $\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}$ in the main body will be a little different from (1.2) but asymptotically the same, see Remark 9.9.]

In order to quantify error bounds, we introduce a filtration $\{\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\leqslant\kappa}\}_{\kappa\in\mathbb{Z}_{\geqslant 0}}$ on $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ as a complex vector space. The filtration is increasing, exhaustive and depends on a non-canonical choice. Roughly speaking, $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\leqslant\kappa}$ is like the span of all monomials of degree $\leqslant \kappa$ when $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ is identified with (a subalgebra of) a polynomial algebra. For each ξ, it is possible to assign a positive integer $m(\xi)$ in terms of the highest weight of ξ. When we say that weight is going to infinity, it means that $m(\xi)$ grows to ∞ in the usual sense.

The main result on error bounds alluded to above is the following. (See Theorems 9.16 and 9.19 for the precise statements and Remarks 9.18 and 9.21 for an explicit choice of constants.) A uniform bound on orbital integrals, cf. (1.9) below, enters the proof of (ii) [but not (i)].

Theorem 1.3 *Let $\mathcal{F} = \mathcal{F}(N,\xi)$ be as above. Consider a prime p, an integer $\kappa \geqslant 1$, and a function $\phi_p \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\leqslant\kappa}$ such that $|\phi_p| \leqslant 1$ on $G(\mathbb{Q}_p)$.*

(i) (*level aspect*) *Suppose that ξ remains fixed. There exist constants $A_{\mathrm{lv}}, B_{\mathrm{lv}}, C_{\mathrm{lv}} > 0$ depending only on G such that for any p, κ, ϕ_p as above and for any N coprime to p,*

$$\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}(\widehat{\phi}_p) - \widehat{\mu}_p^{\mathrm{pl},\mathrm{ur}}(\widehat{\phi}_p) = O(p^{A_{\mathrm{lv}}+B_{\mathrm{lv}}\kappa} N^{-C_{\mathrm{lv}}}).$$

(ii) (*weight aspect*) *Fix a level N. There exist constants $A_{\mathrm{wt}}, B_{\mathrm{wt}}, C_{\mathrm{wt}} > 0$ and a lower bound $c > 0$ depending only on G such that for any $p \geqslant c$, κ, ϕ_p as above with $(p, N) = 1$ and for any ξ,*

$$\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}(\widehat{\phi}_p) - \widehat{\mu}_p^{\mathrm{pl},\mathrm{ur}}(\widehat{\phi}_p) = O(p^{A_{\mathrm{wt}}+B_{\mathrm{wt}}\kappa} m(\xi)^{-C_{\mathrm{wt}}}).$$

Let $\{\mathcal{F}_k = \mathcal{F}(N_k, \xi_k)\}_{k\geqslant 1}$ be either kind of family in Sect. 1.1, namely either $N_k \to \infty$ and ξ_k is fixed or N_k is fixed and $\xi_k \to \infty$. When applied to

$\{\mathcal{F}_k\}_{k \geqslant 1}$, Theorem 1.3 leads to the equidistribution results in the following corollary [cf. cases (ii) and (iii) in the paragraph below Theorem 1.1]. Indeed, (i) of the corollary is immediate. Part (ii) is easily derived from the fact that $\widehat{\mu}_p^{\text{pl,ur}}$ weakly converges to $\widehat{\mu}^{\text{ST}}$ as $p \to \infty$. Although the unramified Hecke algebra at p gives rise to only regular functions on the complex variety $\widehat{T}_c/\Omega$, it is not difficult to extend the results to continuous functions on $\widehat{T}_c/\Omega$. (See Sects. 9.6–9.8 for details.)

Corollary 1.4 *Keep the notation of Theorem 1.3. Let $\widehat{\phi}$ be a continuous function on $\widehat{T}_c/\Omega$. In view of (1.1) $\widehat{\phi}$ can be extended by zero to a function $\widehat{\phi}_p$ on $G(\mathbb{Q}_p)^{\wedge,\text{ur}}$ for each prime p.*

(i) (*Automorphic Plancherel density theorem [99]*)

$$\lim_{k \to \infty} \widehat{\mu}_{\mathcal{F}_k,p}^{\text{count}}(\widehat{\phi}_p) = \widehat{\mu}_p^{\text{pl,ur}}(\widehat{\phi}_p).$$

(ii) (*Sato–Tate theorem for families*) *Let $\{p_k\}_{k \geqslant 1}$ be a sequence of primes tending to ∞. Suppose that $\dfrac{\log p_k}{\log N_k} \to 0$ (resp. $\dfrac{\log p_k}{\log m(\xi_k)} \to 0$) as $k \to \infty$ if ξ_k (resp. N_k) remains fixed as k varies. Then*

$$\lim_{k \to \infty} \widehat{\mu}_{\mathcal{F}_k,p_k}^{\text{count}}(\widehat{\phi}_{p_k}) = \widehat{\mu}^{\text{ST}}(\widehat{\phi}).$$

Theorem 1.3 and Corollary 1.4 remain valid if any finite number of primes are simultaneously considered in place of p or p_k. Moreover (i) of the corollary holds true for more general (and possibly ramified) test functions $\widehat{\phi}_p$ on $G(\mathbb{Q}_p)^{\wedge}$ thanks to Sauvageot's density theorem. It would be interesting to quantify the error bounds in this generality. Finally the above results should be compared with the proposition 4 in [97] and the theorem 1 in [78] for modular forms on GL(2). We also note [90] for Maass forms (which are not considered in the the present paper).

1.5 Random matrices

We provide a brief account of the theory of random matrices. The reader will find more details in Sect. 11.1 and extensive treatments in [59,74].

The Gaussian unitary ensemble and Gaussian orthogonal ensemble were introduced by Wigner in the study of resonances of heavy nucleus. The Gaussian symplectic ensemble was introduced later by Dyson together with his circular ensembles. In this paper we are concerned with the ensembles attached to compact Lie groups which are introduced by Katz–Sarnak and occur in the statistics of L-functions. (See [39] for the precise classification of these ensembles attached to different Riemannian symmetric spaces.)

One considers eigenvalues of matrices in compact groups $\mathcal{G}(N)$ of large dimension endowed with the Haar probability measure. We have three symmetry types $\mathcal{G} = \mathrm{SO}(even)$ (resp. $\mathcal{G} = \mathrm{U}$, $\mathcal{G} = \mathrm{USp}$); the notation says that for all $N \geqslant 1$, the groups are $\mathcal{G}(N) = \mathrm{SO}(2N)$ [resp. $\mathcal{G}(N) = \mathrm{U}(N)$ and $\mathcal{G}(N) = \mathrm{USp}(2N)$].

For all matrices $A \in \mathcal{G}(N)$ we have an associated sequence of normalized angles

$$0 \leqslant \vartheta_1 \leqslant \vartheta_2 \leqslant \cdots \leqslant \vartheta_N \leqslant N. \tag{1.3}$$

For example in the case $\mathcal{G} = \mathrm{U}$, the eigenvalues of $A \in \mathrm{U}(N)$ are given by $e(\frac{\vartheta_j}{N}) = e^{2i\pi \vartheta_j / N}$ for $1 \leqslant j \leqslant N$. The normalization is such that the mean spacing of the (ϑ_i) in (1.3) is about one.

For each $N \geqslant 1$ these angles $(\vartheta_i)_{1 \leqslant i \leqslant N}$ are correlated random variables (a point process). By the Weyl integration formula their joint density is proportional to

$$\prod_{1 \leqslant i < j \leqslant N} \left| \sin \left(\frac{\pi (\vartheta_i - \vartheta_j)}{N} \right) \right|^{\beta} d\vartheta_1 \ldots d\vartheta_N. \tag{1.4}$$

The parameter β is a measure of the repulsion between nearby eigenvalues. We have that $\beta = 1$ (resp. $\beta = 2$, $\beta = 4$) for $\mathcal{G} = \mathrm{SO}(even)$ (resp. $\mathcal{G} = \mathrm{U}$, $\mathcal{G} = \mathrm{USp}$).

A fundamental result of Gaudin–Mehta and Dyson, which has been extended to the above ensembles by Katz–Sarnak, is that when $N \to \infty$ the distribution of the angles $(\vartheta_i)_{1 \leqslant i \leqslant N}$ converges to a determinantal point process.[3] The kernel of the limiting point process when $\mathcal{G} = \mathrm{U}$ is given by the Dyson sine kernel

$$K(x, y) = \frac{\sin \pi (x - y)}{\pi (x - y)}, \quad x, y \in \mathbb{R}_+$$

The kernel for $\mathcal{G} = \mathrm{SO}(even)$ is $K_+(x, y) = K(x, y) + K(-x, y)$ and the kernel for $\mathcal{G} = \mathrm{USp}$ is $K_-(x, y) = K(x, y) - K(-x, y)$.

In particular this means that there is a limiting 1-level density $W(\mathcal{G})$ for the angles $(\vartheta_i)_{1 \leqslant i \leqslant N}$ as $N \to \infty$ (see also Proposition 11.1). It is given by the following formulas:

$$
\begin{aligned}
W(\mathrm{SO}(even))(x) &= K_+(x, x) = 1 + \frac{\sin 2\pi x}{2\pi x}, \\
W(\mathrm{U})(x) &= K(x, x) = 1, \\
W(\mathrm{USp})(x) &= K_-(x, x) = 1 - \frac{\sin 2\pi x}{2\pi x}.
\end{aligned} \tag{1.5}
$$

---

[3] For other values of $\beta \neq 1, 2, 4$, the limiting statistics attached to (1.4) has been determined recently by Valkó–Virág in terms of the Brownian carousel.

1.6 Low-lying zeros

We can now state more precisely our results on families of L-functions. Let $\mathfrak{F} = r_* \mathcal{F}$ be a family of L-functions as defined above in Sects. 1.1–1.2.

For all $\Pi \in \mathfrak{F}_k$ we denote by $\rho_j(\Pi)$, the zeros of the completed L-function $\Lambda(s, \Pi)$, where $j \in \mathbb{Z}$. We write $\rho_j(\Pi) = \frac{1}{2} + i\gamma_j(\Pi)$ and therefore $-\frac{1}{2} < \mathrm{Re}\gamma_j(\Pi) < \frac{1}{2}$ for all j. By the functional equation $\Lambda(\frac{1}{2} + i\gamma, \Pi) = 0$ if and only if $\Lambda(\frac{1}{2} + i\overline{\gamma}, \Pi) = 0$. We do not assume the GRH that would further imply $\gamma_j(Pi) \in \mathbb{R}$ for all j.

In the case that Π is self-dual the zeros occur in complex pairs, namely $L(\frac{1}{2} + i\gamma, \Pi) = 0$ if and only if $\Lambda(\frac{1}{2} - i\gamma, \Pi) = 0$.

Following Iwaniec–Sarnak we associate an analytic conductor $C(\mathfrak{F}_k) \geqslant 1$ to the family, see Sects. 4.2 and 11.5. We assume from now that the family is in the weight aspect, so that for each $k \geqslant 1$, all of $\Pi \in \mathfrak{F}_k$ share the same archimedean factor Π_∞ and we can set $C(\mathfrak{F}_k) := C(\Pi_\infty)$. (For families in the level aspect we obtain similar results, see Sect. 11). Note that $C(\mathfrak{F}_k) \to \infty$ and furthermore we shall make the assumption that $\log C(\mathfrak{F}_k) \asymp \log m(\xi_k)$ as $k \to \infty$.

For a given $\Pi \in \mathfrak{F}_k$ the number of zeros $\gamma_j(\Pi)$ of bounded height is $\asymp \log C(\mathfrak{F}_k)$. The low-lying zeros of $\Lambda(s, \Pi)$ are those within distance $O(1/\log(C(\mathfrak{F}_k))$ to the central point; heuristically there are a bounded number of low-lying zeros for a given $\Pi \in \mathfrak{F}_k$, although this can only be proved on average over the family. For a technical reason related to the fact that the explicit formula counts both the zeros and poles of $\Lambda(s, \Pi)$ (Sect. 4.4), we make an hypothesis on the occurrence of poles of $\Lambda(s, \Pi)$ for $\Pi \in \mathfrak{F}_k$, see Hypothesis 11.2.

The statistics of low-lying zeros of the family are studied via the functional

$$D(\mathfrak{F}_k; \Phi) = \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \sum_j \Phi\left(\frac{\gamma_j(\Pi)}{2\pi} \log C(\mathfrak{F}_k)\right), \qquad (1.6)$$

where Φ is a Paley–Wiener function. This is the 1-level density for the family $\mathfrak{F}_k$. Choosing Φ as a smooth approximation of the characteristic function of an interval $[a, b]$, the sum (1.6) should be thought as a weighted count of all the zeros of the family lying in $[a, b]$:

$$\frac{2a\pi}{\log C(\mathfrak{F}_k)} \leqslant \gamma_j(\Pi) \leqslant \frac{2b\pi}{\log C(\mathfrak{F}_k)}, \quad (j \in \mathbb{Z}, \Pi \in \mathcal{F}_k).$$

We want to compare the asymptotic as $k \to \infty$ with the limiting 1-level density of normalized angles (1.3) of the random matrix ensembles described in Sect. 1.5 above.

Theorem 1.5 *Let $r : {}^L G :\to \mathrm{GL}(d, \mathbb{C})$ be a continuous L-homomorphism which is irreducible and non-trivial when restricted to $\widehat{G}$. There exists $\delta > 0$ depending on $\mathfrak{F}$ such that the following holds. Let $\mathfrak{F} = r_* \mathcal{F}$ be a family of L-functions in the weight aspect as in Sect. 1.2, assuming the functoriality conjecture as in Hypothesis 10.1. Assume Hypothesis 11.2 concerning the poles of $\Lambda(s, \Pi)$ for $\Pi \in \mathfrak{F}_k$. Then for all Paley-Wiener functions Φ whose Fourier transform $\widehat{\Phi}$ has support in $(-\delta, \delta)$:*

(i) *there is a limiting 1-level density for the low-lying zeros, namely there is a density $W(x)$ such that*

$$\lim_{k \to \infty} D(\mathfrak{F}_k; \Phi) = \int_{-\infty}^{\infty} \Phi(x) W(x) dx;$$

(ii) *the density $W(x)$ is determined by the Frobenius–Schur indicator of the irreducible representation r. Precisely,*

$$W = \begin{cases} W(\mathrm{SO}(even)), & \text{if } s(r) = -1, \\ W(\mathrm{U}), & \text{if } s(r) = 0, \\ W(\mathrm{USp}), & \text{if } s(r) = 1. \end{cases} \tag{1.7}$$

The constant $\delta > 0$ depends on the family $\mathfrak{F}$, in other words it depends on the group G, the L-morphism $r : {}^L G \to \mathrm{GL}(d, \mathbb{C})$ and the limit of the ratio $\dfrac{\log C(\mathfrak{F}_k)}{\log m(\xi_k)}$. Its numerical value is directly related to the numerical values of the exponents in the error term occurring in Theorem 1.3. Although we do not attempt to do so in the present paper, it is interesting to produce a value of δ that is as large as possible, see [53] for the case of GL(2). This would require sharp bounds for orbital integrals as can be seen from the outline below. A specific problem would be to optimize the exponents a, b, e in (1.9). (In fact we can achieve $e = 1$, see Sect. 1.7 below.)

Our proofs of Theorems 1.3 and 1.5 are effective in the sense that each constant and each exponent in the statements of the estimates could, in principle, be made explicit. Finally we note that, refining the work of E. Royer, Cogdell and Michel [31] have studied the question of distribution of L-values at the edge in the case of symmetric powers of GL(2) and noted in that context the relevance of the indicator $s(r)$.

1.7 Outline of proofs

A wide range of methods are used in the proof. Among them are the Arthur-Selberg trace formula, the analytic theory of L-functions, representation theory and harmonic analysis on p-adic and real groups, and random matrix theory.

The first main result of our paper is Theorem 1.3, proved in Sect. 9. We already pointed out after stating the theorem that the Sato–Tate equidistribution for families (Corollary 1.4) is derived from Theorem 1.3 and the fact that the Plancherel measure tends to the Sato–Tate measure as the residue characteristic is pushed to ∞.

Let us outline the proof of the theorem. In fact we restrict our attention to part (ii), as (i) is handled by a similar method and only simpler to deal with. Thus we consider $\mathcal{F}$ with fixed level and weight ξ, where ξ is regarded as a variable. Our starting point is to realize that for $\phi_p \in C_c^\infty(G(\mathbb{Q}_p))$, we may interpret $\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}(\widehat{\phi}_p)$ in terms of the spectral side of the trace formula for G evaluated against the function $\phi_p \phi^{\infty,p} \phi_\infty \in C_c^\infty(G(\mathbb{A}))$ for a suitable $\phi^{\infty,p}$ (depending on $\mathcal{F}$ and p; note that p is allowed to vary) and an Euler–Poincaré function ϕ_∞ at ∞ (depending on ξ). Applying the trace formula, which has a simple form thanks to ϕ_∞, we get a geometric expansion for $\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}(\widehat{\phi}_p)$:

$$\widehat{\mu}_{\mathcal{F},p}^{\mathrm{count}}(\widehat{\phi}_p) = \sum_{\substack{M \subset G \\ \mathrm{cusp.Levi}}} \sum_{\substack{\gamma \in M(\mathbb{Q})/\sim \\ \mathbb{R}-ell}} a'_{M,\gamma} \cdot O_\gamma^{M(\mathbb{A}^\infty)}(\phi_M^\infty) \frac{\Phi_M^G(\gamma,\xi)}{\dim \xi}. \quad (1.8)$$

where $a'_{M,\gamma} \in \mathbb{C}$ is a coefficient encoding a certain volume associated with the connected centralizer of γ in M and ϕ_M^∞ is the constant term of ϕ^∞ along (a parabolic subgroup associated with) M. The Plancherel formula identifies the term for $M = G$ and $\gamma = 1$ with $\widehat{\mu}_p^{\mathrm{pl}}(\widehat{\phi}_p)$, which basically dominates the right hand side.

The proof of Theorem 1.3 (ii) boils down to bounding the other terms on the right hand side of (1.8). Here is a rough explanation of how to analyze each component there. The first summation is finite and controlled by G, so we may as well look at the formula for each M. There are finitely many conjugacy classes in the second summation for which the summand is nonzero. The number of such conjugacy classes may be bounded by a power of p where the exponent of p depends only on κ (measuring the "complexity" of ϕ_p). The term $a'_{M,\gamma}$, when unraveled, involves a special value of some Artin L-function. We establish a bound on the special value which suffices to deal with $a'_{M,\gamma}$. The last term $\frac{\Phi_M^G(\gamma,\xi)}{\dim \xi}$ can be estimated by using a character formula for the stable discrete series character $\Phi_M^G(\gamma,\xi)$ as well as the dimension formula for ξ. It remains to take care of $O_\gamma^{M(\mathbb{A}^\infty)}(\phi_M^\infty)$. This turns out to be the most difficult task since Theorem 1.3 asks for a bound that is *uniform as the residue characteristic varies*.

We are led to prove that there exist $a, b, e > 0$, depending only on G, such that for almost all q,

$$\left| O_\gamma^{M(\mathbb{Q}_q)}(\phi_q) \right| \leqslant q^{a+b\kappa} D^M(\gamma)^{-e/2} \qquad (1.9)$$

for all semisimple γ and all ϕ_q with $\phi_q \in \mathcal{H}^{\mathrm{ur}}(M(\mathbb{Q}_q))^{\leqslant \kappa}$ and $|\phi_q| \leqslant 1$, where $D^M(\cdot)$ denotes the Weyl discriminant. The justification of (1.9) takes up the whole of Sect. 7. The problem already appears to be deep for the unit elements of unramified Hecke algebras in which case one can take $\kappa = 0$. (By a different argument based on arithmetic motivic integration, Cluckers, Gordon, and Halupczok establish a stronger uniform bound with $e = 1$. This work is presented in Appendix B.) At the (fixed) finite set of primes where wild ramification occurs, the problem comes down to bounding the orbital integral $|O_\gamma^{M(\mathbb{Q}_q)}(\phi_q)|$ for fixed q and ϕ_q. It is deduced from the Shalika germ theory that the orbital integral is bounded by a constant, if normalized by the Weyl discriminant $D^M(\gamma)^{1/2}$, as γ runs over the set of semisimple elements. See Appendix A by Kottwitz for details.

We continue with Theorem 1.5. The proof relies heavily on Theorem 1.3. The connection between the two statements might not be immediately apparent.

A standard procedure based on the explicit formula (see Sect. 4) expresses the sum (1.6) over zeros of L-function as a sum over prime numbers of Satake parameters. The details are to be found in Sect. 12, and the result is that $D(\mathfrak{F}_k, \Phi)$ can be approximated by

$$\sum_{\mathrm{prime}\ p} \widehat{\mu}_{\mathcal{F}_k, p}^{\mathrm{count}}(\widehat{\phi}_p) \Phi\left(\frac{\log p}{\pi \log C(\mathfrak{F}_k)} \right). \qquad (1.10)$$

Here $\phi_p \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\leqslant \kappa}$ is suitably chosen such that $\widehat{\phi}_p(\pi_p)$ is a sum of powers of the Satake parameters of $r_* \pi$ (see Sects. 2 and 3). The integer κ may be large but it depends only on r so should be considered as fixed. Also the sum is over unramified primes. We have $\log C(\mathcal{F}_k) \asymp \log m(\xi_k)$ (see Sects. 10 and 11). We deduce that the sum is supported on those primes $p \leqslant m(\xi_k)^{A\delta}$ where A is a suitable constant and δ is as in Theorem 1.5.

We apply Theorem 1.3 which has two components: the main term and the error term. We begin with the main term which amounts to substituting $\widehat{\mu}_p^{\mathrm{pl, ur}}(\widehat{\phi}_p)$ for $\widehat{\mu}_{\mathcal{F}_k, p}^{\mathrm{count}}(\widehat{\phi}_p)$ in (1.10). Unlike $\widehat{\mu}_{\mathcal{F}_k, p}^{\mathrm{count}}$, this term is purely local, thus simpler. Indeed $\widehat{\mu}_p^{\mathrm{pl, ur}}(\widehat{\phi}_p)$ can be computed explicitly for low rank groups, see e.g. [48] for all the relevant properties of the Plancherel measure. However we want to establish Theorem 1.5 in general so we proceed differently.

Using certain uniform estimates by Kato [57], we can approximate $\widehat{\mu}_p^{\mathrm{pl, ur}}(\widehat{\phi}_p)$ by a much simpler expression that depends directly on the restriction of r to $\widehat{G} \rtimes W_{\mathbb{Q}_p}$. Then a pleasant computation using the Cebotarev equidistribution theorem, Weyl's unitary trick and the properties of the Frobenius–

Schur indicator shows that the sum over primes of this main term contribute $\frac{-s(r)}{2}\Phi(0)$ to (1.10). This exactly reflects the identities (1.7) in the statement (ii) of Theorem 1.5.

We continue with the error term $O(p^{A_{\text{wt}}+B_{\text{wt}}\kappa}m(\xi_k)^{-C_{\text{wt}}})$ which we need to insert in (1.10). We can see the reasons why the proof of Theorem 1.5 requires the full force of Theorem 1.3 and its error term: the polynomial control by $p^{A_{\text{wt}}+B_{\text{wt}}\kappa}$ implies that the sum over primes is at most $m(\xi_k)^{D\delta}$ for some $D > 0$; the power saving $m(\xi_k)^{-C_{\text{wt}}}$ is exactly what is needed to beat $m(\xi_k)^{D\delta}$ when δ is chosen small enough.

1.8 Notation

We distinguish the letter $\mathcal{F}$ for families of automorphic representations on general reductive groups and $\mathfrak{F} = r_*\mathcal{F}$ for the families of automorphic representations on $\text{GL}(d)$.

Let us describe in words the significance of various constants occurring in the main statements. We often use the convention to write multiplicative constants in lowercase letters and constants in the exponents in uppercase or greek letters.

- The exponent β from Lemma 2.6 is such that for all $\phi \in \mathcal{H}^{\text{ur}}(\text{GL}_d)$ of degree at most κ, the pullback $r^*\phi$ is of degree at most $\leqslant \beta\kappa$.
- The exponent b_G from Lemma 2.14 controls a bound for the constant term $|\phi_M(1)|$ for all Levi subgroups M and $\phi \in \mathcal{H}^{\text{ur}}(G)$ of degree at most κ.
- The exponent $0 < \theta < \frac{1}{2}$ is a nontrivial bound towards Ramanujan-Petersson for $\text{GL}(d, \mathbb{A})$.
- The integer $i \geqslant 1$ in Corollary 6.9 is an upper-bound for the ramification of the Galois group $\text{Gal}(E/F)$.
- The constants B_{Ξ} and c_{Ξ} in Lemma 8.4 and A_3, B_3 in Proposition 8.7 control the number of rational conjugacy classes intersecting a small open compact subgroup.
- The integer $u_G \geqslant 1$ in Lemma 8.11 is a uniform upper bound for the number of $G(F_v)$-conjugacy classes in a stable conjugacy class.
- The integer $n_G \geqslant 0$ is the minimum value for the dimension of the unipotent radical of a proper parabolic subgroup of G over $\overline{F}$.
- The constant $c > 0$ is a bound for the number of connected components $\pi_0(Z(\widehat{I_\gamma})^\Gamma)$ in Corollary 8.12.
- The exponents $A_{\text{lv}}, B_{\text{lv}}, C_{\text{lv}} > 0$ in Theorem 9.16 (see also Theorem 1.3) and $A_{\text{wt}}, B_{\text{wt}}, C_{\text{wt}} > 0$ in Theorem 9.19.
- For families in the weight aspect, the constant $\eta > 0$ which may be chosen arbitrary small enters in the condition (11.5) that the dominant weights attached to ξ_k stay away from the walls.

- The exponent $C_{\mathrm{pole}} > 0$ in the Hypothesis 11.2 concerning the density of poles of L-functions.
- The exponents $0 < C_1 < C_2$ control the analytic conductor $C(\mathfrak{F}_k)$ of the families in the weight aspect [Inequality (11.7)] and $0 < C_3 < C_4$ in the level aspect (Hypothesis 11.4).
- The constant $\delta > 0$ in Theorem 11.5 controls the support of the Fourier transform $\widehat{\Phi}$ of the test function Φ.
- The constant $c(f) > 0$ depending on the test function f is a uniform upper bound for normalized orbital integrals $D^G(\gamma)^{\frac{1}{2}} O_\gamma(f)$ (Appendix A).

Several constants are attached directly to the group G such as the dimension $d_G = \dim G$, the rank $r_G = \mathrm{rk} G$, the order of the Weyl group $w_G = |\Omega|$, the degree s_G of the smallest extension of F over which G becomes split. Also in Lemma 2.14 the constant b_G gives a bound for the constant terms along Levi subgroups. The constants a_G, b_G, e_G in Theorem 7.3 gives a uniform bound for certain orbital integrals. In general we have made effort to keep light and consistent notation throughout the text.

In Sect. 6 we will choose a finite extension E/F which splits maximal tori of subgroups of G. The degree $s_G^{\mathrm{spl}} = [E : F]$ will be controlled by $s_G^{\mathrm{spl}} \leqslant s_G w_G$ (see Lemma 6.5), while the ramification of E/F will vary. In Sect. 5 we consider the finite extension F_1/F such that $\mathrm{Gal}(\overline{F}/F)$ acts on $\widehat{G}$ through the faithful action of $\mathrm{Gal}(F_1/F)$. For example if G is a non-split inner form of a split group then $F_1 = F$. In Sect. 12 we consider a finite extension F_2/F_1 such that the representation r factors through $\widehat{G} \rtimes \mathrm{Gal}(F_2/F)$. For a general G, there might not be any direct relationship between the extensions E/F and $F_2/F_1/F$.

1.9 Structure of the paper

For a quick tour of our main results and the structure of our arguments, one could start reading from Sect. 9 after familiarizing oneself with basic notation, referring to earlier sections for further notation and basic facts as needed.

The first Sects. 2 and 3 are concerned with harmonic analysis on reductive groups over local fields, notably the Satake transform, L-groups and L-morphisms, the properties of the Plancherel measure and the Macdonald formula for the unramified spectrum. We establish bounds for truncated Hecke algebras and for character traces that will play a role in subsequent chapters. In Sect. 4 we recall various analytic properties of automorphic L-functions on $\mathrm{GL}(d)$ and notably isobaric sums, bounds towards Ramanujan–Petersson and the so-called explicit formula for the sum of the zeros. Section 5 introduces the Sato–Tate measure for general groups and Sato–Tate equidistribution for Satake parameters and for families. The next Sect. 6 gathers various back-

ground materials on orbital integral, the Gross motive and Tamagawa measure, discrete series characters and Euler–Poincaré functions, and Frobenius–Schur indicator. We establish bounds for special values of the Gross motive which will enter in the geometric side of the trace formula.

In Sect. 7 we establish a uniform bound for orbital integrals of the type (1.9). In Sect. 8 we establish various bounds on conjugacy classes and level subgroups. How these estimates enter in the trace formula has been detailed in the outline above.

Then we are ready in Sect. 9 to establish our main result, an automorphic Plancherel theorem for families with error terms and its application to the Sato–Tate theorem for families. The theorem is first proved for test functions on the unitary dual coming from Hecke algebras by orchestrating all the previous results in the trace formula. Then our result is improved to allow more general test functions, either in the input to the Sato–Tate theorem or in the prescribed local condition for the family, by means of Sauvageot's density theorem.

The last three Sects. 10, 11 and 12 concern the application to low-lying zeros. In complete generality we need to rely on Langlands global functoriality and other hypothesis that we state precisely. These unproven assumptions are within reach in the context of endoscopic transfer and we will return to it in subsequent works.

Appendix A by Kottwitz establishes the boundedness of normalized orbital integrals from the theory of Shalika germs. Appendix B by Cluckers–Gordon–Halupczok establishes a strong form of (1.9) with $e = 1$ by using recent results in arithmetic motivic integration.

2 Satake transforms

2.1 *L*-groups and *L*-morphisms

We are going to recall some definitions and facts from [9, §1, §2] and [62, §1]. Let F be a local or global field of characteristic 0 with an algebraic closure $\overline{F}$, which we fix. Let W_F denote the Weil group of F and set $\Gamma := \mathrm{Gal}(\overline{F}/F)$. Let H and G be connected reductive groups over F. Let $(\widehat{B}, \widehat{T}, \{X_\alpha\}_{\alpha \in \Delta^\vee})$ be a splitting datum fixed by Γ, from which the *L*-group

$$^L G = \widehat{G} \rtimes W_F$$

is constructed. An *L*-morphism $\eta \colon {}^L H \to {}^L G$ is a continuous map commuting with the canonical surjections $^L H \to W_F$ and $^L G \to W_F$ such that $\eta|_{\widehat{H}}$ is a morphism of complex Lie groups. A representation of $^L G$ is by definition a continuous homomorphism $^L G \to \mathrm{GL}(V)$ for some $\mathbb{C}$-vector space V with $\dim V < \infty$ such that $r|_{\widehat{G}}$ is a morphism of complex Lie groups. Clearly

giving a representation $^LG \to \mathrm{GL}(V)$ is equivalent to giving an L-morphism $^LG \to {^L}\mathrm{GL}(V)$.

Let $f\colon H \to G$ be a normal morphism, which means that $f(H)$ is a normal subgroup of G. Then it gives rise to an L-morphism $^LG \to {^L}H$ as explained in [9, 2.5]. In particular, there is a Γ-equivariant map $Z(\widehat{G}) \to Z(\widehat{H})$, which is canonical (independent of the choice of splittings). Thus an exact sequence of connected reductive groups over F

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

gives rise to a Γ-equivariant exact sequence of $\mathbb{C}$-diagonalizable groups

$$1 \to Z(\widehat{G}_3) \to Z(\widehat{G}_2) \to Z(\widehat{G}_1) \to 1.$$

2.2 Satake transform

From here throughout this section, let F be a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}$ and a uniformizer ϖ. Set $q := |\mathcal{O}/\varpi\mathcal{O}|$. Let G be an *unramified* group over F and $B = TU$ be a Borel subgroup decomposed into the maximal torus and the unipotent radical in B. Let A denote the maximal F-split torus in T. Write Φ_F (resp. Φ) for the set of all F-rational roots (resp. all roots over $\overline{F}$) and Φ_F^+ (resp. Φ^+) for the subset of positive roots. Choose a smooth reductive model of G over $\mathcal{O}$ corresponding to a hyperspecial point on the apartment for A. Set $K := G(\mathcal{O})$. Denote by $X_*(A)^+$ the subset of $X_*(A)$ meeting the closed Weyl chamber determined by B, namely $\lambda \in X_*(A)^+$ if $\alpha(\lambda) \geqslant 0$ for all $\alpha \in \Phi_F^+$. Denote by Ω_F (resp. Ω) the F-rational Weyl group for (G, A) (resp. the absolute Weyl group for (G, T)), and ρ_F (resp. ρ) the half sum of all positive roots in Φ_F^+ (resp. Φ^+). A partial order $\leqslant$ is defined on $X_*(A)$ (resp. $X_*(T)$) such that $\mu \leqslant \lambda$ if $\lambda - \mu$ is a linear combination of F-rational positive coroots (resp. positive coroots) with nonnegative coefficients. The same order extends to a partial order $\leqslant_{\mathbb{R}}$ on $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ defined analogously.

Let F^{ur} denote the maximal unramified extension of F. Let Fr denote the geometric Frobenius element of $\mathrm{Gal}(F^{\mathrm{ur}}/F)$. Define W_F^{ur} to be the unramified Weil group, namely the subgroup $\mathrm{Fr}^{\mathbb{Z}}$ of $\mathrm{Gal}(F^{\mathrm{ur}}/F)$. Since $\mathrm{Gal}(\overline{F}/F)$ acts on $\widehat{G}$ through a finite quotient of $\mathrm{Gal}(F^{\mathrm{ur}}/F)$, one can make sense of $^LG^{\mathrm{ur}} := \widehat{G} \rtimes W_F^{\mathrm{ur}}$.

Throughout this section we write G, T, A for $G(F)$, $T(F)$, $A(F)$ if there is no confusion. Define $\mathcal{H}^{\mathrm{ur}}(G) := C_c^{\infty}(K \backslash G/K)$ and $\mathcal{H}^{\mathrm{ur}}(T) := C_c^{\infty}(T(F)/T(F) \cap K)$. The latter is canonically isomorphic to $\mathcal{H}^{\mathrm{ur}}(A) := C_c^{\infty}(A(F)/A(\mathcal{O}))$ via the inclusion $A \hookrightarrow T$. We can further identify

$$\mathcal{H}^{\mathrm{ur}}(T) \simeq \mathcal{H}^{\mathrm{ur}}(A) \simeq \mathbb{C}[X_*(A)]$$

where the last $\mathbb{C}$-algebra isomorphism matches $\lambda \in X_*(A)$ with $\mathbf{1}_{\lambda(\varpi)(A \cap K)} \in \mathcal{H}^{\mathrm{ur}}(A)$. Let $\lambda \in X_*(A)$. Write

$$\tau_\lambda^G := \mathbf{1}_{K\lambda(\varpi)K} \in \mathcal{H}^{\mathrm{ur}}(G), \quad \tau_\lambda^A := \frac{1}{|\Omega_F|} \sum_{w \in \Omega_F} \mathbf{1}_{w\lambda(\varpi)(A \cap K)} \in \mathcal{H}^{\mathrm{ur}}(A)^{\Omega_F}.$$

The sets $\{\tau_\lambda^G\}_{\lambda \in X_*(A)^+}$ and $\{\tau_\lambda^A\}_{\lambda \in X_*(A)^+}$ are bases for $\mathcal{H}^{\mathrm{ur}}(G)$ and $\mathcal{H}^{\mathrm{ur}}(A)^{\Omega_F}$ as $\mathbb{C}$-vector spaces, respectively. Consider the map

$$\mathcal{H}^{\mathrm{ur}}(G) \to \mathcal{H}^{\mathrm{ur}}(T), \quad f \mapsto \left(t \mapsto \delta_B(t)^{1/2} \int_U f(tu)du \right) \qquad (2.1)$$

composed with $\mathcal{H}^{\mathrm{ur}}(T) \simeq \mathcal{H}^{\mathrm{ur}}(A)$ above. The composite map induces a $\mathbb{C}$-algebra isomorphism

$$\mathcal{S}^G : \mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathcal{H}^{\mathrm{ur}}(A)^{\Omega_F} \qquad (2.2)$$

called the Satake isomorphism. We often write just $\mathcal{S}$ for $\mathcal{S}^G$. We note that in general $\mathcal{S}$ does not map τ_λ^G to τ_λ^A.

Another useful description of $\mathcal{H}^{\mathrm{ur}}(G)$ is through representations of $^L G^{\mathrm{ur}}$. (The latter notion is defined as in Sect. 2.1). Write $(\widehat{G} \rtimes \mathrm{Fr})_{\mathrm{ss-conj}}$ for the set of $\widehat{G}$-conjugacy classes of semisimple elements in $\widehat{G} \rtimes \mathrm{Fr}$. Consider the set

$$\mathrm{ch}\left({}^L G^{\mathrm{ur}} \right) := \left\{ \mathrm{tr}\, r : (\widehat{G} \rtimes \mathrm{Fr})_{\mathrm{ss-conj}} \to \mathbb{C} \,|\, r \text{ is a representation of } {}^L G^{\mathrm{ur}} \right\}.$$

Define $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$ to be the $\mathbb{C}$-algebra generated by $\mathrm{ch}({}^L G^{\mathrm{ur}})$ in the space of functions on $(\widehat{G} \rtimes \mathrm{Fr})_{\mathrm{ss-conj}}$. For each $\lambda \in X_*(A)^+$ define the quotient

$$\chi_\lambda := \frac{\sum_{w \in \Omega_F} \mathrm{sgn}(w) w(\lambda + \rho_F)}{\sum_{w \in \Omega_F} \mathrm{sgn}(w) w\rho_F}, \qquad (2.3)$$

which exists as an element of $\mathbb{C}[X_*(A)]^{\Omega_F}$ and is unique. (One may view χ_λ as the analogue in the disconnected case of the irreducible character of highest weight λ, cf. proof of Lemma 2.1 below.) Then $\{\chi_\lambda\}_{\lambda \in X_*(A)^+}$ is a basis for $\mathbb{C}[X_*(A)]^{\Omega_F}$ as a $\mathbb{C}$-vector space, cf. [57, p. 465]. (Another basis was given by τ_λ^A's above.) There is a canonical $\mathbb{C}$-algebra isomorphism

$$\mathcal{T} : \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \xrightarrow{\sim} \mathcal{H}^{\mathrm{ur}}(A)^{\Omega_F}, \qquad (2.4)$$

determined as follows (see [9, Prop 6.7] for detail): for each irreducible r, $\operatorname{tr} r|_{\widehat{T}}$ is shown to factor through $\widehat{T} \to \widehat{A}$ (induced by $A \subset T$). Hence $\operatorname{tr} r|_{\widehat{T}}$ can be viewed as an element of $\mathbb{C}[X^*(\widehat{A})] = \mathbb{C}[X_*(A)]$, which can be seen to be invariant under Ω_F. Define $\mathcal{T}(\operatorname{tr} r)$ to be the latter element.

Let r_0 be an irreducible representation of $\widehat{G}$ of highest weight $\lambda_0 \in X^*(\widehat{T})^+ = X_*(T)^+$. The group W_F^{ur} acts on $X^*(\widehat{T})^+$. Write $\operatorname{Stab}(\lambda_0) \subset W_F^{\mathrm{ur}}$ for the stabilizer subgroup for λ_0, which has finite index (since a finite power of Fr acts trivially on $\widehat{G}$ and thus also on $\widehat{T}$). Put $r := \operatorname{Ind}_{\widehat{G} \rtimes \operatorname{Stab}(\lambda_0)}^{{}^L G^{\mathrm{ur}}} r_0$ and $\lambda := \sum_{\sigma \in W_F^{\mathrm{ur}}/\operatorname{Stab}(\lambda_0)} \sigma \lambda_0 \in X_*(A)^+$. Clearly r and λ depend only on the W_F^{ur}-orbit of λ_0. Put $i(\lambda_0) := [W_F^{\mathrm{ur}} : \operatorname{Stab}(\lambda_0)]$.

Lemma 2.1 (i) *Suppose that r and λ are obtained from r_0 and λ_0 as above. Then*

$$\mathcal{T}(\operatorname{tr} r) = \chi_\lambda. \tag{2.5}$$

(ii) *In general for any irreducible representation $r' \colon {}^L G^{\mathrm{ur}} \to \operatorname{GL}_d(\mathbb{C})$ such that $r'(W_F^{\mathrm{ur}})$ has relatively compact image, let r_0 be any irreducible sub-representation of $r'|_{\widehat{G}}$. Let r be obtained from r_0 as above. Then for some $\zeta \in \mathbb{C}^\times$ with $|\zeta| = 1$,*

$$\operatorname{tr} r' = \zeta \cdot \operatorname{tr} r.$$

Proof Let us prove (i). For any $i \geqslant 1$, let ${}^L G_i$ denote the finite L-group $\widehat{G} \rtimes \operatorname{Gal}(F_i/F)$ where F_i is the degree i unramified extension of F in $\overline{F}$. It is easy to see that $r(\operatorname{Fr}^{i(\lambda_0)})$ is trivial and that $r = \operatorname{Ind}_{\widehat{G}}^{{}^L G_{i(\lambda_0)}} r_0$. Then (2.5) amounts to Kostant's character formula for a disconnected group [61, Thm 7.5] applied to ${}^L G_{i(\lambda_0)}$. As for (ii), let λ_0 and λ be as in the paragraph preceding the lemma. Let $j \geqslant 1$ be such that G becomes split over a degree j unramified extension of F. (Recall that G is assumed to be unramified.) By twisting r' by a unitary character of W_F^{ur} one may assume that r' factors through ${}^L G_j$. Then both r and r' factor through ${}^L G_j$ and are irreducible constituents of $\operatorname{Ind}_{\widehat{G}}^{{}^L G_j} r_0$. From this it is easy to deduce that r' is a twist of r by a finite character of W_F^{ur} of order dividing j. Assertion (ii) follows. $\qquad\square$

Each $\lambda \in X_*(A)^+$ determines $s_{\lambda,\mu} \in \mathbb{C}$ such that

$$\mathcal{S}^{-1}(\chi_\lambda) = \sum_{\mu \in X_*(A)^+} s_{\lambda,\mu} \tau_\mu^G \tag{2.6}$$

where only finitely many $s_{\lambda,\mu}$ are nonzero. In fact Theorem 1.3 of [57] identifies $s_{\lambda,\mu}$ with $K_{\lambda,\mu}(q^{-1})$ defined in (1.2) of that paper, cf. §4 of [48]. In particular

$s_{\lambda,\lambda} \neq 0$ and $s_{\lambda,\mu} \neq 0$ unless $\mu \leqslant \lambda$. The following information will be useful in Sect. 2.7.

Lemma 2.2 *Let $\lambda, \mu \in X_*(A)^+$. Suppose that $\lambda \star_w \mu := w(\lambda + \rho_F) - (\mu + \rho_F)$ is nontrivial for all $w \in \Omega_F$. For $\kappa \in X_*(A)$ let $p(\kappa) \in \mathbb{Z}_{\geqslant 0}$ be the number of tuples $(c_{\alpha^\vee})_{\alpha^\vee \in (\Phi_F^\vee)^+}$ with $c_{\alpha^\vee} \in \mathbb{Z}_{\geqslant 0}$ such that $\sum_{\alpha^\vee} c_{\alpha^\vee} \cdot \alpha^\vee = \kappa$. Then*

$$|s_{\lambda,\mu}| \leqslant q^{-1} |\Omega_F| \max_{w \in \Omega_F} p(\lambda \star_w \mu).$$

Proof It is easy to see from the description of $K_{\lambda,\mu}(q^{-1})$ in [57, (1.2)] that

$$\left| K_{\lambda,\mu}(q^{-1}) \right| \leqslant |\Omega_F| \max_{w \in \Omega_F} \widehat{\mathscr{P}}\left(w(\lambda + \rho_F) - (\mu + \rho_F); q^{-1} \right).$$

The definition of $\widehat{\mathscr{P}}$ in [57, (1.1)] shows that $0 \leqslant \widehat{\mathscr{P}}(\kappa; q^{-1}) \leqslant p(\kappa)q^{-1}$ if $\kappa \neq 0$. □

2.3 Truncated unramified Hecke algebras

Set $n := \dim T$ and $X_*(T)_{\mathbb{R}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Choose an $\mathbb{R}$-basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of $X_*(T)_{\mathbb{R}}$. For each $\lambda \in X_*(T)_{\mathbb{R}}$, written as $\lambda = \sum_{i=1}^n a_i(\lambda)e_i$ for unique $a_i(\lambda) \in \mathbb{R}$, define

$$|\lambda|_{\mathcal{B}} := \max_{1 \leqslant i \leqslant n} |a_i(\lambda)|, \quad \|\lambda\|_{\mathcal{B}} := \max_{\omega \in \Omega}(|\omega\lambda|_{\mathcal{B}}).$$

When there is no danger of confusion, we will simply write $|\cdot|_{\mathcal{B}}$ or even $|\cdot|$ instead of $|\cdot|_{\mathcal{B}}$, and similarly for $\|\cdot\|_{\mathcal{B}}$. It is clear that $\|\cdot\|_{\mathcal{B}}$ is Ω-invariant and that $|\lambda_1 + \lambda_2|_{\mathcal{B}} \leqslant |\lambda_1|_{\mathcal{B}} + |\lambda_2|_{\mathcal{B}}$ for all $\lambda_1, \lambda_2 \in X_*(T)$. When $\kappa \in \mathbb{Z}_{\geqslant 0}$, define

$$\mathcal{H}^{\mathrm{ur}}(G)^{\leqslant \kappa, \mathcal{B}} := \left\{ \mathbb{C}\text{-subspace of } \mathcal{H}^{\mathrm{ur}}(G) \text{ generated by } \tau_\lambda^G, \ \lambda \in X_*(A)^+, \right.$$
$$\left. \|\lambda\|_{\mathcal{B}} \leqslant \kappa \right\}. \tag{2.7}$$

It is simply written as $\mathcal{H}^{\mathrm{ur}}(G)^{\leqslant \kappa}$ when the choice of $\mathcal{B}$ is clear.

Lemma 2.3 *Let $\mathcal{B}$ and $\mathcal{B}'$ be two $\mathbb{R}$-bases of $X_*(T)_{\mathbb{R}}$. Then there exist constants $c_1, c_2, B_1, B_2, B_3, B_4 > 0$ such that for all $\lambda \in X_*(T)_{\mathbb{R}}$,*

(i) $c_1|\lambda|_{\mathcal{B}'} \leqslant |\lambda|_{\mathcal{B}} \leqslant c_2|\lambda|_{\mathcal{B}'}$,
(ii) $B_1|\lambda|_{\mathcal{B}} \leqslant \|\lambda\|_{\mathcal{B}} \leqslant B_2|\lambda|_{\mathcal{B}}$ *for all $\lambda \in X_*(T)_{\mathbb{R}}$,*
(iii) $B_3\|\lambda\|_{\mathcal{B}'} \leqslant \|\lambda\|_{\mathcal{B}} \leqslant B_4\|\lambda\|_{\mathcal{B}'}$ *for all $\lambda \in X_*(T)_{\mathbb{R}}$ and*
(iv) $\mathcal{H}^{\mathrm{ur}}(G)^{\leqslant B_4^{-1}\kappa, \mathcal{B}'} \subset \mathcal{H}^{\mathrm{ur}}(G)^{\leqslant \kappa, \mathcal{B}} \subset \mathcal{H}^{\mathrm{ur}}(G)^{\leqslant B_3^{-1}\kappa, \mathcal{B}'}$.

Proof Let us verify (i). As the roles of $\mathcal{B}$ and $\mathcal{B}'$ can be changed, it suffices to prove the existence of c_2. For this, it suffices to take $c_2 = \sup_{|\lambda|_{\mathcal{B}} \leqslant 1} |\lambda|_{\mathcal{B}'}$. The latter is finite since $| \cdot |_{\mathcal{B}'}$ is a continuous function on the set of λ such that $|\lambda|_{\mathcal{B}} \leqslant 1$, which is compact. Part (ii) is obtained by applying the lemma to the bases $\mathcal{B}' = \omega\mathcal{B}$ for all $\omega \in \Omega$. Let us check (iii). Let $B_1, B_2 > 0$ (resp. $B_1', B_2' > 0$) be the constants of (ii) for the basis $\mathcal{B}$ (resp. $\mathcal{B}'$). Then

$$c_1 B_1 (B_2')^{-1} \|\lambda\|_{\mathcal{B}'} \leqslant c_1 B_1 |\lambda|_{\mathcal{B}'} \leqslant B_1 |\lambda|_{\mathcal{B}} \leqslant \|\lambda\|_{\mathcal{B}}$$

and similarly $\|\lambda\|_{\mathcal{B}} \leqslant c_2 B_2 (B_1')^{-1} \|\lambda\|_{\mathcal{B}'}$. Finally (iv) immediately follows from (iii). $\qquad \square$

It is natural to wonder whether the definition of truncation in (2.7) changes if a different basis $\{\tau_\lambda^G\}$ or $\{\chi_\lambda\}$ is used. We assert that it changes very little in a way that the effect on κ is bounded by a κ-independent constant. To ease the statement define $\mathcal{H}_i^{\mathrm{ur}}(G)^{\leqslant \kappa, \mathcal{B}}$ for $i = 1$ (resp. $i = 2$) to be the $\mathbb{C}$-subspace of $\mathcal{H}^{\mathrm{ur}}(G)$ generated by $\mathcal{S}^{-1}(\tau_\lambda^A)$ (resp. $\mathcal{S}^{-1}(\chi_\lambda)$) for $\lambda \in X_*(A)^+$ with $\|\lambda\|_{\mathcal{B}} \leqslant \kappa$.

Lemma 2.4 *There exists a constant $C \geqslant 1$ such that for every $\kappa \in \mathbb{Z}_{\geqslant 0}$ and for any $i, j \in \{\emptyset, 1, 2\}$,*

$$\mathcal{H}_i^{\mathrm{ur}}(G)^{\leqslant \kappa, \mathcal{B}} \subset \mathcal{H}_j^{\mathrm{ur}}(G)^{\leqslant C\kappa, \mathcal{B}}.$$

Proof It is enough to prove the lemma for a particular choice of $\mathcal{B}$ by Lemma 2.3. So we may assume that $\mathcal{B}$ extends the set of simple coroots in $\Phi^\vee$ by an arbitrary basis of $X_*(Z(G))_{\mathbb{R}}$. Again by Lemma 2.3 the proof will be done if we show that each of the following generates the same $\mathbb{C}$-subspace:

  (i) the set of τ_λ^G for $\lambda \in X_*(A)^+$ with $|\lambda|_{\mathcal{B}} \leqslant \kappa$,
 (ii) the set of $\mathcal{S}^{-1}(\tau_\lambda^A)$ for $\lambda \in X_*(A)^+$ with $|\lambda|_{\mathcal{B}} \leqslant \kappa$,
(iii) the set of $\mathcal{S}^{-1}(\chi_\lambda)$ for $\lambda \in X_*(A)^+$ with $|\lambda|_{\mathcal{B}} \leqslant \kappa$.

It suffices to show that the matrices representing the change of bases are "upper triangular" in the sense that the (λ, λ) entries are nonzero and (λ, μ) entries are zero unless $\lambda \geqslant \mu$. (Note that $\lambda \geqslant \mu$ implies $|\lambda|_{\mathcal{B}} \geqslant |\mu|_{\mathcal{B}}$ by the choice of $\mathcal{B}$.) We have remarked below (2.3) that $s_{\lambda, \mu}$'s have this property, accounting for (i)↔(iii). For (ii)↔(iii) the desired property can be seen directly from (2.3) by writing χ_λ in terms of τ_μ^A's. $\qquad \square$

2.4 The case of GL_d

The case $G = \mathrm{GL}_d$ is considered in this subsection. Let $A = T$ be the diagonal maximal torus and B the group of upper triangular matrices. For $1 \leqslant i \leqslant d$,

take $Y_i \in X_*(A)$ to be $y \mapsto \text{diag}(1, \ldots, 1, y, 1, \ldots, 1)$ with y in the i-th place. One can naturally identify $X_*(A) \simeq \mathbb{Z}^d$ such that the images of Y_i form the standard basis of $\mathbb{Z}^d$. Then Ω_F is isomorphic to $\mathscr{S}_d$, the symmetric group in d variables acting on $\{Y_1, \ldots, Y_d\}$ via permutation of indices. We have the Satake isomorphism

$$\mathcal{S} \colon \mathcal{H}^{\text{ur}}(\text{GL}_d) \xrightarrow{\sim} \mathcal{H}^{\text{ur}}(T)^{\Omega_F} \simeq \mathbb{C}\left[Y_1^{\pm}, \ldots, Y_d^{\pm}\right]^{\mathscr{S}_d}.$$

For an alternative description let us introduce standard symmetric polynomials $X_1, \ldots, X_d$ by the equation in a formal Z-variable $(Z - Y_1) \ldots (Z - Y_d) = Z^d - X_1 Z^{d-1} + \cdots + (-1)^d X_d$. Then

$$\mathbb{C}\left[Y_1^{\pm}, \ldots, Y_d^{\pm}\right]^{\mathscr{S}_d} = \mathbb{C}\left[X_1, \ldots, X_{d-1}, X_d^{\pm}\right].$$

Let $\kappa \in \mathbb{Z}_{\geqslant 0}$. Define $\mathcal{H}^{\text{ur}}(\text{GL}_d)^{\leqslant \kappa}$, or simply $\mathcal{H}_d^{\leqslant \kappa}$, to be the preimage under $\mathcal{S}$ of the $\mathbb{C}$-vector space generated by

$$\left\{ \sum_{\sigma \in \mathscr{S}_d} Y_{\sigma(1)}^{a_1} Y_{\sigma(2)}^{a_2} \cdots Y_{\sigma(d)}^{a_d} : a_1, \ldots, a_d \in [-\kappa, \kappa] \right\}.$$

The following is standard (cf. [48]).

Lemma 2.5 *Let* $r \in \mathbb{Z}_{\geqslant 1}$. *Let* $\lambda_r := (r, 0, 0, \ldots, 0) \in X_*(A)^+$. *Then*

$$\mathcal{S}^{-1}(Y_1^r + \cdots + Y_d^r) = \sum_{\substack{\mu \in X_*(A)^+ \\ \mu \leqslant \lambda_r}} c_{\lambda_r, \mu} \cdot \tau_\mu^G$$

for $c_{\lambda_r, \mu} \in \mathbb{C}$ *with* $c_{\lambda_r, \lambda_r} = q^{r(1-d)/2}$, *where the sum runs over the set of* $\mu \in X_*(T)^+$ *such that* $\mu \leqslant \lambda_r$. *In particular,*

$$\mathcal{S}^{-1}(Y_1 + \cdots + Y_d) = q^{(1-d)/2} \tau_{(1,0,\ldots,0)}^G,$$
$$\mathcal{S}^{-1}\left(Y_1^2 + \cdots + Y_d^2\right) = q^{1-d}\left(\tau_{(2,0,\ldots,0)}^G + (1-q)\tau_{(1,1,0,\ldots,0)}^G\right).$$

2.5 L-morphisms and unramified Hecke algebras

Assume that H and G are unramified groups over F. Let $\eta \colon {}^L H \to {}^L G$ be an unramified L-morphism, which means that it is inflated from some L-morphism ${}^L H^{\text{ur}} \to {}^L G^{\text{ur}}$ (the notion of L-morphism for the latter is defined as in Sect. 2.1). There is a canonically induced map $\text{ch}({}^L G^{\text{ur}}) \to \text{ch}({}^L H^{\text{ur}})$. Via

(2.2) and (2.4), the latter map gives rise to a $\mathbb{C}$-algebra map $\eta^* \colon \mathcal{H}^{\mathrm{ur}}(G) \to \mathcal{H}^{\mathrm{ur}}(H)$.

We apply the above discussion to an unramified representation

$$r \colon {}^L G \to \mathrm{GL}_d(\mathbb{C}).$$

Viewing r as an L-morphism ${}^L G \to {}^L \mathrm{GL}_d$, we obtain

$$r^* \colon \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d) \to \mathcal{H}^{\mathrm{ur}}(G).$$

Lemma 2.6 *Let $\mathcal{B}$ be an $\mathbb{R}$-basis of $X_*(T)_{\mathbb{R}}$. There exists a constant $\beta > 0$ (depending on $\mathcal{B}$, d and r) such that for all $\kappa \in \mathbb{Z}_{\geqslant 0}$, $r^*(\mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d)^{\leqslant \kappa}) \subset \mathcal{H}^{\mathrm{ur}}(G)^{\leqslant \beta \kappa, \mathcal{B}}$.*

Proof Thanks to Lemma 2.3, it is enough to deal with a particular choice of $\mathcal{B}$. Choose $\mathcal{B}$ by extending the set $\Delta^\vee$ of simple coroots, and write $\mathcal{B} = \Delta^\vee \coprod \mathcal{B}_0$. We begin by proving the following claim: let $\lambda_1, \lambda_2 \in X_*(A)^+$ and expand the convolution product

$$\tau_{\lambda_1}^G * \tau_{\lambda_2}^G = \sum_\mu a_{\lambda_1, \lambda_2}^\mu \tau_\mu^G$$

where only $\mu \in X_*(A)^+$ such that $\mu \leqslant_{\mathbb{R}} \lambda_1 + \lambda_2$ contribute (cf. [18, p. 148]). Only finitely many terms are nonzero. Then the claim is that

$$|\mu|_{\mathcal{B}} \leqslant |\lambda_1 + \lambda_2|_{\mathcal{B}}, \quad \text{whenever } a_{\lambda_1, \lambda_2}^\mu \neq 0.$$

To check the claim, consider $\mu = \sum_{e \in \mathcal{B}} a_e(\mu) \cdot e$ and $\lambda_1 + \lambda_2 = \sum_{e \in \mathcal{B}} a_e(\lambda_1 + \lambda_2) \cdot e$, where the coefficients are in $\mathbb{R}$. The conditions $\mu \leqslant_{\mathbb{R}} \lambda_1 + \lambda_2$ and $\mu \in X_*(T)_{\mathbb{R}, +}$ imply that $a_e(\mu) = a_e(\lambda_1 + \lambda_2)$ if $e \in \mathcal{B}_0$ and $0 \leqslant a_e(\mu) \leqslant a_e(\lambda_1 + \lambda_2)$ if $e \in \Delta^\vee$. Hence $|\mu|_{\mathcal{B}} \leqslant |\lambda_1 + \lambda_2|_{\mathcal{B}}$.

We are ready to prove the lemma. It is explained in Lemma 2.4 and the remark below it that there exists a constant $\beta_1 > 0$ which is independent of κ such that every $\phi \in \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d)^{\leqslant \kappa}$ can be written as a $\mathbb{C}$-linear combination of

$$\sum_{\sigma \in \mathscr{S}_d} Y_{\sigma(1)}^{a_1} Y_{\sigma(2)}^{a_2} \ldots Y_{\sigma(d)}^{a_d}, \quad a_1, \ldots, a_d \in [-\beta_1 \kappa, \beta_1 \kappa].$$

Each element above can be rewritten in terms of the symmetric polynomials X_i's of Sect. 2.4: first, $X_d^{\beta_1 \kappa}$ times $\sum_{\sigma \in \mathscr{S}_d} Y_{\sigma(1)}^{a_1} Y_{\sigma(2)}^{a_2} \ldots Y_{\sigma(d)}^{a_d}$ is a symmetric polynomial of degree $\leqslant 2\beta_1 \kappa$, which in turn is a polynomial in $X_1, \ldots, X_d$ of degree $\leqslant 2\beta_1 \kappa$. We conclude that every $\phi \in \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d)^{\leqslant \kappa}$ is in the span of monomials

$$X_1^{b_1} X_1^{b_2} \ldots X_d^{b_d}, \quad b_1, \ldots, b_d \in [-2\beta_1\kappa, 2\beta_1\kappa]. \tag{2.8}$$

For each $1 \leqslant i \leqslant d$, write $r^*(X_i)$ [resp. $r^*(X_i^{-1})$] as a linear combination of $\tau_{\lambda_{i,j}}^G$ (resp. $\tau_{\lambda_{i,j}^-}^G$) with nonzero coefficients. Define β_0 to be the maximum among all possible $|\lambda_{i,j}|$ and $|\lambda_{i,j}^-|$. The above claim $r^*(X_1^{b_1} X_1^{b_2} \ldots X_d^{b_d})$ as in (2.8) is in the $\mathbb{C}$-span of τ_μ^G satisfying

$$|\mu|_{\mathcal{B}} \leqslant (|b_1| + \cdots + |b_d|)\beta_0 \leqslant 2d\beta_0\beta_1\kappa.$$

So the above span contains $r^*(\phi)$ for $\phi \in \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d)^{\leqslant\kappa}$. By Lemma 2.3 there exists a constant $B_2 > 0$ such that $\|\mu\|_{\mathcal{B}} \leqslant B_2|\mu|_{\mathcal{B}}$ for every $\mu \in X_*(T)$. Hence the lemma holds true with $\beta := 2B_2 d\beta_0\beta_1$. □

The map r also induces a functorial transfer for unramified representations

$$r_*\colon \mathrm{Irr}^{\mathrm{ur}}(G(F)) \to \mathrm{Irr}^{\mathrm{ur}}(\mathrm{GL}_d(F)) \tag{2.9}$$

uniquely characterized by $\operatorname{tr} r_*(\pi)(\phi) = \operatorname{tr} \pi(r^*\phi)$ for all $\pi \in \mathrm{Irr}^{\mathrm{ur}}(G(F))$ and $\phi \in \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d(F))$.

2.6 Partial Satake transform

Keep the assumption that G is unramified over F. Let P be an F-rational parabolic subgroup of G with Levi M and unipotent radical N such that $B = TU$ is contained in P. Let Ω_M (resp. $\Omega_{M,F}$) denote the absolute (resp. F-rational) Weyl group for (M, T). A partial Satake transform is defined as [cf. (2.1)]

$$\mathcal{S}_M^G\colon \mathcal{H}^{\mathrm{ur}}(G) \to \mathcal{H}^{\mathrm{ur}}(M), \quad f \mapsto \left(m \mapsto \delta_P(m)^{1/2} \int_N f(mn)dn\right)$$

It is well known that $\mathcal{S}^G = \mathcal{S}^M \circ \mathcal{S}_M^G$. More concretely, $\mathcal{S}_M^G$ is the canonical inclusion $\mathbb{C}[X_*(A)]^{\Omega_{M,F}} \hookrightarrow \mathbb{C}[X_*(A)]^{\Omega_F}$ if $\mathcal{H}^{\mathrm{ur}}(M)$ and $\mathcal{H}^{\mathrm{ur}}(G)$ are identified with the source and the target via $\mathcal{S}^G$ and $\mathcal{S}^M$, respectively. Since T is a common maximal torus of M and G, an $\mathbb{R}$-basis $\mathcal{B}$ of $X_*(T)_{\mathbb{R}}$ determines truncations on $\mathcal{H}^{\mathrm{ur}}(M)$ and $\mathcal{H}^{\mathrm{ur}}(G)$.

Lemma 2.7 *For any $\kappa \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{S}_M^G(\mathcal{H}^{\mathrm{ur}}(G)^{\leqslant\kappa,\mathcal{B}}) \subset \mathcal{H}^{\mathrm{ur}}(M)^{\leqslant\kappa,\mathcal{B}}$.*

Proof It is enough to note that $\|\lambda\|_{\mathcal{B},M} \leqslant \|\lambda\|_{\mathcal{B},G}$ for all $\lambda \in X_*(A)$, which holds since the Ω_M-orbit of λ is contained in the Ω-orbit of λ. □

Remark 2.8 Let $\eta\colon {}^L M \to {}^L G$ be the embedding of [9, §3], well defined up to $\widehat{G}$-conjugacy. Then $\mathcal{S}_M^G$ coincides with $\eta^*\colon \mathcal{H}^{\mathrm{ur}}(G) \to \mathcal{H}^{\mathrm{ur}}(M)$ of Sect. 2.5

2.7 Some explicit test functions

Assume that $r \colon {}^L G = \widehat{G} \rtimes W_F \to \mathrm{GL}_d(\mathbb{C})$ is an *irreducible* representation arising from an unramified L-morphism ${}^L G^{\mathrm{ur}} \to {}^L \mathrm{GL}_d^{\mathrm{ur}}$ such that $r(W_F)$ is relatively compact. For later applications it is useful to study the particular element $r^*(Y_1 + \cdots + Y_d)$ in $\mathcal{H}^{\mathrm{ur}}(G)$.

Lemma 2.9 *Let $\phi = r^*(Y_1 + \cdots + Y_d)$. Then*

(i) *Suppose that $r \colon {}^L G^{\mathrm{ur}} \to \mathrm{GL}_d(\mathbb{C})$ does not factor through W_F^{ur} (or equivalently that $r|_{\widehat{G}}$ is not the trivial representation). Then*

$$|\phi(1)| \leqslant |\Omega_F| \max_{w \in \Omega_F} p(\lambda \star_w 0) \cdot q^{-1}.$$

(ii) *Suppose that $r|_{\widehat{G}}$ is trivial. Then $\phi(1) = r(\mathrm{Fr})$.*

Proof Let us do some preparation. By twisting r by an unramified unitary character of W_F (viewed as a character of ${}^L G$) we may assume that $r = \mathrm{Ind}_{\widehat{G}}^{{}^L G_j} r_0$ for some irreducible representation r_0 of $\widehat{G}$, cf. the proof of Lemma 2.1 (ii). Let λ_0 be the highest weight of r_0 and define $\lambda \in X_*(A)^+$ as in the paragraph preceding Lemma 2.1. The lemma tells us that $\mathcal{S}(\phi) = \zeta \chi_\lambda \in \mathbb{C}[X_*(A)]^{\Omega_F}$ with $|\zeta| = 1$.

In the case of (ii), r is just an unramified unitary character of W_F (with $d = 1$), and it is easily seen that $\chi_\lambda = \tau_0^A$, $\zeta = r(\mathrm{Fr})$, and so $\phi(1) = r(\mathrm{Fr})$. Let us put ourselves in the case (i) so that $\lambda \neq 0$. Note that $\phi(1)$ is just the coefficient of τ_0^G when $\phi = \zeta \mathcal{S}^{-1}(\chi_\lambda)$ is written with respect to the basis $\{\tau_\mu^G\}$. Such a coefficient equals $\zeta \cdot s_{\lambda,0}$ according to (2.6), so $|\phi(1)| = |s_{\lambda,0}|$. Now Lemma 2.2 concludes the proof. [Observe that $\lambda \star_w 0 \neq 0$ whenever $0 \neq \lambda \in X_*(A)^+$.] $\qquad\square$

2.8 Examples in the split case

When G is split, it is easy to see that $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$ is canonically identified with $\mathbb{C}[\mathrm{ch}(\widehat{G})]$ which is generated by finite dimensional characters in the space of functions on $\widehat{G}$. So we may use $\mathbb{C}[\mathrm{ch}(\widehat{G})]$ in place of $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$.

Example 2.10 (When $G = \mathrm{Sp}_{2n}$, $n \geqslant 1$)

Take $r \colon \widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ to be the standard representation. Then

$$Y_1 + \cdots + Y_{2n+1} = \mathrm{tr}\,(\mathrm{Std}) \in \mathbb{C}[\mathrm{ch}(\mathrm{GL}_{2n+1})]$$

is mapped to $\mathrm{tr}\,(r) \in \mathbb{C}[\mathrm{ch}(\mathrm{SO}_{2n+1})]$ and

$$Y_1^2 + \cdots + Y_{2n+1}^2 = \mathrm{tr}\,(\mathrm{Sym}^2(\mathrm{Std}) - \wedge^2(\mathrm{Std})) \in \mathbb{C}[\mathrm{ch}(\mathrm{GL}_{2n+1})]$$

is mapped to $\mathrm{tr}\,(r) \in \mathbb{C}[\mathrm{ch}(\mathrm{SO}_{2n+1})]$. Then $\mathrm{Sym}^2(V)$ breaks into $\mathbb{C}$ and an irreducible representation of $\widehat{G}$ of highest weight $(2, 0, \ldots, 0)$ in the standard parametrization. When $n > 1$, $\wedge^2(V)$ is irreducible of highest weight $(1, 1, 0, \ldots, 0)$. When $n = 1$, $\wedge^2(V) \simeq V^\vee$, i.e. isomorphic to $(\mathrm{Std})^\vee$. (See [41, §19.5].) Let us systematically write Λ_λ for the irreducible representation of SO_{2n+1} with highest weight λ. Then

$$r^*(Y_1 + \cdots + Y_{2n+1}) = \mathrm{tr}\,\Lambda_{(1,0,\ldots,0)},$$
$$r^*\left(Y_1^2 + \cdots + Y_{2n+1}^2\right) = \mathrm{tr}\,(\mathbb{C} + \Lambda_{(2,0,\ldots,0)} - \Lambda_{(1,1,0,\ldots,0)}). \quad (2.10)$$

if $n \geqslant 2$. If $n = 1$, the same is true if $\Lambda_{(1,1,0,\ldots,0)}$ is replaced with $\Lambda_{(-1)}$. For $i = 1, 2$, define

$$\phi^{(i)} := \mathcal{S}^{-1}\left(r^*\left(Y_1^i + \cdots + Y_{2n+1}^i\right)\right).$$

Then one computes

$$\phi^{(1)} = q^{\frac{1-2n}{2}} \mathbf{1}_{K\mu_{(1,0,\ldots,0)}(\varpi_v)K},$$
$$\phi^{(2)} = \mathbf{1}_K + q^{1-2n} \mathbf{1}_{K\mu_{(2,0,\ldots,0)}(\varpi_v)K} - q^{1-2n}(q-1)\mathbf{1}_{K\mu_{(1,1,0,\ldots,0)}(\varpi_v)K}.$$

where μ_λ is the cocharacter of a maximal torus given by λ in the standard parametrization. In particular, $\phi^{(1)}(1) = 0$ and $\phi^{(2)}(1) = 1$.

Example 2.11 (When $G = \mathrm{SO}_{2n}$, $n \geqslant 2$)
    Take $r \colon \widehat{G} = \mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ to be the standard representation. Similarly as before, $\mathrm{Sym}^2(V)$ breaks into $\mathbb{C}$ and an irreducible representation of $\widehat{G}$ of highest weight $(2, 0, \ldots, 0)$. When $n > 1$, $\wedge^2(V)$ is irreducible of highest weight $(1, 1, 0, \ldots, 0)$. When $n = 1$, $\wedge^2(V) \simeq \mathbb{C}$. (See [41, §19.5].) The same formulas as (2.10) hold in this case. Defining

$$\phi^{(i)} := \mathcal{S}^{-1}\left(r^*\left(Y_1^i + \cdots + Y_{2n}^i\right)\right), \quad (2.11)$$

we can compute $\phi^{(1)}$, $\phi^{(2)}$ and see that $\phi^{(1)}(1) = 0$ and $\phi^{(2)}(1) = 1$.

Example 2.12 (When $G = \mathrm{SO}_{2n+1}$)
    Take $r \colon \widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ to be the standard representation. Then

$$Y_1 + \cdots + Y_{2n} = \mathrm{tr}\,(\mathrm{Std}) \in \mathbb{C}[\mathrm{ch}(\mathrm{GL}_{2n})]$$

is mapped to tr $(r \circ \mathrm{Std}) \in \mathbb{C}[\mathrm{ch}(Sp_{2n})]$ and Then

$$Y_1^2 + \cdots + Y_{2n}^2 = \mathrm{tr}\,(\mathrm{Sym}^2(\mathrm{Std}) - \wedge^2(\mathrm{Std})) \in \mathbb{C}[\mathrm{ch}(\mathrm{GL}_{2n})]$$

is mapped to tr $(r \circ \mathrm{Std}) \in \mathbb{C}[\mathrm{ch}(Sp_{2n})]$. If $n \geqslant 2$ then $\wedge^2(V)$ breaks into $\mathbb{C}$ and an irreducible representation of $\widehat{G}$ of highest weight $(1, 1, 0, \ldots, 0)$. (See [41, §17.3].) We have

$$r^*(Y_1 + \cdots + Y_{2n+1}) = \mathrm{tr}\,\Lambda_{(1,0,\ldots,0)},$$
$$r^*\left(Y_1^2 + \cdots + Y_{2n+1}^2\right) = \mathrm{tr}\,(\Lambda_{(2,0,\ldots,0)} - \Lambda_{(1,1,0,\ldots,0)} - \mathbb{C}).$$

As in Example 2.10, Λ designates a highest weight representation (now of Sp_{2n}). Define $\phi^{(i)}$ as in (2.11). By a similar computation as above, $\phi^{(1)}(1) = 0$, $\phi^{(2)}(1) = -1$.

2.9 Bounds for truncated unramified Hecke algebras

Let F, G, A, T and K be as in Sect. 2.2. Throughout this subsection, an $\mathbb{R}$-basis $\mathcal{B}$ of $X_*(T)_{\mathbb{R}}$ will be fixed once and for all. Denote by $\rho \in X^*(T) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ half the sum of all $\alpha \in \Phi^+$.

Lemma 2.13 *For any $\mu \in X_*(A)$, $[K\mu(\varpi)K : K] \leqslant q^{d_G + r_G + \langle \rho, \mu \rangle}$.*

Proof Let vol denote the volume for the Haar measure on $G(F)$ such that $\mathrm{vol}(K) = 1$. Let $I \subset K$ be an Iwahori subgroup of $G(F)$. Then $I = (I \cap U)(I \cap T)(I \cap \overline{U})$. We follow the argument of [106, pp. 241–242], freely using Waldspurger's notation. Our $I, U, \overline{U}$, and T will play the roles of his H, U_0, $\overline{U}_0$ and M_0, respectively. For all $m \in \overline{M}_0^+$ (in his notation), it is not hard to verify that $c'_{U_0}(m) = c_{\overline{U}_0}(m) = c_{M_0}(m) = 1$. Then Waldspurger's argument shows

$$\mathrm{vol}(K\mu(\varpi)K) \leqslant [K : I]^2 \,\mathrm{vol}(I\mu(\varpi)I) \leqslant [K : I]^2 q^{\langle \rho, \mu \rangle}\,\mathrm{vol}(I)$$
$$= [K : I]q^{\langle \rho, \mu \rangle}.$$

Finally observe that $[K : I] \leqslant |G(\mathbb{F}_q)| \leqslant q^{d_G}(1 + \frac{1}{q})^{r_G} \leqslant q^{d_G + r_G}$. (The middle inequality is easily derived from Steinberg's formula. cf. [47, (3.1)].) $\square$

The following lemma will play a role in studying the level aspect in Sect. 9.

Lemma 2.14 *Let M be an F-rational Levi subgroup of G. There exists a constant $b_G > 0$ (depending only on G) such that for all $\kappa \in \mathbb{Z}_{>0}$ and all $\phi \in \mathcal{H}^{\mathrm{ur}}(G)^{\leqslant \kappa, \mathcal{B}}$ such that $|\phi| \leqslant 1$, we have $|\phi_M(1)| = O(q^{d_G + r_G + b_G \kappa})$ (the implicit constant being independent of κ and ϕ), where we put $\phi_M := \mathcal{S}_M^G(\phi)$.*

Proof When $M = G$, the lemma is obvious (with $b_G = 0$). Henceforth we assume that $M \subsetneq G$. In view of Lemma 2.3, it suffices to treat one $\mathbb{R}$-basis $\mathcal{B}$. Fix a $\mathbb{Z}$-basis $\{e_1, \ldots, e_{\dim A}\}$ of $X_*(A)$, and choose any $\mathcal{B}$ which extends that $\mathbb{Z}$-basis. It is possible to write

$$\phi = \sum_{\|\mu\| \leqslant \kappa} a_\mu \cdot \mathbf{1}_{K\mu(\varpi)K}$$

for $|a_\mu| \leqslant 1$. Thus

$$|\phi_M(1)| = \left| \int_{N(F)} \phi(n)dn \right| \leqslant \sum_{\|\mu\| \leqslant \kappa} \left| \int_{N(F)} \mathbf{1}_{K\mu(\varpi)K}(n)dn \right|.$$

For each μ, $K\mu(\varpi)K$ is partitioned into left K-cosets. On each coset γK,

$$\left| \int_{N(F)} \mathbf{1}_{\gamma K}(n)dn \right| \leqslant \mathrm{vol}(K \cap N(F)) = 1.$$

Hence, together with Lemma 2.13,

$$|\phi_M(1)| \leqslant \sum_{\|\mu\| \leqslant \kappa} [K\mu(\varpi)K : K] \leqslant \sum_{\|\mu\| \leqslant \kappa} q^{d_G + r_G + \langle \rho, \mu \rangle}.$$

Write b_0 for the maximum of $|\langle \rho, e_i \rangle|$ for $i = 1, \ldots, \dim A$. Take $b_G := b_0 \dim A + 2 \dim A$. If $\|\mu\| \leqslant \kappa$ then $\mu = \sum_{i=1}^{\dim A} a_i e_i$ for $a_i \in \mathbb{Z}$ with $-\kappa \leqslant a_i \leqslant \kappa$. Hence the right hand side is bounded by $(2\kappa + 1)^{\dim A} q^{d_G + r_G + b_0 \kappa \dim A} \leqslant q^{d_G + r_G + b_G \kappa}$ since $2\kappa + 1 \leqslant 2^{2\kappa} \leqslant q^{2\kappa}$. $\qquad\square$

An elementary matrix computation shows the bound below, which will be used several times.

Lemma 2.15 *Let* $s = \mathrm{diag}(s_1, \ldots, s_m) \in \mathrm{GL}_m(\overline{F}_v)$ *and* $u = (u_{ij})_{i,j=1}^m \in \mathrm{GL}_m(\overline{F}_v)$. *Define* $v_{\min}(u) := \min_{i,j} v(u_{ij})$ *and similarly* $v_{\min}(u^{-1})$. *Then for any eigenvalue* λ *of* $su \in \mathrm{GL}_m(F_v)$,

$$v(\lambda) \in \left[v_{\min}(u) + \min_i v(s_i), -v_{\min}(u^{-1}) + \max_i v(s_i) \right].$$

Remark 2.16 The lemma will be typically applied when $u \in \mathrm{GL}_m(\overline{\mathcal{O}}_v)$ where $\overline{\mathcal{O}}_v$ is the integer ring of $\overline{F}_v$. In this case $v_{\min}(u) = v_{\min}(u^{-1}) = 0$.

Proof Let V be the underlying $\overline{F}_v$-vector space with standard basis $\{e_1, \ldots, e_m\}$. Let $\mathcal{B}_j = \{\mathbf{i} = (i_1, \ldots, i_j)|1 \leqslant i_1 < \cdots < i_j \leqslant m\}$. Then $\wedge^j V$ has a basis $\{e_{i_1} \wedge \cdots \wedge e_{i_j}\}_{\mathbf{i} \in \mathcal{B}_j}$. We claim that

$$v(\text{tr}\,(su|\wedge^j V)) \geqslant j \cdot \min_i v(s_i).$$

Let us verify this. Let $(u_{\mathbf{i},\mathbf{i}'})_{\mathbf{i},\mathbf{i}' \in \mathcal{B}_j}$ denote the matrix entries for the u-action on $\wedge^j V$ with respect to the above basis. Observe that $v(u_{\mathbf{i},\mathbf{i}'}) \geqslant j \cdot v_{\min}(u)$ for all $\mathbf{i}, \mathbf{i}' \in \mathcal{B}_j$. Then

$$v(\text{tr}\,(su|\wedge^j V)) = v\left(\sum_{\mathbf{i} \in \mathcal{B}_j} s_{i_1} s_{i_2} \ldots s_{i_j} \cdot u_{\mathbf{i},\mathbf{i}}\right)$$

$$\geqslant \min_{\mathbf{i}} v(s_{i_1} s_{i_2} \ldots s_{i_j} \cdot u_{\mathbf{i},\mathbf{i}}) \geqslant j \cdot \min_i v(s_i) + \min_{\mathbf{i}} v(u_{\mathbf{i},\mathbf{i}})$$

$$\geqslant j(\min_i v(s_i) + v_{\min}(u)).$$

The coefficients of the characteristic polynomial for $su \in \text{GL}_m(F_v)$ are given by $\text{tr}\,(su|\wedge^j V)$ up to sign. The above claim and an elementary argument with the Newton polygon show that any root λ satisfies $v(\lambda) \geqslant v_{\min}(u) + \min_i v(s_i)$. Finally, applying the argument so far to s^{-1} and u^{-1}, we obtain the upper bound for $v(\lambda)$. □

As before, the smooth reductive model for G over $\mathcal{O}$ such that $G(\mathcal{O}) = K$ will still be denoted G.

Lemma 2.17 *Let* $\Xi\colon G \hookrightarrow \text{GL}_m$ *be an embedding of algebraic groups over* $\mathcal{O}$. *Then there exists a* $GL_m(\mathcal{O})$-*conjugate of* Ξ *which maps* A (*a fixed maximal split torus of* G) *into the diagonal maximal torus of* GL_m.

Proof Note that the maximal F-split torus A naturally extend to $A \subset G$ over $\mathcal{O}$, cf. [103, §3.5]. The representation of A on a free $\mathcal{O}$-module of rank m via Ξ defines a weight decomposition of $\mathcal{O}^m$ into free $\mathcal{O}$-modules. Choose any refinement of the decomposition to write $\mathcal{O}^m = L_1 \oplus \cdots \oplus L_m$, as the direct sum of rank 1 free $\mathcal{O}$-submodules. Let v_i be an $\mathcal{O}$-generator of L_i for $1 \leqslant i \leqslant m$. Conjugating Ξ by the matrix representing the change of basis from $\{v_1, \ldots, v_m\}$ to the standard basis for $\mathcal{O}^m$, one can achieve that $\Xi(A)$ lies in the diagonal maximal torus. □

Let $\gamma \in G(F)$ be a semisimple element and choose a maximal torus T_γ of G defined over F such that $\gamma \in T_\gamma(F)$. Denote by $\Phi(G, T_\gamma)$ the set of roots for T_γ in G.

Lemma 2.18 *Suppose that there exists an embedding of algebraic groups* $\Xi : G \hookrightarrow \mathrm{GL}_m$ *over* $\mathcal{O}$. *There exists a constant* $B_5 > 0$ *such that for every* $\kappa \in \mathbb{Z}_{\geq 0}$, *every* $\mu \in X_*(A)$ *satisfying* $\|\mu\| \leqslant \kappa$, *every semisimple* $\gamma \in K\mu(\varpi)K$ *and every* $\alpha \in \Phi_\gamma$ (*for any choice of* T_γ *as above*), *we have* $-B_5\kappa \leqslant v(\alpha(\gamma)) \leqslant B_5\kappa$. *In particular,* $|1 - \alpha(\gamma)| \leqslant q^{B_5\kappa}$.

Remark 2.19 Later Ξ will be provided by Proposition 8.1.

Proof We may assume that $\Xi(A)$ is contained in the diagonal torus of GL_m, denoted by $\mathbb{T}$, thanks to Lemma 2.17. Write T for the maximal torus of G which is the centralizer of A so that $\Xi(T) \subset \mathbb{T}$. We have a surjection $X^*(\mathbb{T}) \twoheadrightarrow X^*(T)$ induced by Ξ. For each α in the set of roots $\Phi(G, T)$, we fix a lift $\widetilde{\alpha} \in X^*(\mathbb{T})$ once and for all. Set $c_1 := \max_{\alpha \in \Phi(G,T)} \|\widetilde{\alpha}\|_{\mathrm{GL}_m}$.

Let $c_2 := \max_{\|\mu\| \leqslant 1} \|\Xi \circ \mu\|_{\mathrm{GL}_m}$ where $\mu \in X_*(A)_\mathbb{R}$ runs over elements with $\|\mu\| \leqslant 1$. Then for any $\kappa \in \mathbb{Z}_{\geq 0}$, $\|\mu\| \leqslant \kappa$ implies $\|\Xi \circ \mu\|_{\mathrm{GL}_m} \leqslant c_2\kappa$. Hence $\Xi(\mu(\varpi))$ is a diagonal matrix in which each entry x satisfies $-c_2\kappa \leqslant v(x) \leqslant c_2\kappa$.

We can write $\gamma = k_1\mu(\varpi)k_2$ for some $k_1, k_2 \in G(\mathcal{O})$. Then $\Xi(\gamma) = k_1'\Xi(\mu(\varpi))k_2'$ for $k_1', k_2' \in \mathrm{GL}_m(\mathcal{O})$, and $\Xi(\gamma)$ is conjugate to $\Xi(\mu(\varpi))k_2'(k_1')^{-1}$. It follows from Lemma 2.15 that for every eigenvalue λ of $\Xi(\gamma)$, we have $-c_2\kappa \leqslant v(\lambda) \leqslant c_2\kappa$.

Choose any T_γ as above. There exists an isomorphism $T \simeq T_\gamma$ over $\overline{F}$ induced by a conjugation action $t \mapsto gtg^{-1}$ given by some $g \in G(\overline{F})$. The isomorphism is well defined only up to the Weyl group action but induces a bijection from $\Phi(G, T)$ onto $\Phi(G, T_\gamma)$. Put $\mathbb{T}_\gamma := \Xi(g)\mathbb{T}\Xi(g)^{-1}$. The conjugation by $\Xi(g)$ induces an isomorphism $\mathbb{T} \simeq \mathbb{T}_\gamma$ over $\overline{F}$ and a bijection from $\Phi(\mathrm{GL}_m, \mathbb{T})$ onto $\Phi(\mathrm{GL}_m, \mathbb{T}_\gamma)$. Let $\alpha_\gamma \in \Phi(G, T_\gamma)$ (resp. $\widetilde{\alpha}_\gamma \in \Phi(\mathrm{GL}_m, \mathbb{T}_\gamma)$) denote the image of α (resp. $\widetilde{\alpha}$) under the bijections. By construction, the composition $T_\gamma \simeq T \overset{\Xi}{\to} \mathbb{T} \simeq \mathbb{T}_\gamma$ coincides with the restriction of Ξ to T_γ. Hence the induced map $X^*(\mathbb{T}_\gamma) \to X^*(T_\gamma)$ maps $\widetilde{\alpha}_\gamma$ to α_γ.

Using the isomorphisms $\mathbb{T}_\gamma(\overline{F}) \simeq \mathbb{T}(\overline{F}) \simeq (\overline{F}^\times)^m$, let $(\lambda_1, \ldots, \lambda_m) \in (\overline{F}^\times)^m$ be the image of $\Xi(\gamma)$ under the composition isomorphism. We may write $\widetilde{\alpha}_\gamma$ as a character $(\overline{F}^\times)^m \to \overline{F}^\times$ given by $(t_1, \ldots, t_m) \mapsto t_1^{a_1} \ldots t_m^{a_m}$ with $a_1, \ldots, a_m \in \mathbb{Z}$ such that $-c_1 \leqslant a_i \leqslant c_1$ for every $1 \leqslant i \leqslant m$. We have

$$\alpha_\gamma(\gamma) = \widetilde{\alpha}_\gamma(\Xi(\gamma)) = \lambda_1^{a_1} \ldots \lambda_m^{a_m},$$

so $v(\alpha_\gamma(\gamma)) = \sum_{i=1}^m a_i v(\lambda_i)$. Hence $-mc_1c_2\kappa \leqslant v(\alpha_\gamma(\gamma)) \leqslant mc_1c_2\kappa$, proving the first assertion of the lemma. From this the last assertion is obvious. $\square$

Remark 2.20 Suppose that F runs over the completions of a number field $\mathbf{F}$ at non-archimedean places v, that G over F comes from a fixed reductive group

G over **F**, and that Ξ comes from an embedding $\mathbf{G} \hookrightarrow GL_m$ over the integer ring of **F** (at least for every v where **G** is unramified). Then B_5 of the lemma can be chosen to be independent of v (and dependent only on the data over **F**). This is easy to see from the proof.

3 Plancherel measure on the unramified spectrum

3.1 Basic setup and notation

Let F be a finite extension of $\mathbb{Q}_p$. Suppose that G is unramified over F. Fix a hyperspecial subgroup K of G. Recall the notation from the start of Sect. 2.2. In particular Ω (resp. Ω_F) denotes the Weyl group for $(G_{\overline{F}}, T_{\overline{F}})$ [resp. (G, A)]. There is a natural $\mathrm{Gal}(\overline{F}/F)$-action on Ω, under which $\Omega^{\mathrm{Gal}(\overline{F}/F)} = \Omega_F$. (See [9, §6.1].) Since G is unramified, $\mathrm{Gal}(\overline{F}/F)$ factors through a finite unramified Galois group. Thus there is a well-defined action of Fr on Ω, and $\Omega^{\mathrm{Fr}} = \Omega_F$.

The unitary dual $G(F)^{\wedge}$ of $G(F)$, or simply $G^{\wedge}$ if there is no danger of ambiguity, is equipped with Fell topology. (This notation should not be confused with the dual group $\widehat{G}$). Let $G^{\wedge,\mathrm{ur}}$ denote the unramified spectrum in $G^{\wedge}$, and $G^{\wedge,\mathrm{ur,temp}}$ its tempered sub-spectrum. The Plancherel measure $\widehat{\mu}^{\mathrm{pl}}$ on $G^{\wedge}$ is supported on the tempered spectrum $G^{\wedge,\mathrm{temp}}$. The restriction of $\widehat{\mu}^{\mathrm{pl}}$ to $G^{\wedge,\mathrm{ur}}$ will be written as $\widehat{\mu}^{\mathrm{pl,ur}}$. The latter is supported on $G^{\wedge,\mathrm{ur,temp}}$. Harish-Chandra's Plancherel formula (cf. [106]) tells us that $\widehat{\mu}^{\mathrm{pl}}(\widehat{\phi}) = \phi(1)$ for all $\phi \in \mathcal{H}(G(F))$. In particular, $\widehat{\mu}^{\mathrm{pl,ur}}(\widehat{\phi}) = \phi(1)$ for all $\phi \in \mathcal{H}^{\mathrm{ur}}(G(F))$.

3.2 The unramified tempered spectrum

An unramified L-parameter $W_F^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$ is defined to be an L-morphism ${}^L H^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$ (Sect. 2.5) with $H = \{1\}$. Two such parameters φ_1 and φ_2 are considered equivalent if $\varphi_1 = g\varphi_2 g^{-1}$ for some $g \in \widehat{G}$. Consider the following sets:

(i) Irreducible unramified representations π of $G(F)$ up to isomorphism.
(ii) Group homomorphisms $\chi : T(F)/T(F) \cap K \to \mathbb{C}^{\times}$ up to Ω_F-action.
(iii) Unramified L-parameters $\varphi : W_F^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$ up to equivalence.
(iv) Elements of $(\widehat{G} \rtimes \mathrm{Fr})_{\mathrm{ss-conj}}$; this set was defined in Sect. 2.2.
(v) Ω^{Fr}-orbits in $\widehat{T}/(\mathrm{Fr} - \mathrm{id})\widehat{T}$.
(vi) Ω_F-orbits in $\widehat{A}$.
(viii) $\mathbb{C}$-algebra morphisms $\theta : \mathcal{H}^{\mathrm{ur}}(G) \to \mathbb{C}$.

Let us describe canonical maps among them in some directions.

- (i) $\to$ (vii) Choose any $0 \neq v \in \pi^K$. Define $\theta(\phi)$ by $\theta(\phi)v = \int_{G(F)} \phi(g)\pi(g)v\,dg$.

- (ii) → (i) π is the unique unramified subquotient of n-ind$_{B(F)}^{G(F)}\chi$.
- (ii) ↔ (vi) Induced by $\mathrm{Hom}(T(F)/T(F) \cap K, \mathbb{C}^\times) \simeq \mathrm{Hom}(A(F)/A(F) \cap K, \mathbb{C}^\times)$

$$\simeq \mathrm{Hom}(X_*(A), \mathbb{C}^\times) \simeq \mathrm{Hom}(X^*(\widehat{A}), \mathbb{C}^\times) \simeq X_*(\widehat{A}) \otimes_\mathbb{Z} \mathbb{C}^\times \simeq \widehat{A} \quad (3.1)$$

  where the second isomorphism is induced by $X_*(A) \to A(F)$ sending μ to $\mu(\varpi)$.
- (iii) → (iv) Take $\varphi(\mathrm{Fr})$.
- (v) → (iv) Induced by the inclusion $t \mapsto t \rtimes \mathrm{Fr}$ from $\widehat{T}$ to $\widehat{G} \rtimes \mathrm{Fr}$.
- (v) → (vi) Induced by the surjection $\widehat{T} \twoheadrightarrow \widehat{A}$, which is the dual of $A \hookrightarrow T$. (Recall $\Omega^{\mathrm{Fr}} = \Omega_F$.)
- (vii) → (vi) Via $\mathcal{S} \colon \mathcal{H}^{\mathrm{ur}}(G) \simeq \mathbb{C}[X^*(\widehat{A})]^{\Omega_F}$, θ determines an element of [cf. (3.1)]

$$\Omega_F \backslash \mathrm{Hom}(X^*(\widehat{A}), \mathbb{C}^\times) \simeq \Omega_F \backslash \widehat{A}.$$

Lemma 3.1 *Under the above maps, the sets corresponding to (i)–(vii) are in bijection with each other.*

Proof See §6, §7 and §10.4 of [9]. □

Let F' be the finite unramified extension of F such that $\mathrm{Gal}(\overline{F}/F)$ acts on $\widehat{G}$ through the faithful action of $\mathrm{Gal}(F'/F)$. Write $^L G_{F'/F} := \widehat{G} \rtimes \mathrm{Gal}(F'/F)$. Let $\widehat{K}$ be a maximal compact subgroup of $\widehat{G}$ which is Fr-invariant. Denote by $\widehat{T}_c$ (resp. $\widehat{A}_c$) the maximal compact subtorus of $\widehat{T}$ (resp. $\widehat{A}$).

Lemma 3.2 *The above bijections restrict to the bijections among the sets consisting of the following objects.*

(i)$_t$ *irreducible unramified tempered representations π of $G(F)$ up to isomorphism.*

(ii)$_t$ *unitary group homomorphisms $\chi \colon T(F)/T(F) \cap K \to U(1)$ up to Ω_F-action.*

(iii)$_t$ *unramified L-parameters $\varphi \colon W_F^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$ with bounded image up to equivalence.*

(iv)$_t$ *$\widehat{G}$-conjugacy classes in $\widehat{K} \rtimes \mathrm{Fr}$ (viewed in $^L G_{F'/F}$).*

(iv)$_t'$ *$\widehat{K}$-conjugacy classes in $\widehat{K} \rtimes \mathrm{Fr}$ (viewed in $\widehat{K} \rtimes \mathrm{Gal}(F'/F)$).*

(v)$_t$ *Ω^{Fr}-orbits in $\widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c$.*

(vi)$_t$ *Ω_F-orbits in $\widehat{A}_c$.*

[*The boundedness in (iii)$_t$ means that the projection of $\mathrm{Im}\,\varphi$ into $^L G_{F'/F}$ is contained in a maximal compact subgroup of $^L G_{F'/F}$.*]

Proof $(i)_t \leftrightarrow (ii)_t$ is standard and $(iii)_t \leftrightarrow (iv)_t$ is obvious. Also straightforward is $(ii)_t \leftrightarrow (vi)_t$ in view of (3.1).

Let us show that $(v)_t \leftrightarrow (vi)_t$. Choose a topological isomorphism of complex tori $\widehat{T} \simeq (\mathbb{C}^\times)^d$ with $d = \dim T$. Using $\mathbb{C}^\times \simeq U(1) \times \mathbb{R}_{>0}^\times$, we can decompose $\widehat{T} = \widehat{T}_c \times \widehat{T}_{nc}$ such that $\widehat{T}_{nc}$ is carried over to $(\mathbb{R}_{>0}^\times)^d$ under the isomorphism. The decomposition of $\widehat{T}$ is canonical in that it is preserved under any automorphism of $\widehat{T}$. By the same reasoning, there is a canonical decomposition $\widehat{A} = \widehat{A}_c \times \widehat{A}_{nc}$ with $\widehat{A}_{nc} \simeq (\mathbb{R}_{>0}^\times)^{\dim A}$. The canonical surjection $\widehat{T} \to \widehat{A}$ carries $\widehat{T}_c$ onto $\widehat{A}_c$ and $\widehat{T}_{nc}$ onto $\widehat{A}_{nc}$. [This reduces to the assertion in the case of $\mathbb{C}^\times$, namely that any maps $U(1) \to \mathbb{R}_{>0}^\times$ and $\mathbb{R}_{>0}^\times \to U(1)$ induced by an algebraic map $\mathbb{C}^\times \to \mathbb{C}^\times$ of $\mathbb{C}$-tori are trivial. This is easy to check.] Therefore the isomorphism $\widehat{T}/(\mathrm{Fr} - \mathrm{id})\widehat{T} \to \widehat{A}$ of Lemma 3.2 induces an isomorphism $\widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c \to \widehat{A}_c$ (as well as $\widehat{T}_{nc}/(\mathrm{Fr} - \mathrm{id})\widehat{T}_{nc} \to \widehat{A}_{nc}$).

Next we show that $(iv)_t \leftrightarrow (v)_t$. It is clear that $t \mapsto t \rtimes \mathrm{Fr}$ maps $(v)_t$ into $(iv)_t$. Since $(v)_t$ and $(iv)_t$ are the subsets of (v) and (iv), which are in bijective correspondence, we deduce that $(v)_t \to (iv)_t$ is injective. To show surjectivity, pick any $k \in \widehat{K}$. There exists $t \in \widehat{T}$ such that the image of t in (iv) corresponds under $(iv) \leftrightarrow (v)$ to the $\widehat{G}$-conjugacy class of $\widehat{k} \rtimes \mathrm{Fr}$. It is enough to show that we can choose $t \in \widehat{T}_c$. Consider the subgroup $\widehat{T}_c(t)$ of

$$\widehat{T}/(\mathrm{Fr} - \mathrm{id})\widehat{T} = \widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c \times \widehat{T}_{nc}/(\mathrm{Fr} - \mathrm{id})\widehat{T}_{nc}$$

generated by $\widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c$ and the image of t. The isomorphism $(iv) \leftrightarrow (v)$ maps $\widehat{T}_c(t)$ into $(v)_t$ by the assumption on t. Since $(v)_t$ form a compact set, the group $\widehat{T}_c(t)$ must be contained in a compact subset of $\widehat{T}/(\mathrm{Fr} - \mathrm{id})\widehat{T}$. This forces the image of t in $\widehat{T}_{nc}/(\mathrm{Fr} - \mathrm{id})\widehat{T}_{nc}$ to be trivial. (Indeed, the latter quotient is isomorphic as a topological group to a quotient of $\mathbb{R}^{\dim T}$ modulo an $\mathbb{R}$-subspace via the exponential map. So any subgroup generated by a nontrivial element is not contained in a compact set.) Therefore t can be chosen in $\widehat{T}_c$.

It remains to verify that $(iv)_t$, $(iv)'_t$ and $(v)_t$ are in bijection. Clearly $(iv)'_t \to (iv)_t$ is onto. As we have just seen that $(iv)_t \leftrightarrow (v)_t$, it suffices to observe that $(v)_t \to (iv)'_t$ is onto, which is a standard fact [for instance in the context of the (twisted) Weyl integration formula for $\widehat{K} \rtimes \mathrm{Fr}$]. $\qquad \square$

3.3 Plancherel measure on the unramified spectrum

Lemma 3.2 provides a bijection $G^{\wedge,\mathrm{ur,temp}} \simeq \Omega_F \backslash \widehat{A}_c$, which is in fact a topological isomorphism. The Plancherel measure $\widehat{\mu}^{\mathrm{pl,ur}}$ on $G^{\wedge,\mathrm{ur}}$ is supported on $G^{\wedge,\mathrm{ur,temp}}$. We would like to describe its pullback measure on $\widehat{A}_c$, to be denoted $\widehat{\mu}_0^{\mathrm{pl,ur,temp}}$. Note that $\widehat{A}_c$ is topologically isomorphic to $\widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c$.

(This is induced by the natural surjection $\widehat{T}_c \twoheadrightarrow \widehat{A}_c$.) Fix a measure $d\bar{t}$ on the latter which is a push forward from a Haar measure on $\widehat{T}_c$.

Proposition 3.3 *The measure* $\widehat{\mu}_0^{\mathrm{pl,ur,temp}}$ *pulled back to* $\widehat{T}_c/(\mathrm{Fr} - \mathrm{id})\widehat{T}_c$ *is*

$$\widehat{\mu}_0^{\mathrm{pl,ur,temp}}(\bar{t}) = C \cdot \frac{\det(1 - \mathrm{ad}(t \rtimes \mathrm{Fr})|\mathrm{Lie}\,(\widehat{G})/\mathrm{Lie}\,(\widehat{T}^{\mathrm{Fr}}))}{\det(1 - q^{-1}\mathrm{ad}(t \rtimes \mathrm{Fr})|\mathrm{Lie}\,(\widehat{G})/\mathrm{Lie}\,(\widehat{T}^{\mathrm{Fr}}))} d\bar{t}$$

for some constant $C \in \mathbb{C}^\times$, *depending on the normalization of Haar measures. Here* $t \in \widehat{T}_c$ *is any lift of* $\bar{t}$. (*The right hand side is independent of the choice of* t.)

Proof The formula is due to Macdonald [72]. For our purpose, it is more convenient to follow the formulation as in the conjecture of [98, p. 281] (which also discusses the general conjectural formula of the Plancherel measure due to Langlands). By that conjecture (known in the unramified case),

$$\widehat{\mu}_0^{\mathrm{pl,ur,temp}}(\bar{t}) = C' \cdot \frac{L(1, \sigma^{-1}(\bar{t}), r)}{L(0, \sigma(\bar{t}), r)} \frac{L(1, \sigma(\bar{t}), r)}{L(0, \sigma^{-1}(\bar{t}), r)} d\bar{t}$$

where $C' \in \mathbb{C}^\times$ is a constant, $\sigma(\bar{t}) \colon T(F) \to \mathbb{C}^\times$ is the character corresponding to $\bar{t}$ [via (ii) $\leftrightarrow$ (v) of Lemma 3.1], and $r \colon {}^L T \to \mathrm{GL}(\mathrm{Lie}\,({}^L U))$ is the adjoint representation. Here ${}^L U$ is the L-group of U (viewed in ${}^L B$). By unraveling the local L-factors, obtain

$$\widehat{\mu}_0^{\mathrm{pl,ur,temp}}(\bar{t}) = C' \cdot \frac{\det(1 - \mathrm{ad}(t \rtimes \mathrm{Fr})|\mathrm{Lie}\,(\widehat{G})/\mathrm{Lie}\,(\widehat{T}))}{\det(1 - q^{-1}\mathrm{ad}(t \rtimes \mathrm{Fr})|\mathrm{Lie}\,(\widehat{G})/\mathrm{Lie}\,(\widehat{T}))} d\bar{t}. \quad (3.2)$$

Finally, observe that $\det(1 - q^{-s}\mathrm{ad}(t \rtimes \mathrm{Fr})|\mathrm{Lie}\,(\widehat{T})/\mathrm{Lie}\,(\widehat{T}^{\mathrm{Fr}}))$ is independent of $\bar{t}$ (and t). Therefore the right hand sides are the same up to constant in (3.2) and the proposition. □

Remark 3.4 Note that the choice of a Haar measure on $G(F)$ determines the measure $\widehat{\mu}_0^{\mathrm{pl,ur,temp}}$. For example if the Haar measure on $G(F)$ assigns volume 1 to K then $G^{\wedge,\mathrm{ur,temp}}$ has total volume 1 with respect to $\widehat{\mu}_0^{\mathrm{pl,ur,temp}}(\bar{t})$ as implied by the Plancherel formula for $\mathbf{1}_K$. Hence the product $C \cdot d\bar{t}$.

4 Automorphic L-functions

According to Langlands conjectures, the most general L-functions should be expressible as products of the principal L-functions $L(s, \Pi)$ associated to cuspidal automorphic representations Π of $\mathrm{GL}(d)$ over number fields (for

varying d). The analytic properties and functional equation of such L-functions were first established by Godement–Jacquet for general $d \geqslant 1$. This involves the Godement–Jacquet integral representation. The other known methods are the Rankin–Selberg integrals, the doubling method and the Langlands–Shahidi method. The purpose of this section is to recall these analytic properties and to set-up notation. More detailed discussions may be found in [32,55,75], [86, §2] and [52, §5].

In this section and some of the later sections we use the following notation.

- F is a number field, i.e. a finite extension of $\mathbb{Q}$.
- G is a connected reductive group over F (not assumed to be quasi-split).
- $Z = Z(G)$ is the center of G.
- $\mathcal{V}_F$ (resp. $\mathcal{V}_F^\infty$) is the set of all (resp. all finite) places of F.
- $S_\infty := \mathcal{V}_F \backslash \mathcal{V}_F^\infty$.
- A_G is the maximal F-split subtorus in the center of $\mathrm{Res}_{F/\mathbb{Q}} G$, and $A_{G,\infty} := A_G(\mathbb{R})^0$.

4.1 Automorphic forms

Let $\chi \colon A_{G,\infty} \to \mathbb{C}^\times$ be a continuous homomorphism. Denote by $L_\chi^2(G(F) \backslash G(\mathbb{A}_F))$ the space of all functions f on $G(\mathbb{A}_F)$ which are square-integrable modulo $A_{G,\infty}$ and satisfy $f(g\gamma z) = \chi(z) f(\gamma)$ for all $g \in G(F), \gamma \in G(\mathbb{A}_F)$ and $z \in A_{G,\infty}$. There is a spectral decomposition into discrete and continuous parts

$$L_\chi^2(G(F) \backslash G(\mathbb{A}_F)) = L_{\mathrm{disc},\chi}^2 \oplus L_{\mathrm{cont},\chi}^2, \quad L_{\mathrm{disc},\chi}^2 = \widehat{\bigoplus_\pi} m_{\mathrm{disc},\chi}(\pi) \cdot \pi$$

where the last sum is a Hilbert direct sum running over the set of all irreducible representations of $G(\mathbb{A}_F)$ up to isomorphism. Write $\mathcal{AR}_{\mathrm{disc},\chi}(G)$ for the set of isomorphism classes of all irreducible representations π of $G(\mathbb{A}_F)$ such that $m_{\mathrm{disc},\chi}(\pi) > 0$. Any $\pi \in \mathcal{AR}_{\mathrm{disc},\chi}(G)$ is said to be a discrete automorphic representation of $G(\mathbb{A}_F)$. If χ is trivial (in particular if $A_{G,\infty} = \{1\}$) then we write m_{disc} for $m_{\mathrm{disc},\chi}$.

The above definitions allow a modest generalization. Let $\mathfrak{X}_G$ be a closed subgroup of $Z(\mathbb{A}_F)$ containing $A_{G,\infty}$ and $\omega \colon Z(\mathbb{A}_F) \cap \mathfrak{X}_G \backslash \mathfrak{X}_G \to \mathbb{C}^\times$ be a continuous (quasi-)character. Then $L_\omega^2, L_{\mathrm{disc},\omega}^2, m_{\mathrm{disc},\omega}$ etc can be defined analogously. In fact the Arthur-Selberg trace formula applies to this setting. (See [4, Ch 3.1].)

For the rest of Sect. 4 we are concerned with $G = \mathrm{GL}(d)$. Take $\mathfrak{X}_G = Z(\mathbb{A}_F)$ so that ω is a quasi-character of $Z(F) \backslash Z(\mathbb{A}_F)$. Note that $A_{G,\infty} = Z(F_\infty)^\circ$ in this case. We denote by $\mathcal{A}_\omega(\mathrm{GL}(d, F))$ the space consisting of automorphic functions on $\mathrm{GL}(d, F) \backslash \mathrm{GL}(d, \mathbb{A}_F)$ which satisfy $f(zg) = \omega(z) f(g)$

for all $z \in Z(\mathbb{A}_F)$ and $g \in \mathrm{GL}(d, \mathbb{A}_F)$ (see Borel and Jacquet [10] for the exact definition and the growth condition). We denote by $\mathcal{A}_{\mathrm{cusp},\omega}(\mathrm{GL}(d, F))$ the subspace of cuspidal functions (i.e. the functions with vanishing period against all nontrivial unipotent subgroups).

An automorphic representation Π of $\mathrm{GL}(d, \mathbb{A}_F)$ is by definition an irreducible admissible representation of $\mathrm{GL}(d, \mathbb{A}_F)$ which is a constituent of the regular representation on $\mathcal{A}_\omega(\mathrm{GL}(d, F))$. Then ω is the central character of Π. The subspace $\mathcal{A}_{\mathrm{cusp},\omega}(\mathrm{GL}(d, F))$ decomposes discretely and an irreducible component is a cuspidal automorphic representation. The notion of cuspidal automorphic representations is the same if the space of cuspidal functions in $L^2_\omega(GL(d, F)\backslash GL(d, \mathbb{A}_F))$ is used in the definition in place of $\mathcal{A}_{\mathrm{cusp},\omega}(\mathrm{GL}(d, F))$, cf. [10, §4.6].

When ω is unitary we can work with the completed space $L^2_\omega(\mathrm{GL}(d, F)\backslash \mathrm{GL}(d, \mathbb{A}_F))$ of square-integrable functions modulo $Z(\mathbb{A}_F)$ and with unitary automorphic representations. Note that a cuspidal automorphic representation is unitary if and only if its central character is unitary. We recall the Langlands decomposition of $L^2_\omega(\mathrm{GL}(d, F)\backslash \mathrm{GL}(d, \mathbb{A}_F))$ into the cuspidal, residual and continuous spectra. What will be important in the sequel is the notion of isobaric representations which we review in Sect. 4.3.

In the context of L-functions, the functional equation involves the contragredient representation $\widetilde{\Pi}$. An important fact is that the contragredient of a unitary automorphic representation of $\mathrm{GL}(d, \mathbb{A}_F)$ is isomorphic to its complex conjugate.

4.2 Principal L-functions

Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}(d, \mathbb{A}_F)$ with unitary central character. The principal L-function associated to Π is denoted

$$L(s, \Pi) = \prod_{v \in \mathcal{V}_F^\infty} L(s, \Pi_v).$$

The Euler product is absolutely convergent when $\Re e\, s > 1$. The completed L-function is denoted $\Lambda(s, \Pi)$, the product now running over all places $v \in \mathcal{V}_F$. For each finite place $v \in \mathcal{V}_F^\infty$, the inverse of the local L-function $L(s, \Pi_v)$ is a Dirichlet polynomial in q_v^{-s} of degree $\leqslant d$. Write

$$L(s, \Pi_v) = \prod_{i=1}^{d} \left(1 - \alpha_i(\Pi_v) q_v^{-s}\right)^{-1}.$$

Note that when Π_v is unramified, $\alpha_i(\Pi_v)$ is non-zero for all i and corresponds to the eigenvalues of a semisimple conjugacy class in $\mathrm{GL}_d(\mathbb{C})$ associated to

Π_v, but when Π_v is ramified the Langlands parameters are more sophisticated and we allow some (or even all of) the $\alpha_i(\Pi_v)$ to be equal to zero. In this way we have a convenient notation for all local L-factors.

For each archimedean v, the local L-function $L(s, \Pi_v)$ is a product of d Gamma factors

$$L(s, \Pi_v) = \prod_{i=1}^{d} \Gamma_v(s - \mu_i(\Pi_v)), \tag{4.1}$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Note that $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ by the doubling formula, so when v is complex, $L(s, \Pi_v)$ may as well be expressed as a product of $2d$ $\Gamma_{\mathbb{R}}$ factors.

The completed L-function $\Lambda(s, \Pi) := L(s, \Pi) \prod_{v|\infty} L(s, \Pi_v)$ has the following analytic properties. It has a meromorphic continuation to the complex plane. It is entire except when $d = 1$ and $\Pi = |.|^{it}$ for some $t \in \mathbb{R}$, in which case $L(s, \Pi) = \zeta_F(s + it)$ is (a shift of) the Dedekind zeta function of the ground field F with simple poles at $s = -it$ and $s = 1 - it$. It is bounded in vertical strips and satisfies the functional equation

$$\Lambda(s, \Pi) = \epsilon(s, \Pi)\Lambda(1 - s, \widetilde{\Pi}), \tag{4.2}$$

where $\epsilon(s, \Pi)$ is the epsilon factor and $\widetilde{\Pi}$ is the contragredient automorphic representation. The epsilon factor has the form

$$\epsilon(s, \Pi) = \epsilon(\Pi)q(\Pi)^{\frac{1}{2}-s} \tag{4.3}$$

for some positive integer $q(\Pi) \in \mathbb{Z}_{\geq 1}$ and root number $\epsilon(\Pi)$ of modulus one.

Note that $q(\Pi) = q(\widetilde{\Pi})$, $\epsilon(\widetilde{\Pi}) = \overline{\epsilon(\Pi)}$ and for all $v \in \mathcal{V}_F$, $L(s, \widetilde{\Pi}_v) = \overline{L(\overline{s}, \Pi_v)}$. For instance this follows from the fact [42] that $\widetilde{\Pi}$ is isomorphic to the complex conjugate $\overline{\Pi}$ (obtained by taking the complex conjugate of all forms in the vector space associated to the representation Π).

The conductor $q(\Pi)$ is the product over all finite places $v \in \mathcal{V}_F^\infty$ of the conductor $q(\Pi_v)$ of Π_v. Recall that $q(\Pi_v)$ equals one whenever Π_v is unramified. It is convenient to introduce as well the conductor of admissible representations at archimedean places. When v is real we let $C(\Pi_v) = \prod_{i=1}^{d}(2 + |\mu_i(\Pi_v)|)$ and when v is complex we let $C(\Pi_v) = \prod_{i=1}^{d}(2 + |\mu_i(\Pi_v)|^2)$. Then we let $C(\Pi)$ be the analytic conductor which is the product of all the local conductors

$$C(\Pi) := \prod_{v|\infty} C(\Pi_v) \prod_{v \in \mathcal{V}_F^\infty} q(\Pi_v) = C(\Pi_\infty)q(\Pi).$$

Note that $C(\Pi) \geq 2$ always.

There is $0 \leqslant \theta < \frac{1}{2}$ such that

$$\Re \mu_i(\Pi_v) \leqslant \theta, \quad \text{resp. } \log_{q_v} |\alpha_i(\Pi_v)| \leqslant \theta \tag{4.4}$$

for all archimedean v (resp. finite v) and $1 \leqslant i \leqslant d$. When Π_v is unramified we ask for

$$|\Re \mu_i(\Pi_v)| \leqslant \theta, \quad \text{resp. } \left|\log_{q_v} |\alpha_i(\Pi_v)|\right| \leqslant \theta. \tag{4.5}$$

The value $\theta = \frac{1}{2} - \frac{1}{d^2+1}$ is admissible by an argument of Serre and Luo–Rudnick–Sarnak based on the analytic properties of the Rankin–Selberg convolution $L(s, \Pi \times \widetilde{\Pi})$. Note that for all v, the local L-functions $L(s, \Pi_v)$ are entire on $\Re s > \theta$ and this contains the central line $\Re s = \frac{1}{2}$.

The generalized Ramanujan conjecture asserts that all Π_v are tempered (see [88] and the references herein). This is equivalent to having $\theta = 0$ in the inequalities (4.4) and (4.5). In particular we expect that when Π_v is unramified, $|\alpha_i(\Pi_v)| = 1$.

4.3 Isobaric sums

We need to consider slightly more general L-functions associated to non-cuspidal automorphic representations on $\mathrm{GL}(d, \mathbb{A}_F)$. These L-functions are products of the L-functions associated to cuspidal representations and studied in the previous Sect. 4.2. Closely related to this construction it is useful to introduce, following Langlands [70], the notion of isobaric sums of automorphic representations. The concept of isobaric representations is natural in the context of L-functions and the Langlands functoriality conjectures.

Let Π be an irreducible automorphic representation of $\mathrm{GL}(d, \mathbb{A}_F)$. Then a theorem of Langlands [10] states that there are integers $r \geqslant 1$ and $d_1, \ldots, d_r \geqslant 1$ with $d = d_1 + \cdots + d_r$ and cuspidal automorphic representations $\Pi_1, \ldots, \Pi_r$ of $\mathrm{GL}(d_1, \mathbb{A}_F), \cdots, \mathrm{GL}(d_r, \mathbb{A}_F)$ such that Π is a constituent of the induced representation of $\Pi_1 \otimes \cdots \otimes \Pi_r$ (from the Levi subgroup $\mathrm{GL}(d_1) \times \cdots \times \mathrm{GL}(d_r)$ of $\mathrm{GL}(d)$). A cuspidal representation is unitary when its central character is unitary. When all of Π_j are unitary then Π is unitary. But the converse is not true: note that even if Π is unitary, the representation Π_j need not be unitary in general.

We recall the generalized strong multiplicity one theorem of Jacquet and Shalika [54]. Suppose Π and Π' are irreducible automorphic representations of $\mathrm{GL}(d, \mathbb{A}_F)$ such that Π_v is isomorphic to Π'_v for almost all $v \in \mathcal{V}_F$ (we say that Π and Π' are weakly equivalent) and suppose that Π (resp. Π') is a constituent of the induced representation of $\Pi_1 \otimes \cdots \otimes \Pi_r$ (resp. $\Pi'_1 \otimes \cdots \otimes \Pi'_{r'}$). Then $r = r'$ and up to permutation the sets of cuspidal representations $\{\Pi_j\}$ and $\{\Pi'_j\}$ coincide. Note that this generalizes the strong multiplicity one

theorem of Piatetski-Shapiro which corresponds to the case where Π and Π' are cuspidal.

Conversely suppose $\Pi_1, \ldots, \Pi_r$ are cuspidal representations of $\mathrm{GL}(d_1, \mathbb{A}_F), \ldots, \mathrm{GL}(d_r, \mathbb{A}_F)$. Then from the theory of Eisenstein series there is a unique constituent of the induced representation of $\Pi_1 \otimes \cdots \otimes \Pi_r$ whose local components coincide at each place $v \in \mathcal{V}_F$ with the Langlands quotient of the local induced representation [70, §2]. It is denoted $\Pi_1 \boxplus \cdots \boxplus \Pi_r$ and called an isobaric representation (automorphic representations which are not isobaric are called anomalous). The above results of Langlands and Jacquet–Shalika may now be summarized by saying that an irreducible automorphic representation of $\mathrm{GL}(d, \mathbb{A}_F)$ is weakly equivalent to a unique isobaric representation.

We now turn to L-functions. The completed L-function associated to an isobaric representation $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is by definition

$$\Lambda(s, \Pi) = \prod_{j=1}^{r} \Lambda(s, \Pi_j).$$

All notation from the previous subsection will carry over to $\Lambda(s, \Pi)$. Namely we have the local L-factors $L(s, \Pi_v)$, the local Satake parameters $\alpha_i(\Pi_v)$ and $\mu_i(\Pi_v)$, the epsilon factor $\epsilon(s, \Pi)$, the root number $\epsilon(\Pi)$, the local conductors $q(\Pi_v)$, $C(\Pi_v)$ and the analytic conductor $C(\Pi)$. The Euler product converges absolutely for $\Re e \, s$ large enough.

One important difference concerns the bounds for local Satake parameters. Even if we assume that Π has unitary central character the inequalities (4.4) may not hold. We shall therefore require a stronger condition on Π.

Proposition 4.1 *Let Π be an isobaric representation of $\mathrm{GL}(d, \mathbb{A}_F)$. Assume that the archimedean component Π_∞ is tempered. Then the bounds towards Ramanujan are satisfied. Namely there is a positive constant $\theta < \frac{1}{2}$ such that for all $1 \leqslant i \leqslant d$ and all archimedean (resp. non-archimedean) places v,*

$$\Re e \, \mu_i(\Pi_v) \leqslant \theta, \quad resp. \quad \log_{q_v} |\alpha_i(\Pi_v)| \leqslant \theta. \tag{4.6}$$

Proof Let $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ be the isobaric decomposition with Π_j cuspidal. Then we will show that all Π_j have unitary central character, which implies Proposition 4.1.

By definition we have that Π_∞ is a Langlands quotient of the induced representation of $\Pi_{1\infty} \otimes \cdots \otimes \Pi_{r\infty}$. Since Π_∞ is tempered, this implies that all $\Pi_{j\infty}$ are tempered, and in particular have unitary central character. Then the (global) central character of Π_j is unitary as well. $\qquad \square$

Remark 4.2 In analogy with the local case, an isobaric representation $\Pi_1 \boxplus \cdots \boxplus \Pi_r$ where all cuspidal representations Π_j have unitary central character is called "tempered" in [70]. This terminology is fully justified only under the generalized Ramanujan conjecture for $GL(d, \mathbb{A}_F)$. To avoid confusion we use the adjective "tempered" for $\Pi = \otimes_v \Pi_v$ only in the strong sense that the local representations Π_v are tempered for all $v \in \mathcal{V}_F$.

Remark 4.3 In the proof of Proposition 4.1 we see the importance of the notion of isobaric representations and Langlands quotients. For instance a discrete series representation of $GL(2, \mathbb{R})$ is a constituent (but not a Langlands quotient) of an induced representation of a non-tempered character of $GL(1, \mathbb{R}) \times GL(1, \mathbb{R})$.

4.4 An explicit formula

Let Π be a unitary cuspidal representation of $GL(d, \mathbb{A}_F)$. Let $\rho_j(\Pi)$ denote the zeros of $\Lambda(s, \Pi)$ counted with multiplicities. These are also the non-trivial zeros of $L(s, \Pi)$. The method of Hadamard and de la Vallée Poussin generalizes from the Riemann zeta function to automorphic L-functions, and implies that $0 < \Re \rho_j(\Pi) < 1$ for all j. There is a polynomial $p(s)$ such that $p(s)\Lambda(s, \Pi)$ is entire and of order 1 ($p(s) = 1$ except when $d = 1$ and $\Pi = |.|^{it}$, in which case we choose $p(s) = (s - it)(1 - it - s)$).

The Hadamard factorization shows that there are $a = a(\Pi)$ and $b = b(\Pi)$ such that

$$p(s)\Lambda(s, \Pi) = e^{a+bs} \prod_j \left(1 - \frac{s}{\rho_j(\Pi)}\right) e^{s/\rho_j(\Pi)}.$$

The product is absolutely convergent in compact subsets away from the zeros $\rho_j(\Pi)$. The functional equation implies that

$$\sum_j \Re\left(\rho_j(\Pi)^{-1}\right) = -\Re b(\Pi).$$

The number of zeros of bounded imaginary part is bounded above uniformly:

$$\left|\{j, \left|\Im \rho_j(\Pi)\right| \leqslant 1\}\right| \ll \log C(\Pi).$$

Changing Π into $\Pi \otimes |.|^{it}$ we have an analogous uniform estimate for the number of zeros with $\left|\Im \rho_j(\Pi) - T\right| \leqslant 1$ (in particular this is $\ll_\Pi \log T$).

Let $N(T, \Pi)$ be the number of zeros with $\left|\Im \rho_j(\Pi)\right| \leqslant T$. Then the following estimate holds uniformly in $T \geqslant 1$ (Weyl's law):

$$N(T, \Pi) = \frac{T}{\pi}\left(d \log\left(\frac{T}{2\pi e}\right) + \log C(\Pi)\right) + O_\Pi(\log T).$$

The error term could be made uniform in Π, see [52, §5.3] for more details.[4] The main term can be interpreted as the variation of the argument of $C(\Pi)^{s/2}L(s, \Pi_\infty)$ along certain vertical segments.

We are going to discuss an explicit formula [see (4.8) below] expressing a weighted sum over the zeros of $\Lambda(s, \Pi)$ as a contour integral. It is a direct consequence of the functional Eq. (4.2) and Cauchy formula. The explicit formula is traditionally stated using the Dirichlet coefficients of the L-function $L(s, \Pi)$. For our purpose it is more convenient to maintain the Euler product factorization.

Definte $\gamma_j(\Pi)$ by $\rho_j(\Pi) = \frac{1}{2} + i\gamma_j(\Pi)$. We know that $\left|\Im \gamma_j(\Pi)\right| < \frac{1}{2}$ and under the GRH, $\gamma_j(\Pi) \in \mathbb{R}$ for all j.

It is convenient to denote by $\frac{1}{2} + ir_j(\Pi)$ the (eventual) poles of $\Lambda(s, \Pi)$ counted with multiplicity. We have seen that poles only occur when $\Pi = |.|^{it}$ in which case the poles are simple and $\{r_j(\Pi)\} = \{t + \frac{i}{2}, -t - \frac{i}{2}\}$.

The above discussion applies with little change to isobaric representations. If we also assume that Π_∞ is tempered then we have seen in the proof of Proposition 4.1 that $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ with Π_i unitary cuspidal representations of $GL(d_i, \mathbb{A})$ for all $1 \leqslant i \leqslant r$. In particular the bounds towards Ramanujan apply and $\left|\Im \gamma_j(\Pi)\right| < \frac{1}{2}$ for all j.

Let Φ be a Paley–Wiener function whose Fourier transform

$$\widehat{\Phi}(y) := \int_{-\infty}^{+\infty} \Phi(x)e^{-2\pi ixy}dx \tag{4.7}$$

has compact support. Note that Φ may be extended to an entire function on $\mathbb{C}$.

Proposition 4.4 *Let Π be an isobaric representation of* $GL(d, \mathbb{A})$ *satisfying the bounds towards Ramanujan* (4.4). *With notation as above and for $\sigma > \frac{1}{2}$, the following identity holds*

---

[4] One should be aware that Theorem 5.8 in [52] does not apply directly to our setting because it is valid under certain further assumptions on Π such as $\mu_i(\Pi_v)$ being real for archimedean places v.

$$\sum_j \Phi(\gamma_j(\Pi)) = \sum_j \Phi(r_j(\Pi)) + \frac{\log q(\Pi)}{2\pi}\widehat{\phi}(0)$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\Lambda'}{\Lambda}\left(\frac{1}{2}+\sigma+ir,\Pi\right)\Phi(r-i\sigma)\right.$$

$$\left.+ \overline{\frac{\Lambda'}{\Lambda}\left(\frac{1}{2}+\sigma+ir,\Pi\right)}\Phi(r+i\sigma)\right]dr. \tag{4.8}$$

There is an important remark about the explicit formula that we will use frequently. Therefore we insert it here before going into the proof. The line of integration in (4.8) is away from the zeros and poles because $\sigma > 1/2$. In particular the line of integration cannot be moved to $\sigma = 0$ directly. But we can do the following which is a natural way to produce the sum over primes. First we replace $\Lambda(s,\Pi)$ by its Euler product which is absolutely convergent in the given region ($\Re e\, s > 1$). Then for each of the term we may move the line of integration to $\sigma = 0$ because we have seen that $\frac{L'}{L}(s,\Pi_v)$ has no pole for $\Re e\, s > \theta$. Thus we have

$$\int_{-\infty}^{\infty}\frac{\Lambda'}{\Lambda}\left(\frac{1}{2}+\sigma+ir,\Pi\right)\Phi(r-i\sigma)dr$$

$$= \sum_{v\in\mathcal{V}_F}\int_{-\infty}^{\infty}\times\frac{L'}{L}\left(\frac{1}{2}+ir,\Pi_v\right)\Phi(r)dr. \tag{4.9}$$

The latter expression is convenient to use in practice. The integral in the right-hand side of (4.9) is absolutely convergent because Φ is rapidly decreasing and the sum over $v \in \mathcal{V}_F$ is actually finite since the support of $\widehat{\Phi}$ is compact. [5]

Proof The first step is to work with the Mellin transform rather than the Fourier transform. Namely we set

$$H\left(\frac{1}{2}+is\right) = \Phi(s), \quad s \in \mathbb{C}.$$

Note that H is an entire function which is rapidly decreasing on vertical strips. This justifies all shifting of contours below.

We form the integral

$$\int_{(2)}\frac{\Lambda'}{\Lambda}(s,\Pi)H(s)\frac{ds}{2i\pi}.$$

---

[5] Note however that it is never allowed to switch the sum and integration symbols in (4.9). This is because the L-function is evaluated at the center of the critical strip in which the Euler product does not converge absolutely.

We shift the contour to $\Re\,s = -1$ crossing zeros and eventual poles of $\dfrac{\Lambda'}{\Lambda}$ inside the critical strip. The sum over the zeros reads

$$\sum_j H(\rho_j(\Pi)) = \sum_j \Phi(\gamma_j(\Pi))$$

and the sum over the poles reads

$$-\sum_j \Phi(r_j(\Pi)).$$

Note that since $\epsilon(s, \Pi) = \epsilon(\Pi)q(\Pi)^{\frac{1}{2}-s}$ we have

$$\frac{\epsilon'}{\epsilon}(s, \Pi) = -\log q(\Pi), \quad s \in \mathbb{C}.$$

We obtain as consequence of the functional Eq. (4.2) that

$$\int_{(-1)} \frac{\Lambda'}{\Lambda}(s, \Pi)H(s)\frac{ds}{2i\pi} = \int_{(2)} \frac{\Lambda'}{\Lambda}(1-s, \Pi)H(1-s)\frac{ds}{2i\pi}$$

$$= -\int_{(2)}\left(\log q(\Pi) + \frac{\Lambda'}{\Lambda}(s, \widetilde{\Pi})\right)H(1-s)\frac{ds}{2i\pi}.$$

Now we observe that

$$\int_{(2)} H(s)\frac{ds}{2i\pi} = \frac{1}{2\pi}\widehat{\phi}(0)$$

and also

$$\int_{(2)} \frac{\Lambda'}{\Lambda}(s, \Pi)H(s)\frac{ds}{2i\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi\left(r - \frac{3i}{2}\right)\frac{\Lambda'}{\Lambda}(2 + ir, \Pi)dr.$$

and

$$\int_{(2)} \frac{\Lambda'}{\Lambda}(s, \widetilde{\Pi})H(1-s)\frac{ds}{2i\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi\left(r + \frac{3i}{2}\right)\frac{\Lambda'}{\Lambda}(2 - ir, \widetilde{\Pi})dr$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi\left(r + \frac{3i}{2}\right)\overline{\frac{\Lambda'}{\Lambda}(2 + ir, \Pi)}dr.$$

Since $\Lambda(s, \widetilde{\Pi}) = \overline{\Lambda(\bar{s}, \Pi)}$ this concludes the proof of the proposition by collecting all the terms above. Precisely this yields the formula when $\sigma = 3/2$, and then we can make $\sigma > 1/2$ arbitrary by shifting the line of integration. □

We conclude this section with a couple of remarks on symmetries. The first observation is that the functional equation implies that if ρ is a zero (resp. pole) of $\Lambda(s, \Pi)$ then so is $1 - \bar{\rho}$ (reflexion across the central line). Thus the set $\{\gamma_j(\Pi)\}$ (resp. $\{r_j(\Pi)\}$) is invariant by the reflexion across the real axis (namely γ goes into $\overline{\gamma}$). Note that this is compatible with the GRH which predicts that $\mathfrak{Re} \, \rho_j(\Pi) = \frac{1}{2}$ and $\gamma_j(\Pi) \in \mathbb{R}$.

Assuming Φ is real-valued the explicit formula is an identity between real numbers. Indeed the Schwartz reflection principle gives $\Phi(s) = \overline{\Phi(\bar{s})}$ for all $s \in \mathbb{C}$. Because of the above remark the sum over the zeros (resp. poles) in (4.8) is a real; the integrand is real-valued as well for all $r \in (-\infty, \infty)$.

The situation when Π is self-dual occurs often in practice. The zeros $\gamma_j(\Pi)$ satisfy another symmetry which is the reflexion across the origin. Assuming Π is cuspidal and non-trivial there is no pole. The explicit formula (4.8) simplifies and may be written

$$\sum_j \Phi\left(\gamma_j(\Pi)\right) = \frac{\log q(\Pi)}{2\pi} \widehat{\Phi}(0) + \frac{1}{\pi} \sum_{v \in \mathcal{V}_F} \int_{-\infty}^{\infty} \frac{L'}{L}\left(\frac{1}{2} + ir, \Pi_v\right) \Phi(r) dr.$$

5 Sato–Tate equidistribution

Let G be a connected reductive group over a number field F as in the previous section. The choice of a $\mathrm{Gal}(\overline{F}/F)$-invariant splitting datum $(\widehat{B}, \widehat{T}, \{X_\alpha\}_{\alpha \in \Delta^\vee})$ as in Sect. 2.1 induces a composite map $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Out}(\widehat{G}) \hookrightarrow \mathrm{Aut}(\widehat{G})$ with open kernel. Let F_1 be the unique finite extension of F in $\overline{F}$ such that

$$\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \mathrm{Gal}(F_1/F) \hookrightarrow \mathrm{Aut}(\widehat{G}).$$

5.1 Definition of the Sato–Tate measure

Set $\Gamma_1 := \mathrm{Gal}(F_1/F)$. Let $\widehat{K}$ be a maximal compact subgroup of $\widehat{G}$ which is Γ_1-invariant. (It is not hard to see that such a $\widehat{K}$ exists, cf. [2].) Set $\widehat{T}_c := \widehat{T} \cap \widehat{K}$. (The subscript c stands for "compact" as it was in Sect. 3.3.) Denote by Ω_c the Weyl group for $(\widehat{K}, \widehat{T}_c)$.

Let $\theta \in \Gamma_1$. Define $\Omega_{c,\theta}$ to be the subset of θ-invariant elements of Ω_c. Consider the topological quotient $\widehat{K}_\theta^\natural$ of $\widehat{K} \rtimes \theta$ by the $\widehat{K}$-conjugacy equivalence relation. Set $\widehat{T}_{c,\theta} := \widehat{T}_c/(\theta - \mathrm{id})\widehat{T}_c$. Note that the action of $\Omega_{c,\theta}$ on $\widehat{T}_c$ induces an action on $\widehat{T}_{c,\theta}$. The inclusion $\widehat{T}_c \hookrightarrow \widehat{K}$ induces a canonical topological isomorphism (cf. Lemma 3.2)

$$\widehat{K}_\theta^\natural \simeq \widehat{T}_{c,\theta}/\Omega_{c,\theta}. \tag{5.1}$$

The Haar measure on $\widehat{K}$ (resp. on $\widehat{T}_c$) with total volume 1 is written as $\mu_{\widehat{K}}$ (resp. $\mu_{\widehat{T}_c}$). Then $\mu_{\widehat{K}}$ on $\widehat{K} \rtimes \theta$ induces the quotient measure $\mu_{\widehat{K}_\theta^\natural}$ (so that for any continuous function $f^\natural$ on $\widehat{K}_\theta^\natural$ and its pullback f on $\widehat{K}$, $\int f^\natural \mu_{\widehat{K}_\theta^\natural} = \int f \mu_{\widehat{K}}$) thus also a measure $\mu_{\widehat{T}_{c,\theta}}$ on $\widehat{T}_{c,\theta}$.

Definition 5.1 The θ-Sato–Tate measure $\widehat{\mu}_\theta^{\mathrm{ST}}$ on $\widehat{T}_{c,\theta}/\Omega_{c,\theta}$ is the measure transported from $\mu_{\widehat{K}_\theta^\natural}$ via (5.1).

Lemma 5.2 *Let $\widehat{\mu}_{\theta,0}^{\mathrm{ST}}$ denote the measure on $\widehat{T}_{c,\theta}$ pulled back from $\widehat{\mu}_\theta^{\mathrm{ST}}$ on $\widehat{T}_{c,\theta}/\Omega_{c,\theta}$ [so that $\int f \widehat{\mu}_{\theta,0}^{\mathrm{ST}} = \int \overline{f} \widehat{\mu}_\theta^{\mathrm{ST}}$ for every continuous $\overline{f}$ on $\widehat{T}_{c,\theta}/\Omega_{c,\theta}$ and its pullback f]. Then*

$$\widehat{\mu}_{\theta,0}^{\mathrm{ST}} = \frac{1}{|\Omega_{c,\theta}|} D_\theta(t) \mu_{\widehat{T}_{c,\theta}},$$

where $D_\theta(t) = \det(1 - \mathrm{ad}(t \rtimes \theta)|\mathrm{Lie}\,(\widehat{K})/\mathrm{Lie}\,(\widehat{T}_c^\theta))$ and t signifies a parameter on $\widehat{T}_{c,\theta}$.

Proof The twisted Weyl integration formula tells us that for a continuous $f : \widehat{K} \to \mathbb{C}$,

$$\int_{\widehat{K}} f(k)\mu_{\widehat{K}} = \frac{1}{|\Omega_{c,\theta}|} \int_{\widehat{T}_{c,\theta}^{\mathrm{reg}}} D_\theta(t) \int_{\widehat{K}_{t\theta}\backslash\widehat{K}} f(x^{-1}tx^\theta) \cdot dxdt.$$

Notice that $\widehat{K}_{t\theta}$ is the twisted centralizer group of t in $\widehat{K}$ (or, the centralizer group of $t\theta$ in $\widehat{K}$). On the right hand side, $\mu_{\widehat{T}_{c,\theta}}$ is used for integration. When f is a pullback from $\widehat{K}_\theta^\natural$, the formula simplifies as

$$\int_{\widehat{K}_\theta^\natural} f(k)\mu_{\widehat{K}_\theta^\natural} = \frac{1}{|\Omega_{c,\theta}|} \int_{\widehat{T}_{c,\theta}^{\mathrm{reg}}} D_\theta(t) f(t) \cdot \mu_{\widehat{T}_{c,\theta}}$$

and the left hand side is equal to $\int_{\widehat{T}_{c,\theta}} f(t)\widehat{\mu}_{\theta,0}^{\mathrm{ST}}$ by definition. $\qquad\square$

5.2 Limit of the Plancherel measure versus the Sato–Tate measure

Let $\theta, \tau \in \Gamma_1$. Then clearly $\Omega_{c,\theta} = \Omega_{c,\tau\theta\tau^{-1}}$, $\widehat{K}_\theta^\natural \simeq \widehat{K}_{\tau\theta\tau^{-1}}^\natural$ via $k \mapsto \tau(k)$ and $\widehat{T}_{c,\theta} \simeq \widehat{T}_{c,\tau\theta\tau^{-1}}$ via $t \mapsto \tau(t)$. Accordingly $\widehat{\mu}_\theta^{\mathrm{ST}}$ and $\widehat{\mu}_{\theta,0}^{\mathrm{ST}}$ are identified with $\widehat{\mu}_{\tau\theta\tau^{-1}}^{\mathrm{ST}}$ and $\widehat{\mu}_{\tau\theta\tau^{-1},0}^{\mathrm{ST}}$, respectively.

Fix once and for all a set of representatives $\mathscr{C}(\Gamma_1)$ for conjugacy classes in Γ_1. For $\theta \in \mathscr{C}(\Gamma_1)$, denote by $[\theta]$ its conjugacy class. For each finite place v

such that G is unramified over F_v, the geometric Frobenius $\mathrm{Fr}_v \in \mathrm{Gal}(F_v^{\mathrm{ur}}/F_v)$ gives a well-defined conjugacy class $[\mathrm{Fr}_v]$ in Γ_1. The set of all finite places v of F where G is unramified is partitioned into

$$\{\mathcal{V}_F(\theta)\}_{\theta \in \mathscr{C}(\Gamma_1)}$$

such that $v \in \mathcal{V}_F(\theta)$ if and only if $[\mathrm{Fr}_v] = [\theta]$.

For each finite place v of F, the unitary dual of $G(F_v)$ and its Plancherel measure are written as $G_v^\wedge$ and $\widehat{\mu}_v^{\mathrm{pl}}$. Similarly adapt the notation of Sect. 3.1 by appending the subscript v. Now fix $\theta \in \mathscr{C}(\Gamma_1)$ and suppose that G is unramified at v and that $v \in \mathcal{V}_F(\theta)$. We choose $\overline{F} \hookrightarrow \overline{F}_v$ such that Fr_v has image θ in Γ_1 (rather than some other conjugate). This rigidifies the identification in the second map below. (If Fr_v maps to $\tau\theta\tau^{-1}$ then the second map is twisted by τ.)

$$G(F_v)^{\wedge,\mathrm{ur},\mathrm{temp}} \overset{\text{canonical}}{\simeq} \widehat{T}_{c,\mathrm{Fr}_v}/\Omega_{c,\mathrm{Fr}_v} = \widehat{T}_{c,\theta}/\Omega_{c,\theta}. \tag{5.2}$$

By abuse of notation let $\widehat{\mu}_v^{\mathrm{pl},\mathrm{ur},\mathrm{temp}}$ [a measure on $G(F_v)^{\wedge,\mathrm{ur},\mathrm{temp}}$] also denote the transported measure on $\widehat{T}_{c,\theta}/\Omega_{c,\theta}$. Let C_v denote the constant of Proposition 3.3, which we normalize such that $\widehat{\mu}_{v,0}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}}$ has total volume 1. Note that $\widehat{\mu}_\theta^{\mathrm{ST}}$ also has total volume 1.

Proposition 5.3 *Fix any $\theta \in \mathscr{C}(\Gamma_1)$. As $v \to \infty$ in $\mathcal{V}_F(\theta)$, we have weak convergence $\widehat{\mu}_v^{\mathrm{pl},\mathrm{ur},\mathrm{temp}} \to \widehat{\mu}_\theta^{\mathrm{ST}}$ as $v \to \infty$.*

Proof It is enough to show that $\widehat{\mu}_{v,0}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}} \to \widehat{\mu}_{\theta,0}^{\mathrm{ST}}$ on $\widehat{T}_{c,\theta}$ as v tends to ∞ in $\mathcal{V}_F(\theta)$. Consider the measure $\widehat{\mu}_{v,1}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}} := C_v^{-1}\widehat{\mu}_{v,0}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}}$. It is clear from the formula of Proposition 3.3 that $\widehat{\mu}_{v,1}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}} \to \widehat{\mu}_\theta^{\mathrm{ST}}$ as $v \to \infty$ in $\mathcal{V}_F(\theta)$. In particular, the total volume of $\widehat{\mu}_{v,1}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}}$ tends to 1, hence $C_v \to 1$ as $v \to \infty$ in $\mathcal{V}_F(\theta)$. We conclude that $\widehat{\mu}_{v,0}^{\mathrm{pl},\mathrm{ur},\mathrm{temp}} \to \widehat{\mu}_{\theta,0}^{\mathrm{ST}}$ as desired. $\square$

Remark 5.4 The above proposition was already noticed by Sarnak for $G = SL(n)$ in [90, §4].

5.3 The generalized Sato–Tate problem

Let π be a cuspidal[6] tempered automorphic representation of $G(\mathbb{A}_F)$ satisfying

Hypothesis. The conjectural global L-parameter φ_π for π has Zariski dense image in ${}^L G_{F_1/F}$.

---

[6] If π is not cuspidal then the hypothesis is never supposed to be satisfied.

Of course this hypothesis is more philosophical than practical. The global Langlands correspondence between (L-packets of) automorphic representations and global L-parameters of $G(\mathbb{A}_F)$ is far from established. A fundamental problem here is that global L-parameters cannot be defined unless the conjectural global Langlands group is defined. (Some substitutes have been proposed by Arthur in the case of classical groups. The basic idea is that a cuspidal automorphic representation of GL_n can be put in place of an irreducible n-dimensional representation of the global Langlands group.) Nevertheless, the above hypothesis can often be replaced with another condition, which should be equivalent but can be stated without reference to conjectural objects. For instance, when π corresponds to a Hilbert modular form of weight $\geqslant 2$ at all infinite places, one can use the hypothesis that it is not a CM form (i.e. not an automorphic induction from a Hecke character over a CM field).

Let us state a general form of the Sato–Tate conjecture. Let q_v denote the cardinality of the residue field cardinality at a finite place v of F. Define $\mathcal{V}_F(\theta, \pi)^{\leqslant x} := \{v \in \mathcal{V}_F(\theta, \pi) : q_v \leqslant x\}$ for $x \in \mathbb{R}_{\geqslant 1}$.

Conjecture 5.5 *Assume the above hypothesis. For each $\theta \in \mathscr{C}(\Gamma_1)$, let $\mathcal{V}_F(\theta, \pi)$ be the subset of $v \in \mathcal{V}_F(\theta)$ such that π_v is unramified. Then $\{\pi_v\}_{v \in \mathcal{V}_F(\theta, \pi)}$ are equidistributed according to $\widehat{\mu}_\theta^{\mathrm{ST}}$. More precisely*

$$\frac{1}{|\mathcal{V}_F(\theta, \pi)^{\leqslant x}|} \sum_{v \in \mathcal{V}_F(\theta, \pi)^{\leqslant x}} \delta_{\pi_v} \to \widehat{\mu}_\theta^{\mathrm{ST}} \quad as \ x \to \infty.$$

The above conjecture is deemed plausible in that it is essentially a consequence of the Langlands functoriality conjecture at least when G is (an inner form of) a split group. Namely if we knew that the L-function $L(s, \pi, \rho)$ for any irreducible representation $^L G \to \mathrm{GL}_d$ were a cuspidal automorphic L-function for GL_d then the desired equidistribution is implied by Theorem 1 of [92, AppA.2].

Remark 5.6 In general when the above hypothesis is dropped, it is likely that π comes from an automorphic representation on a smaller group than G. [If φ_π factors through an injective L-morphism $^L H_{F_1/F} \to {}^L G_{F_1/F}$ then the Langlands functoriality predicts that π arises from an automorphic representation of $H(\mathbb{A}_F)$.] Suppose that the Zariski closure of $\mathrm{Im}(\varphi_\pi)$ in $^L G_{F_1/F}$ is isomorphic to $^L H_{F_1/F}$ for some connected reductive group H over F. (In general the Zariski closure may consist of finitely many copies of $^L H_{F_1/F}$.) Then $\{\pi_v\}_{v \in \mathcal{V}_F(\theta, \pi)}$ should be equidistributed according to the Sato–Tate measure belonging to H in order to be consistent with the functoriality conjecture.

One can also formulate a version of the conjecture where v runs over the set of *all* finite places where π_v are unramified by considering conjugacy classes

in $^L G_{F_1/F}$ rather than those in $\widehat{G} \rtimes \theta$ for a fixed θ. For this let $\widehat{K}^{\natural}$ denote the quotient of $\widehat{K}$ by the equivalence relation coming from the conjugation by $\widehat{K} \rtimes \Gamma_1$. Since $\widehat{K}^{\natural}$ is isomorphic to a suitable quotient of $\widehat{T}_c$, the Haar measure on $\widehat{K}$ gives rise to a measure, to be denoted $\widehat{\mu}^{ST}$, on the quotient of $\widehat{T}_c$. Let $\mathcal{V}_F(\pi)^{\leqslant x}$ (where $x \in \mathbb{R}_{\geqslant 1}$) denote the set of finite places of F such that π_v are unramified and $q_v \leqslant x$. By writing $v \to \infty$ we mean that q_v tends to infinity.

Conjecture 5.7 *Assume the above hypothesis. Then as $x \to \infty$ the set $\{\pi_v : v \in \mathcal{V}_F(\pi)^{\leqslant x}\}$ is equidistributed according to $\widehat{\mu}_\theta^{ST}$. Namely*

$$\frac{1}{|\mathcal{V}_F(\pi)^{\leqslant x}|} \sum_{v \in \mathcal{V}_F(\pi)^{\leqslant x}} \delta_{\pi_v} \to \widehat{\mu}^{ST} \quad as \ x \to \infty.$$

Remark 5.8 Unlike Conjecture 5.5 it is unnecessary to choose embeddings $\overline{F} \hookrightarrow \overline{F}_v$ to rigidify (5.2) since the ambiguity in the rigidification is absorbed in the conjugacy classes in $^L G_{F_1/F}$. The formulation of Conjecture 5.7 might be more suitable than the previous one in the motivic setting where we would not want to fix $\overline{F} \hookrightarrow \overline{F}_v$. The interested reader may compare Conjecture 5.7 with the motivic Sato–Tate conjecture of [96, 13.5].

The next subsection will discuss the analogue of Conjecture 5.5 for automorphic families. Conjecture 5.7 will not be considered any more in our paper. It is enough to mention that the analogue of the latter conjecture for families of algebraic varieties makes sense and appears to be interesting.

5.4 The Sato–Tate conjecture for families

The Sato–Tate conjecture has been proved for Hilbert modular forms in [6,7]. Analogous equidistribution theorems in the function field setting are due to Deligne and Katz. (See [59, Thm9.2.6] for instance.) Despite these fantastic developments, we have little unconditional theoretical evidence for the Sato–Tate conjecture for general reductive groups over number fields. On the other hand, it has been noticed that the analogue of the Sato–Tate conjecture for families of automorphic representations is more amenable to attack. Indeed there was some success in the case of holomorphic modular forms and Maass forms [34, Thm2] and [53,83,97]. The conjecture has the following coarse form, which should be thought of as a guiding principle rather than a rigorous conjecture. Compare with some precise results in Sect. 9.7.

Heuristic 5.9 *Let $\{\mathcal{F}_k\}_{k \geqslant 1}$ be a "general" sequence of finite families of automorphic representations of $G(\mathbb{A}_F)$ such that $|\mathcal{F}_k| \to \infty$ as $k \to \infty$. Then $\{\pi_v \in \mathcal{F}_k\}$ are equidistributed according to $\widehat{\mu}_\theta^{ST}$ as k and v tend to infinity*

subject to the conditions that $v \in \mathcal{V}_F(\theta)$ and that all members of $\mathcal{F}_k$ are unramified at v.

We are not going to make precise what "general" means, but merely remark that it should be the analogue of the condition that the hypothesis of Sect. 5.3 holds for the "generic fiber" of the family when the family has a geometric meaning (see also [87]). In practice one would verify the conjecture for many interesting families while simply ignoring the word "general". Some relation between k and v holds when taking limit: k needs to grow fast enough compared to v (or more precisely $|\mathcal{F}_k|$ needs to grow fast enough compared to q_v).

It is noteworthy that the unpleasant hypothesis of Sect. 5.3 can be avoided for families. Also note that the temperedness assumption is often unnecessary due to the fact that the Plancherel measure is supported on the tempered spectrum. This is an indication that most representations in a family are globally tempered, which we will return to in a subsequent work.

Later we will verify the conjecture for many families in Sect. 9.7 as a corollary to the automorphic Plancherel theorem proved earlier in Sect. 9. Our families arise as the sets of all automorphic representations with increasing level or weight, possibly with prescribed local conditions at finitely many fixed places.

6 Background materials

This section collects background materials in the local and global contexts. Sections 6.1 and 6.3 are concerned with p-adic groups while Sects. 6.4, 6.5 and 6.8 are with real and complex Lie groups. The rest is about global reductive groups.

6.1 Orbital integrals and constant terms

We introduce some notation in the p-adic context.

- F is a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}$ and multiplicative valuation $|\cdot|$.
- G is a connected reductive group over F.
- A is a maximal F-split torus of G, and put $M_0 := Z_G(A)$.
- K is a maximal compact subgroup of G corresponding to a special point in the apartment for A.
- $P = MN$ is a parabolic subgroup of G over F, with M and N its Levi subgroup and unipotent radical, such that $M \supset M_0$.
- $\gamma \in G(F)$ is a semisimple element. (The case of a non-semisimple element is not needed in this paper.)

- I_γ is the neutral component of the centralizer of γ in G. Then I_γ is a connected reductive group over F.
- μ_G (resp. μ_{I_γ}) is a Haar measure on $G(F)$ (resp. $I_\gamma(F)$).
- $\frac{\mu_G}{\mu_{I_\gamma}}$ is the quotient measure on $I_\gamma(F)\backslash G(F)$ induced by μ_G and μ_{I_γ}.
- $\phi \in C_c^\infty(G(F))$.
- $D^G(\gamma) := \prod_\alpha |1 - \alpha(\gamma)|$ for a semisimple $\gamma \in G(F)$, where α runs over the set of roots of G (with respect to any maximal torus in the connnected centralizer of γ in G) such that $\alpha(\gamma) \neq 1$. Let M be an F-rational Levi subgroup of G. For a semisimple $\gamma \in G(F)$, we define $D_M^G(\gamma)$ similarly by further excluding those α in the set of roots of M.

Define the orbital integral

$$O_\gamma^{G(F)}(\phi, \mu_G, \mu_{I_\gamma}) := \int_{I_\gamma(F)\backslash G(F)} \phi(x^{-1}\gamma x)\frac{\mu_G}{\mu_{I_\gamma}}.$$

When the context is clear, we use $O_\gamma(\phi)$ as a shorthand notation.

We recall the theory of constant terms (cf. [105, p. 236]). Choose Haar measures μ_K, μ_M, μ_N, on $K, M(F), N(F)$, respectively, such that $\mu_G = \mu_K\mu_M\mu_N$ holds with respect to $G(F) = KM(F)N(F)$. Define the (normalized) constant term $\phi_M \in C_c^\infty(M(F))$ by

$$\phi_M(m) = \delta_P^{1/2}(m) \int_{N(F)} \int_K \phi(kmnk^{-1})\mu_K\mu_N. \qquad (6.1)$$

Although the definition of ϕ_M involves not only M but P, the following lemma shows that the orbital integrals of ϕ_M depend only on M by the density of regular semisimple orbital integrals, justifying our notation.

Lemma 6.1 *For all (G, M)-regular semisimple $\gamma \in M(F)$,*

$$O_\gamma(\phi_M, \mu_M, \mu_{I_\gamma}) = D_M^G(\gamma)^{1/2}O_\gamma(\phi, \mu_G, \mu_{I_\gamma}).$$

Proof [105, Lem 9]. (Although the lemma is stated for regular elements $\gamma \in G$, it suffices to require γ to be (G, M)-regular. See Lemma 8 of loc. cit.) □

It is standard that the definition and facts we have recollected above extend to the adelic case. (Use [63, §§7–8], for instance). We will skip rewriting the analogous definition in the adelic setting.

Now we restrict ourselves to the local unramified case. Suppose that G is unramified over F. Let $B \subset P \subset G$ be Borel and parabolic subgroups defined over F. Write $B = TU$ and $P = MN$ where T and M are Levi subgroups such that $T \subset M$ and U and N are unipotent radicals.

Lemma 6.2 *Let $\phi \in \mathcal{H}^{\mathrm{ur}}(G)$. Then $\mathcal{S}_M^G(\phi) = \phi_M$, in particular $\mathcal{S}^G(\phi) = \mathcal{S}^M(\mathcal{S}_M^G\phi) = \phi_T$.*

Proof Straightforward from (2.1) and (6.1). ∎

6.2 Gross's motives

Now let F be a finite extension of $\mathbb{Q}$ (although Gross's theory applies more generally). Let G be a connected reductive group over F and consider its quasi-split inner form G^*. Let T^* be the centralizer of a maximal F-split torus of G^*. Denote by Ω the Weyl group for (G^*, T^*) over $\overline{F}$. Set $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Gross [47] attaches to G an Artin–Tate motive

$$\mathrm{Mot}_G = \bigoplus_{d \geqslant 1} \mathrm{Mot}_{G,d}(1 - d)$$

with coefficients in $\mathbb{Q}$. Here $(1 - d)$ denotes the Tate twist. The Artin motive $\mathrm{Mot}_{G,d}$ (denoted V_d by Gross) may be thought of as a Γ-representation on a $\mathbb{Q}$-vector space whose dimension is $\dim \mathrm{Mot}_{G,d}$. Define

$$L(\mathrm{Mot}_G) := L(0, \mathrm{Mot}_G)$$

to be the Artin L-value of $L(s, \mathrm{Mot}_G)$ at $s = 0$. We recall some properties of Mot_G from Gross's article.

Proposition 6.3 (i) $\mathrm{Mot}_{G,d}$ *is self-dual for each $d \geqslant 1$.*
(ii) $\sum_{d \geqslant 1} \dim \mathrm{Mot}_{G,d} = r_G = \mathrm{rk}G$.
(iii) $\sum_{d \geqslant 1}(2d - 1) \dim \mathrm{Mot}_{G,d} = \dim G$.
(iv) $|\Omega| = \prod_{d \geqslant 1} d^{\dim \mathrm{Mot}_{G,d}}$.
(vi) *If T^* splits over a finite extension E of F then the Γ-action on Mot_G factors through $\mathrm{Gal}(E/F)$.*

The Artin conductor $\mathfrak{f}(\mathrm{Mot}_{G,d})$ is defined as follows. Let F' be the fixed field of the kernel of the Artin representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V_d)$ associated to $\mathrm{Mot}_{G,d}$. For each finite place v of F, let w be any place of F' above v. Let $\Gamma(v)_i := \mathrm{Gal}(F'_w/F_v)_i$ $(i \geqslant 0)$ denote the i-th ramification subgroups. Set

$$f(G_v, d) := \sum_{i \in \mathbb{Z}_{\geqslant 0}} \frac{|\Gamma(v)_i|}{|\Gamma(v)_0|} \dim \left(V_d / V_d^{\Gamma(v)_i} \right), \tag{6.2}$$

which is an integer independent of the choice of w. Write $\mathfrak{p}_v$ for the prime ideal of $\mathcal{O}_F$ corresponding to v. If v is unramified in E then $f(G_v, d) = 0$.

Thus the product makes sense in the following definition.

$$\mathfrak{f}(\mathrm{Mot}_{G,d}) := \prod_{v \nmid \infty} \mathfrak{p}_v^{f(G_v,d)}$$

Let E be the splitting field of T^* (which is an extension of F) and set $s_G^{\mathrm{spl}} := [E : F]$.

Lemma 6.4 *For every finite place v of F,*

$$f(G_v, d) \leqslant (\dim \mathrm{Mot}_{G,d}) \cdot \left(s_G^{\mathrm{spl}} \left(1 + e_{F_v/\mathbb{Q}_p} \log_p s_G^{\mathrm{spl}} \right) - 1 \right).$$

Proof Let F', w and V_d be as in the preceding paragraph. Then $F \subset F' \subset E$. Set $s_v := [F'_w : F_v]$ so that $s_v \leqslant s_G^{\mathrm{spl}}$. The case $s_v = 1$ is obvious (in which case $f(G_v, d) = 0$), so we may assume $s_v \geqslant 2$. From (6.2) and Corollary 6.9 below,

$$f(G_v, d) \leqslant \sum_{i \geqslant 0} \dim \left(V_d / V_d^{\Gamma(v)_i} \right) \leqslant (\dim V_d)(s_v(1 + e_{F_v/\mathbb{Q}_p} \log_p s_v) - 1).$$

$\square$

Recall that $w_G = |\Omega|$ is the cardinality of the absolute Weyl group. Let s_G be the degree of the smallest extension of F over which G becomes split. The following useful lemma implies in particular that $s_G^{\mathrm{spl}} \leqslant w_G s_G$.

Lemma 6.5 [56, Lem 2.2] *For any maximal torus T of G defined over F, there exists a finite Galois extension E of F such that $[E : F] \leqslant w_G s_G$ and T splits over E.*

6.3 Lemmas on ramification

This subsection is meant to provide an ingredient of proof (namely Corollary 6.9) for Lemma 6.4.

Fix a prime p. Let E and F be finite extensions of $\mathbb{Q}_p$ with uniformizers ϖ_E and ϖ_F, respectively. Normalize valuations $v_E : E^\times \to \mathbb{Z}$ and $v_F : F^\times \to \mathbb{Z}$ such that $v_E(\varpi_E) = v_F(\varpi_F) = 1$. Write $e_{E/F} \in \mathbb{Z}_{\geqslant 1}$ for the ramification index and $\mathfrak{D}_{E/F}$ for the different. For a nonzero principal ideal $\mathfrak{a}$ of $\mathcal{O}_E$, we define $v_E(\mathfrak{a})$ to be $v_E(a)$ for any generator a of $\mathfrak{a}$. This is well defined.

Lemma 6.6 *Let E be a totally ramified Galois extension of F with $[E : F] = p^n$ for $n \geqslant 0$. Then*

$$v_E(\mathfrak{D}_{E/F}) \leqslant p^n(1 + n \cdot e_{F/\mathbb{Q}_p}) - 1.$$

Remark 6.7 In fact the inequality is sharp. There are totally ramified extensions E/F for which the above equality holds as shown by Öre. See also [95, §1] for similar results.

Proof The lemma is trivial when $n = 0$. Next assume $n = 1$ but allow E/F to be a non-Galois extension. Let $f(x) = \sum_{i=0}^{p} a_i x^i \in \mathcal{O}_F[x]$ (with $a_p = 1$ and $v_F(a_i) \geqslant 1$ for $i < p$) be the Eisenstein polynomial having ϖ_E as a root. By [94, III.6, Cor 2], $v_E(\mathfrak{D}_{E/F}) = v_E(f'(\varpi_E))$. The latter equals

$$v_E\left(\sum_{i=1}^{p} i a_i \varpi_E^{i-1}\right) = \min_{1 \leqslant i \leqslant p} v_E\left(i a_i \varpi_E^{i-1}\right) \leqslant v_E\left(p \varpi_E^{p-1}\right) = e_{E/\mathbb{Q}_p} + p - 1.$$

This prepares us to tackle the case of arbitrary n. Choose a sequence of subextensions $E = F_0 \supset F_1 \supset \cdots \supset F_n = F$ such that $[F_m : F_{m+1}] = p$ (where F_m/F_{m+1} may not be a Galois extension). By above, $v_{F_m}(\mathfrak{D}_{F_m/F_{m+1}}) \leqslant e_{F_m/\mathbb{Q}_p} + p - 1$ for $0 \leqslant m \leqslant n - 1$. Hence

$$v_E(\mathfrak{D}_{E/F}) = \sum_{m=0}^{n-1} v_E(\mathfrak{D}_{F_m/F_{m+1}}) \leqslant \sum_{m=0}^{n-1} p^m(e_{F_m/\mathbb{Q}_p} + p - 1)$$
$$= n p^n e_{F/\mathbb{Q}_p} + p^n - 1.$$

$\square$

Lemma 6.8 *Let E be a finite Galois extension of F. Then*

$$v_E(\mathfrak{D}_{E/F}) \leqslant [E : F](1 + e_{F/\mathbb{Q}_p} \log_p[E : F]) - 1.$$

Proof Let E^t (resp. E^{ur}) be the maximal tame (resp. unramified) extension of F in E. Then $v_{E^t}(\mathfrak{D}_{E^t/E^{\mathrm{ur}}}) = [E^t : E^{\mathrm{ur}}] - 1$ by [94, III.6, Prop 13]. Clearly $v_{E^{\mathrm{ur}}}(\mathfrak{D}_{E^{\mathrm{ur}}/F}) = 0$. Together with Lemma 6.6, we obtain

$$v_E(\mathfrak{D}_{E/F}) = v_E(\mathfrak{D}_{E/E^t}) + [E : E^t] v_{E^t}(\mathfrak{D}_{E^t/E^{\mathrm{ur}}})$$
$$\leqslant [E : E^t]\left(1 + e_{E^t/\mathbb{Q}_p} \log_p[E : E^t]\right)$$
$$\quad -1 + [E : E^t]([E^t : E^{\mathrm{ur}}] - 1)$$
$$= [E : E^{\mathrm{ur}}](1 + e_{F/\mathbb{Q}_p} \log_p[E : E^t]) - 1$$
$$\leqslant [E : F](1 + e_{F/\mathbb{Q}_p} \log_p[E : F]) - 1.$$

$\square$

Corollary 6.9 *Let E be a finite Galois extension of F. Then the ith ramification group $\mathrm{Gal}(E/F)_i$ is trivial for $i = [E : F](1 + e_{F/\mathbb{Q}_p} \log_p[E : F]) - 1$.*

Proof In the notation of section IV.1 of [94], we have $\mathrm{Gal}(E/F)_m = 1$ by definition if $m = \max_{1 \neq s \in \mathrm{Gal}(E/F)} i_G(s)$. But the proposition 4 in that section implies that $m \leqslant v_E(\mathfrak{D}_{E/F})$, so Lemma 6.8 finishes the proof.

6.4 Stable discrete series characters

In Sects. 6.4 and 6.5 we specialize to the situation of real groups.

- G is a connected reductive group over $\mathbb{R}$.
- $A_{G,\infty} = A_G(\mathbb{R})^0$ where A_G is the maximal split torus in the center of G.
- K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ and $K'_\infty := K_\infty A_{G,\infty}$.
- $q(G) := \frac{1}{2} \dim_{\mathbb{R}} G(\mathbb{R})/K'_\infty \in \mathbb{Z}_{\geqslant 0}$.
- T is an $\mathbb{R}$-elliptic maximal torus in G. (Assume that such a T exists.)
- B is a Borel subgroup of G over $\mathbb{C}$ containing T.
- I_γ denotes the connected centralizer of $\gamma \in G(\mathbb{R})$.
- Φ^+ (resp. Φ) is the set of positive (resp. all) roots of T in G over $\mathbb{C}$.
- Ω is the Weyl group for (G, T) over $\mathbb{C}$, and Ω_c is the compact Weyl group.
- $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.
- ξ is an irreducible finite dimensional algebraic representation of $G(\mathbb{R})$.
- $\lambda_\xi \in X^*(T)$ is the B-dominant highest weight for ξ.
- $m(\xi) := \min_{\alpha \in \Phi^+} \langle \lambda_\xi + \rho, \alpha \rangle$. We always have $m(\xi) > 0$.
- $\Pi_{\mathrm{disc}}(\xi)$ is the set of irreducible discrete series representations of $G(\mathbb{R})$ with the same infinitesimal character and the same central character as ξ. [This is an L-packet for $G(\mathbb{R})$.]
- $D_\infty^G(\gamma) := \prod_\alpha |1 - \alpha(\gamma)|$ for $\gamma \in T(\mathbb{R})$, where α runs over elements of Φ such that $\alpha(\gamma) \neq 1$. [If γ is in the center of $G(\mathbb{R})$, $D_\infty^G(\gamma) = 1$.]

If M is a Levi subgroup of G over $\mathbb{C}$ containing T, the following are defined in the obvious manner as above: $\Phi_M^+, \Phi_M, \Omega_M, \rho_M, D_\infty^M$. Define $\Omega^M := \{\omega \in \Omega : \omega^{-1}\Phi_M^+ \subset \Phi^+\}$, which is a set of representatives for Ω/Ω_M. For each *regular* $\gamma \in T(\mathbb{R})$, let us define (cf. [3, (4.4)])

$$\Phi_M^G(\gamma, \xi) := (-1)^{q(G)} D_\infty^G(\gamma)^{1/2} D_\infty^M(\gamma)^{-1/2} \sum_{\pi \in \Pi_{\mathrm{disc}}(\xi)} \Theta_\pi(\gamma)$$

where Θ_π is the character function of π. It is known that the function $\Phi_M^G(\gamma, \xi)$ continuously extends to an Ω_M-invariant function on $T(\mathbb{R})$, thus also to a function on $M(\mathbb{R})$ which is invariant under $M(\mathbb{R})$-conjugation and supported on elliptic elements ([3, Lem 4.2], cf. [45, Lem 4.1]). When $M = G$, simply $\Phi_M^G(\gamma, \xi) = \mathrm{tr}\, \xi(\gamma)$.

We would like to have an upper bound for $|\Phi_M^G(\gamma, \xi)|$ that we will need in Sect. 9.5. This is a refinement of [99, Lem 4.8].

Lemma 6.10 (i) $\dim \xi = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda_\xi + \rho \rangle}{\langle \alpha, \rho \rangle}$.

(ii) *There exists a constant $c > 0$ independent of ξ such that for every elliptic $\gamma \in G(\mathbb{R})$ and ξ,*

$$\frac{|\operatorname{tr} \xi(\gamma)|}{\dim \xi} \leqslant c \frac{D_\infty^G(\gamma)^{-1/2}}{m(\xi)^{|\Phi^+| - |\Phi_{I_\gamma}^+|}}.$$

Proof Part (i) is the standard Weyl dimension formula. Let us prove (ii). The formula right above the corollary 1.12 in [19] implies that

$$|\operatorname{tr} \xi(\gamma)| \leqslant D_\infty^G(\gamma)^{-1/2} \times \sum_{\omega \in \Omega^{I_\gamma}} \left(\prod_{\alpha \in \Phi_{I_\gamma}^+} \frac{\langle \omega^{-1} \alpha, \lambda_\xi + \rho \rangle}{\langle \alpha, \rho_{I_\gamma} \rangle} \right).$$

Note that their M is our I_γ and that $|\alpha(\gamma)| = 1$ for all $\alpha \in \Phi$ and all elliptic $\gamma \in G(\mathbb{R})$. Hence by (i),

$$\frac{|\operatorname{tr} \xi(\gamma)|}{\dim \xi} \leqslant D_\infty^G(\gamma)^{-1/2} \sum_{\omega \in \Omega^{I_\gamma}} \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}{\prod_{\alpha \in \Phi_{I_\gamma}^+} \langle \alpha, \rho_{I_\gamma} \rangle} \left(\prod_{\alpha \in \Phi^+ \setminus \omega^{-1} \Phi_{I_\gamma}^+} \langle \lambda_\xi + \rho, \alpha \rangle \right)^{-1}$$

$$\leqslant D_\infty^G(\gamma)^{-1/2} |\Omega^{I_\gamma}| \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}{\prod_{\alpha \in \Phi_{I_\gamma}^+} \langle \alpha, \rho_{I_\gamma} \rangle} m(\xi)^{-(|\Phi^+| - |\Phi_{I_\gamma}^+|)}.$$

$\square$

Lemma 6.11 *Assume that M is a Levi subgroup of G over $\mathbb{R}$ containing an elliptic maximal torus. There exists a constant $c_0 > 0$ independent of ξ such that for every elliptic $\gamma \in M(\mathbb{R})$,*

$$\frac{\left| \Phi_M^G(\gamma, \xi) \right|}{\dim \xi} \leqslant c_0 \frac{D_\infty^M(\gamma)^{-1/2}}{m(\xi)^{|\Phi^+| - |\Phi_{I_\gamma^M}^+|}}.$$

Proof As the case $M = G$ is already proved by Lemma 6.10 (ii), we assume that $M \subsetneq G$. Fix an elliptic maximal torus $T \subset M$. Since every elliptic element has a conjugate in $T(\mathbb{R})$ and both sides of the inequality are conjugate-invariant, it is enough to verify the lemma for $\gamma \in T(\mathbb{R})$. In this proof we borrow some notation and facts from [45, pp. 494–498] as well as [3, pp. 272–274]. For the purpose of proving Lemma 6.11, we may restrict to $\gamma \in \Gamma^+$, corresponding to a closed chamber for the root system of $T(\mathbb{R})$ in $G(\mathbb{R})$. (See page 497 of [45] for the precise definition.) The proof of [45, Lem 4.1] shows that

$$\Phi_M^G(\gamma, \xi) = \sum_{\omega \in \Omega^M} c(\omega, \xi) \cdot \mathrm{tr}\, \xi_\omega^M(\gamma)$$

where ξ_ω^M is the irreducible representation of $M(\mathbb{R})$ of highest weight $\omega(\xi + \rho) - \rho_M$. We claim that there is a constant $c_1 > 0$ independent of ξ such that

$$|c(\omega, \xi)| \leqslant c_1$$

for all ω and ξ. The coefficients $c(\omega, \xi)$ can be computed by rewriting the right hand side of [3, (4.8)] as a linear combination of $\mathrm{tr}\, \xi_\omega^M(\gamma)$ using the Weyl character formula. In order to verify the claim, it suffices to point out that $\overline{c}(Q_{ys\lambda}^+, R_H^+)$ in Arthur's (4.8) takes values in a finite set which is independent of ξ (or τ in Arthur's notation). This is obvious: as $Q_{ys\lambda}^+ \subset \Phi^\vee$ and $R_H^+ \subset \Phi$, there are finitely many possibilities for $Q_{ys\lambda}^+$ and R_H^+.

Now by Lemma 6.10 (i),

$$\frac{\dim \xi_\omega^M}{\dim \xi} = \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}{\prod_{\alpha \in \Phi_M^+} \langle \alpha, \rho_M \rangle} \prod_{\alpha \in \Phi^+ \backslash \Phi_M^+} \langle \alpha, \lambda_\xi + \rho \rangle^{-1} \leqslant c_2 m(\xi)^{-(|\Phi^+| - |\Phi_M^+|)}$$

with $c_2 = (\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle)(\prod_{\alpha \in \Phi_M^+} \langle \alpha, \rho_M \rangle)^{-1} > 0$. According to Lemma 6.10 (ii), there exists a constant $c_3 > 0$ such that

$$\frac{\left| \mathrm{tr}\, \xi_\omega^M(\gamma) \right|}{\dim \xi_\omega^M} \leqslant c_3 \frac{D_\infty^M(\gamma)^{-1/2}}{m(\xi)^{|\Phi_M^+| - |\Phi_{I_\gamma^M}^+|}}.$$

To conclude the proof, multiply the last two formulas. □

6.5 Euler–Poincaré functions

We continue to use the notation of Sect. 6.4. Let $\overline{\mu}_\infty^{\mathrm{EP}}$ denote the Euler–Poincaré measure on $G(\mathbb{R})/A_{G,\infty}$ (so that its induced measure on the compact inner form has volume 1). There exists a unique Haar measure μ_∞^{EP} on $G(\mathbb{R})$ which is compatible with $\overline{\mu}_\infty^{\mathrm{EP}}$ and the standard Haar measure on $A_{G,\infty}$. Write ω_ξ for the central character of ξ on $A_{G,\infty}$. Let $\Pi(\omega_\xi^{-1})$ denote the set of irreducible admissible representations of $G(\mathbb{R})$ whose central characters on $A_{G,\infty}$ are ω_ξ^{-1}. For $\pi \in \Pi(\omega_\xi^{-1})$, define

$$\chi_{\mathrm{EP}}(\pi \otimes \xi) := \sum_{i \geqslant 0} (-1)^i \dim H^i \left(\mathrm{Lie}\, G(\mathbb{R}), K_\infty', \pi \otimes \xi \right).$$

Clozel and Delorme [21] constructed a bi-K_∞-finite function $\phi_\xi \in C^\infty(G(\mathbb{R}))$ which transforms under $A_{G,\infty}$ by ω_ξ and is compactly supported modulo $A_{G,\infty}$, such that

$$\forall \pi \in \Pi\left(\omega_\xi^{-1}\right), \quad \operatorname{tr} \pi \left(\phi_\xi, \mu_\infty^{\mathrm{EP}}\right) = \chi_{\mathrm{EP}}(\pi \otimes \xi).$$

The following are well-known:

- $\chi_{\mathrm{EP}}(\pi \otimes \xi) = 0$ unless $\pi \in \Pi(\omega_\xi^{-1})$ has the same infinitesimal character as $\xi^\vee$.
- If the highest weight of ξ is regular then $\chi_{\mathrm{EP}}(\pi \otimes \xi) \neq 0$ if and only if $\pi \in \Pi_{\mathrm{disc}}(\xi^\vee)$.
- If $\pi \in \Pi(\omega_\xi^{-1})$ is a discrete series and $\chi_{\mathrm{EP}}(\pi \otimes \xi) \neq 0$ then $\pi \in \Pi_{\mathrm{disc}}(\xi^\vee)$ and $\chi_{\mathrm{EP}}(\pi \otimes \xi) = (-1)^{q(G)}$. More precisely, $\dim H^i(\operatorname{Lie} G(\mathbb{R}), K'_\infty, \pi \otimes \xi)$ equals 1 if $i = q(G)$ and 0 if not.

6.6 Canonical measures and Tamagawa measures

We return to the global setting so that F and G are as in Sect. 6.2. Let $G_\infty := (\operatorname{Res}_{F/\mathbb{Q}} G) \times_\mathbb{Q} \mathbb{R}$, to which the contents of Sects. 6.4 and 6.5 apply. In particular we have a measure μ_∞^{EP} on $G_\infty(\mathbb{R})$. For each finite place v of F, define $\mu_v^{\mathrm{can}} := \Lambda(\operatorname{Mot}_{G_v}^\vee(1)) \cdot |\omega_{G_v}|$ in the notation of [47] where $|\omega_{G_v}|$ is the "canonical" Haar measure on $G(F_v)$ as in §11 of that article. When G is unramified over F_v, the measure μ_v^{can} assigns volume 1 to a hyperspecial subgroup of $G(F_v)$. In particular,

$$\mu^{\mathrm{can},\mathrm{EP}} := \prod_{v \nmid \infty} \mu_v^{\mathrm{can}} \times \mu_\infty^{\mathrm{EP}}$$

is a well-defined measure on $G(\mathbb{A}_F)$.

Let $\overline{\mu}^{\mathrm{Tama}}$ denote the Tamagawa measure on $G(F)\backslash G(\mathbb{A}_F)/A_{G,\infty}$, so that its volume is the Tamagawa number (cf. [64, p. 629])

$$\begin{aligned}
\tau(G) &:= \overline{\mu}^{\mathrm{Tama}}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\infty}) \\
&= |\pi_0(Z(\widehat{G})^{\operatorname{Gal}(\overline{F}/F)})| \cdot |\ker^1(F, Z(\widehat{G}))|^{-1}.
\end{aligned} \tag{6.3}$$

The Tamagawa measure μ^{Tama} on $G(\mathbb{A}_F)$ of [47] is compatible with $\overline{\mu}^{\mathrm{Tama}}$ if $G(F)$ and $A_{G,\infty}$ are equipped with the point-counting measure and the Lebesgue measure, respectively. The ratio of two Haar measures on $G(\mathbb{A}_F)$ is computed as:

Proposition 6.12 [47, 10.5]

$$\frac{\mu^{\mathrm{can,EP}}}{\mu^{\mathrm{Tama}}} = \frac{L(\mathrm{Mot}_G) \cdot |\Omega|/|\Omega_c|}{e(G_\infty) 2^{\mathrm{rk}_\mathbb{R} G_\infty}}.$$

The following notion will be useful in that the Levi subgroups contributing to the trace formula in Sect. 9 turn out to be the cuspidal ones.

Definition 6.13 We say that G is cuspidal if $G_0 := \mathrm{Res}_{F/\mathbb{Q}} G$ satisfies the condition that $A_{G_0} \times_\mathbb{Q} \mathbb{R}$ is the maximal split torus in the center of $G_0 \times_\mathbb{Q} \mathbb{R}$.

Assume that G is cuspidal, so that $G(\mathbb{R})/A_{G,\infty}$ contains a maximal $\mathbb{R}$-torus which is anisotropic.

Corollary 6.14

$$\frac{\overline{\mu}^{\mathrm{can,EP}}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\infty})}{\overline{\mu}_\infty^{\mathrm{EP}}(\overline{G}(F_\infty)/A_{G,\infty})} = \frac{\tau(G) \cdot L(\mathrm{Mot}_G) \cdot |\Omega|/|\Omega_c|}{e(G_\infty) 2^{[F:\mathbb{Q}]r_G}}.$$

Proof It suffices to remark that the Euler–Poincaré measure on a compact Lie group has total volume 1, hence $\overline{\mu}_\infty^{\mathrm{EP}}(\overline{G}(F_\infty)/A_{G,\infty}) = 1$.

6.7 Bounds for Artin L-functions

For later use we estimate the L-value $L(\mathrm{Mot}_G)$ in Corollary 6.14.

Proposition 6.15 *Let $s \geqslant 1$ and E be a Galois extension of F of degree $[E : F] \leqslant s$.*

(i) *For all $\epsilon > 0$ there exists a constant $c = c(\epsilon, s, F) > 0$ which depends only on ϵ, s and F such that the following holds: For all non-trivial irreducible representations ρ of $\mathrm{Gal}(E/F)$,*

$$cd_E^{-\epsilon} \leqslant L(1, \rho) \leqslant cd_E^{\epsilon}.$$

(ii) *The same inequalities hold for the residue $\mathrm{Res}_{s=1}\zeta_E(s)$ of the Dedekind zeta function of E.*

(iii) *There is a constant $A_1 = A_1(s, F) > 0$ which depends only on s and F such that for all faithful irreducible representation ρ of $\mathrm{Gal}(E/F)$,*

$$d_{E/F}^{A_1} \leqslant \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}_\rho) \leqslant d_{E/F}^{1/\dim(\rho)},$$

where $d_{E/F} = \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})$ is the relative discriminant of E/F; recall that $d_E = d_F^{[E:F]} d_{E/F}$.

Proof The assertion (ii) is Brauer–Siegel theorem [14, Theorem 2]. We also note the implication (i) $\Rightarrow$ (ii) which follows from the formula

$$\zeta_E(s) = \prod_\rho L(s, \rho)^{\dim \rho}. \tag{6.4}$$

where ρ ranges over all irreducible representations of $\mathrm{Gal}(E/F)$.

The proof of assertion (i) is reduced to the 1-dimensional case by Brauer induction as in [14]. In this reduction one uses the fact that if E'/F' is a subextension of E/F then the absolute discriminant $d_{E'}$ of E' divides the absolute discriminant d_E of E. Also we may assume that E'/F' is cyclic. For a character χ of $\mathrm{Gal}(E'/F')$ we have the convexity bound $L(1, \chi) \leqslant c d_{E'}^\epsilon$ (Landau). The lower bound for $L(1, \chi)$ follows from (ii) and the product formula (6.4).

In the assertion (iii) the right inequality follows from the discriminant-conductor formula which implies that $\mathfrak{f}_\rho^{\dim(\rho)} | \mathfrak{D}_{E/F}$. The left inequality follows from local considerations. Let v be a finite place of F dividing $\mathfrak{D}_{E/F}$; since ρ is faithful, its restriction to the inertia group above v is non-trivial and therefore v divides $\mathfrak{f}_\rho$. Since $v(\mathfrak{D}_{E/F})$ is bounded above by a constant $A_1(s, F)$ depending only on $[E : F] \leqslant s$ and F by Lemma 6.8, we have $v(\mathfrak{D}_{E/F}) \leqslant A_1 v(\mathfrak{f}_\rho)$ which concludes the proof. $\qquad\square$

Corollary 6.16 *For all integers $R, D, s \in \mathbb{Z}_{\geqslant 1}$, and $\epsilon > 0$ there is a constant $c_1 = c_1(\epsilon, R, D, s, F) > 0$ (depending on R, D, s, F and ϵ) with the following property*

(i) *For any G such that $r_G \leqslant R$, $\dim G \leqslant D$, $Z(G)$ is F-anisotropic, and G splits over a Galois extension of F of degree $\leqslant s$,*

$$|L(\mathrm{Mot}_G)| \leqslant c_1 \prod_{d=1}^{\lfloor \frac{d_G+1}{2} \rfloor} \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}(\mathrm{Mot}_{G,d}))^{d-\frac{1}{2}+\epsilon}.$$

(ii) *There is a constant $A_{20} = A_{20}(R, D, s, F)$ such that for any G as in (i),*

$$|L(\mathrm{Mot}_G)| \leqslant c_1 \prod_{v \in \mathrm{Ram}(G)} q_v^{A_{20}}.$$

The choice $A_{20} = \frac{(D+1)Rs}{2} \max_{\substack{\text{prime } p}} (1 + e_{F_v/\mathbb{Q}_p} \log s)$ is admissible.

Proof The functional equation for Mot_G reads

$$L(\mathrm{Mot}_G) = L(\mathrm{Mot}_G^\vee(1))\epsilon(\mathrm{Mot}_G) \cdot \frac{L_\infty(\mathrm{Mot}_G^\vee(1))}{L_\infty(\mathrm{Mot}_G)}$$

where $\epsilon(\mathrm{Mot}_G) = |\Delta_F|^{d_G/2} \prod_{d \geqslant 1} \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}(\mathrm{Mot}_{G,d}))^{d-\frac{1}{2}}$.

The (possibly reducible) Artin representation for $\mathrm{Mot}_{G,d}$ factors through $\mathrm{Gal}(E/F)$ with $[E : F] \leqslant s$ by the assumption. Let $A_1 = A_1(s, F)$ be as in (iii) of Proposition 6.15. For all $\epsilon > 0$, (i) of Proposition 6.15 implies that there is a constant $c = c(\epsilon, s, F) > 1$ depending only on s and F such that

$$\left|L\left(\mathrm{Mot}_G^\vee(1)\right)\right| \leqslant \prod_{d \geqslant 1} \left(c\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}(\mathrm{Mot}_{G,d}))^{A_1\epsilon}\right)^{\dim \mathrm{Mot}_{G,d}}$$

$$\leqslant c^{r_G} \prod_{d \geqslant 1} \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}(\mathrm{Mot}_{G,d}))^{\epsilon A_1 r_G}.$$

Formula (7.7) of [47], the first equality below, leads to the following bound since only $1 \leqslant d \leqslant \lfloor \frac{d_G+1}{2} \rfloor$ can contribute in view of Proposition 6.3 (iii).

$$\left|\frac{L_\infty\left(\mathrm{Mot}_G^\vee(1)\right)}{L_\infty(\mathrm{Mot}_G)}\right| = 2^{-[F:\mathbb{Q}]r_G} \prod_{d \geqslant 1} \left(\frac{(d-1)!}{(2\pi)^d}\right)^{\dim \mathrm{Mot}_{G,d}}$$

$$\leqslant 2^{-[F:\mathbb{Q}]r_G} \left(\left\lfloor \frac{d_G-1}{2} \right\rfloor!\right)^{r_G}.$$

Set $c_1(R, D, s, F, \epsilon) := |\Delta_F|^{D/2} 2^{-[F:\mathbb{Q}]R} \left(\lfloor \frac{D-1}{2} \rfloor!\right)^R$. Then we see that

$$|L(\mathrm{Mot}_G)| \leqslant c_1 \prod_{d=1}^{\lfloor \frac{d_G+1}{2} \rfloor} \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{f}(\mathrm{Mot}_{G,d}))^{d-\frac{1}{2}+\epsilon}$$

$$= c_1 \prod_{v \in \mathrm{Ram}(G)} \prod_{d=1}^{\lfloor \frac{d_G+1}{2} \rfloor} q_v^{(d-\frac{1}{2}+\epsilon) \cdot f(G_v, d)}.$$

This concludes the proof of (i).

According to Lemma 6.4, the exponent in the right hand side is bounded by

$$df(G_v, d) \leqslant \frac{D+1}{2} \dim \mathrm{Mot}_{G,d} \cdot (s(1 + e_{F_v/\mathbb{Q}_p} \log s) - 1).$$

(we have chosen $\epsilon = \frac{1}{2}$). The proof of (ii) is concluded by the fact that

$$\sum_{d \geqslant 1} \dim \mathrm{Mot}_{G,d} = r_G \leqslant R,$$

see Proposition 6.3 (ii).

Corollary 6.17 *Let G be a connected cuspidal reductive group over F with anisotropic center. Then there exist constants $c_2 = c_2(G, F) > 0$ and $A_2(G, F) > 0$ depending only on G and F such that: for any cuspidal F-Levi subgroup M of G and any semisimple $\gamma \in M(F)$ which is elliptic in $M(\mathbb{R})$,*

$$\left| L(\mathrm{Mot}_{I_\gamma^M}) \right| \leqslant c_2 \prod_{v \in \mathrm{Ram}(I_\gamma^M)} q_v^{A_2}$$

where I_γ^M denote the connected centralizer of γ in M. The following choice is admissible:

$$A_2 = \frac{(d_G + 1)r_G w_G s_G}{2} \max_{prime\ p} (1 + e_{F_v/\mathbb{Q}_p} \log w_G s_G).$$

Proof According to Lemma 6.5, $s_{I_\gamma^M}^{\mathrm{spl}} \leqslant w_G s_G$. Apply Corollary 6.16 for each I_γ^M with $R = r_G$, $D = d_G$ and $s = w_G s_G$ to deduce the first assertion, which obviously implies the last assertion. Note that $\mathrm{rk}\, I_\gamma^M \leqslant r_G$ and that $\dim I_\gamma^M \leqslant d_G$.

Instead of using the Brauer–Siegel theorem which is ineffective, we could use the estimates by Zimmert [109] for the size of the regulator of number fields. This yields an effective estimate for the constants c_2 and c_3 above, at the cost of enlarging the value of the exponents A_1 and A_2.

6.8 Frobenius–Schur indicator

The Frobenius–Schur indicator is an invariant associated to an irreducible representation. It may take the three values $1, 0, -1$. This subsection gathers several well-known facts and recalls some familiar constructions.

The Frobenius–Schur indicator can be constructed in greater generality but the following setting will suffice for our purpose. We will only consider finite dimensional representations on vector spaces over $\mathbb{C}$ or $\mathbb{R}$. The representations are continuous (and unitary) from compact Lie groups or algebraic from linear algebraic groups (these are in fact closely related by the classical "unitary trick" of Hurwitz and Weyl).

Let G be a compact Lie group and denote by μ the Haar probability measure on G. Let (V, r) be a continuous irreducible representation of G. Denote by $\chi(g) = \mathrm{Tr}(r(g))$ its character.

Definition 6.18 The Frobenius–Schur indicator of an irreducible representation (V, r) of G is defined by

$$s(r) := \int_G \chi(g^2) d\mu(g).$$

We have that $s(r) \in \{-1, 0, 1\}$ always.

Remark 6.19 More generally if G is an arbitrary group but V is still finite dimensional, then $s(r)$ is defined as the multiplicity of the trivial representation in the virtual representation on $\mathrm{Sym}^2 V - \wedge^2 V$. This is consistent with the above definition.

Remark 6.20 (i) Let $(V^\vee, r^\vee)$ be the dual representation of G in the dual $V^\vee$. It is easily seen that $s(r) = s(r^\vee)$.
(ii) If $G = G_1 \times G_2$ and r is the irreducible representation of G on $V = V_1 \otimes V_2$ where (V_1, r_1) and (V_2, r_2) are irreducible representations of G_1 (resp. G_2), then $s(r) = s(r_1)s(r_2)$.

The classical theorem of Frobenius and Schur says that r is a real, complex or quaternionic representation if and only if $s(r) = 1, 0$ or -1 respectively. We elaborate on that dichotomy in the following three lemma.

Lemma 6.21 (Real representation) *Let (V, r) be an irreducible representation of G. The following assertions are equivalent:*

(i) $s(r) = 1$;
(ii) *r is self-dual and defined over $\mathbb{R}$ in the sense that $V \simeq V_0 \otimes_{\mathbb{R}} \mathbb{C}$ for some irreducible representation on a real vector space V_0. (Such an r is said to be a* real representation;)
(iii) *r has an invariant real structure. Namely there is a G-invariant anti-linear map $j : V \to V$ which satisfies $j^2 = 1$.*
(vi) *r is self-dual and any bilinear form on V that realizes the isomorphism $r \simeq r^\vee$ is symmetric;*
(v) *$\mathrm{Sym}^2 V$ contains the trivial representation (then the multiplicity is exactly one).*

We don't repeat the proof here (see e.g. [93]) and only recall some of the familiar constructions. We have a direct sum decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V.$$

The character of the representation $V \otimes V$ is $g \mapsto \chi(g)^2$. By Schur lemma the trivial representation occurs in $V \otimes V$ with multiplicity at most one. In other words the subspace of invariant vectors of $V^\vee \otimes V^\vee$ is at most one. Note that this subspace is identified with $\mathrm{Hom}_G(V, V^\vee)$ which is also the subspace of invariant bilinear forms on V.

The character of the representation $\mathrm{Sym}^2 V$ (resp. $\wedge^2 V$) is

$$\frac{1}{2}(\chi(g)^2 + \chi(g^2)) \quad \text{resp.} \quad \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

From that the equivalence of (i) with (v) follows because the multiplicity of the trivial representation in $\mathrm{Sym}^2 V$ (resp. $\wedge^2 V$) is the mean of its character. The equivalence of (iv) and (v) is clear because a bilinear form on V is an element of $V^\vee \otimes V^\vee$ and it is symmetric if and only if it belongs to $\mathrm{Sym}^2 V^\vee$.

The equivalence of (ii) and (iii) follows from the fact that j is induced by complex conjugation on $V_0 \otimes_\mathbb{R} \mathbb{C}$ and conversely V_0 is the subspace of fixed points by j. Note that a real representation is isomorphic to its complex conjugate representation because j may be viewed equivalently as a G-isomorphism $V \to \overline{V}$. Since V is unitary the complex conjugate representation $\overline{r}$ is isomorphic to the dual representation $r^\vee$. In assertion (ii) one may note that the endomorphism ring of V_0 is isomorphic to $\mathbb{R}$.

Lemma 6.22 (Complex representation) *Let (V, r) be an irreducible representation of G. The following assertions are equivalent:*

   (i) *$s(r) = 0$;*
  (ii) *r is not self-dual;*
 (iii) *r is not isomorphic to $\overline{r}$; (such an r is called a* complex representation;*)*
 (iv) *$V \otimes V$ does not contain the trivial representation.*

We note that for a complex representation, the restriction $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} V$ (obtained by viewing V as a real vector space) is an irreducible real representation of twice the dimension of V. Its endomorphism ring is isomorphic to $\mathbb{C}$.

Lemma 6.23 (Quaternionic/symplectic representation) *Let (V, r) be an irreducible representation of G. The following assertions are equivalent:*

   (i) *$s(r) = -1$;*
  (ii) *r is self-dual and cannot be defined over $\mathbb{R}$.*
 (iii) *r has an invariant quaternionic structure. Namely there is a G-invariant anti-linear map $j : V \to V$ which satisfies $j^2 = -1$. (Such an r is called a* quaternionic representation.*)*

(iv) *r is self-dual and the bilinear form on V that realizes the isomorphism r $\simeq$ r$^\vee$ is antisymmetric. (Such an r is said to be a* symplectic representation;*)*

(v) $\bigwedge^2 V$ *contains the trivial representation* (*the multiplicity is exactly one*).

The equivalence of (iii) and (iv) again comes from the fact that V is unitarizable (because G is a compact group). In that context the notion of symplectic representation is identical to the notion of quaternionic representation. Note that for a quaternionic representation, the restriction $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} V$ is an irreducible real representation of twice the dimension of V. Furthermore its ring of endomorphisms is isomorphic to the quaternion algebra $\mathbb{H}$. Indeed the endomorphism ring contains the (linear) action by i because V is a representation over the complex numbers and together with j and $k = ij$ this is the standard presentation of $\mathbb{H}$.

From the above discussions we see that the Frobenius–Schur indicator can be used to classify irreducible representations over the reals. The endomorphism ring of an irreducible real representation is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and we have described a correspondence with associated complex representations.

7 A uniform bound on orbital integrals

This section is devoted to showing an apparently new result on the uniform bound on orbital integrals evaluated at semisimple conjugacy classes and basis elements of unramified Hecke algebras. Our bound is uniform in the finite place v of a number field (over which the group is defined), the "size" of (the support of) the basis element for the unramified Hecke algebra at v as well as the conjugacy class at v.

The main result is Theorem 7.3, which is invoked in Sect. 9.5. The main local input for Theorem 7.3 is Proposition 7.1. The technical heart in the proof of the proposition is postponed to Sect. 7.3, which the reader may want to skip in the first reading. In Appendix B we discuss an alternative approach to Theorem 7.3 via motivic integration.

7.1 The main local result

We begin with a local assertion with a view toward Theorem 7.3 below. Let G be a connected reductive group over a finite extension F of $\mathbb{Q}_p$ with a maximal F-split torus A. As usual $\mathcal{O}$, ϖ, k_F denote the integer ring, a uniformizer and the residue field. Let $\mathbf{G}$ be the Chevalley group for $G \times_F \overline{F}$, defined over $\mathbb{Z}$. Let $\mathbf{B}$ and $\mathbf{T}$ be a Borel subgroup and a maximal torus of $\mathbf{G}$ such that $\mathbf{B} \supset \mathbf{T}$. We assume that

- G is unramified over F,
- char $k_F > w_G s_G$ and char k_F does not divide the finitely many constants in the Chevalley commutator relations [namely C_{ij} of (7.34)].

(We assume char $k_F > w_G s_G$ to ensure that any maximal torus of G splits over a finite tame extension, cf. Sect. 7.3 below. The latter assumption on char k_F depends only on $\mathbf{G}$.) Fix a smooth reductive model over $\mathcal{O}$ so that $K := G(\mathcal{O})$ is a hyperspecial subgroup of $G(F)$. Fix a Borel subgroup B of G whose Levi factor is the centralizer of A in G. Denote by $v \colon F^\times \to \mathbb{Q}$ the discrete valuation normalized by $v(\varpi) = 1$ and by D^G the Weyl discriminant function, cf. (13.1) below. Set $q_v := |k_F|$.

Suppose that there exists a closed embedding of algebraic groups $\Xi^{\mathrm{spl}} : \mathbf{G} \hookrightarrow \mathrm{GL}_m$ defined over $\mathcal{O}$ such that $\Xi^{\mathrm{spl}}(\mathbf{T})$ [resp. $\Xi^{\mathrm{spl}}(\mathbf{B})$] lies in the group of diagonal (resp. upper triangular) matrices. This assumption will be satisfied by Lemma 2.17 and Proposition 8.1, or alternatively as explained at the start of Sect. 7.4. The assumption may not be strictly necessary but is convenient to have for some later arguments. In the setup of Sect. 7.2 such a Ξ^{spl} will be chosen globally over $\mathbb{Z}[1/Q]$ (i.e. away from a certain finite set of primes), which gives rise to an embedding over $\mathcal{O}$ if v does not divide Q.

Proposition 7.1 *There exist* $a_{G,v}, b_{G,v}, e_{G,v} \geq 0$ (*depending on F, G and* Ξ^{spl}) *such that*

- *for every semisimple* $\gamma \in G(F)$,
- *for every* $\lambda \in X_*(A)$ *and* $\kappa \in \mathbb{Z}_{\geq 0}$ *such that* $\|\lambda\| \leq \kappa$,

$$0 \leq O_\gamma \left(\tau_\lambda^G, \mu_G^{\mathrm{can}}, \mu_{I_\gamma}^{\mathrm{can}} \right) \leq q_v^{a_{G,v} + b_{G,v}\kappa} \cdot D^G(\gamma)^{-e_{G,v}/2}. \tag{7.1}$$

Remark 7.2 We chose the notation $a_{G,v}$ etc rather than $a_{G,F}$ etc in anticipating the global setup of the next subsection where F is the completion of a number field at the place v.

Proof For simplicity we will omit the measures chosen to compute orbital integrals when there is no danger of confusion. Let us argue by induction on the semisimple rank r_G^{ss} of G. In the rank zero case, namely when G is a torus, the proposition is true since $O_\gamma(\tau_\lambda^G)$ is equal to 0 or 1. Now assume that $r_G^{\mathrm{ss}} \geq 1$ and that the proposition is known for all groups whose semisimple ranks are less than r_G^{ss}. In the proof we write a_G, b_G, e_G instead of $a_{G,v}, b_{G,v}, e_{G,v}$ for simplicity.

Step 1. Reduce to the case where $Z(G)$ is anisotropic.
Let A_G denote the maximal split torus in $Z(G)$. Set $\overline{G} := G/A_G$. The goal of Step 1 is to show that if the proposition for $\overline{G}$ then it also holds for G. We have

an exact sequence of algebraic groups over $\mathcal{O}$: $1 \to A_G \to G \to \overline{G} \to 1$. By taking F-points one obtains an exact sequence of groups

$$1 \to A_G(F) \to G(F) \to \overline{G}(F) \to 1,$$

where the surjectivity is implied by Hilbert 90 for A_G. [In fact $G(\mathcal{O}) \to \overline{G}(\mathcal{O})$ is surjective since it is surjective on k_F-points and $G \to \overline{G}$ is smooth, cf. [63, p. 386], but we do not need this.] For any semisimple $\gamma \in G(F)$, denote its image in $\overline{G}(F)$ by $\overline{\gamma}$. The connected centralizer of $\overline{\gamma}$ is denoted $\overline{I}_{\overline{\gamma}}$. There is an exact sequence

$$1 \to A_G(F) \to I_\gamma(F) \to \overline{I}_{\overline{\gamma}}(F) \to 1.$$

We see that $G(F) \to \overline{G}(F)$ induces a bijection $I_\gamma(F) \backslash G(F) \simeq \overline{I}_{\overline{\gamma}}(F) \backslash \overline{G}(F)$. Let A be a maximal F-split torus of G, and $\overline{A}$ be its image in $\overline{G}$. For any $\lambda \in X_*(A)$, denote its image in $X_*(\overline{A})$ by $\overline{\lambda}$. Then

$$O_\gamma^{G(F)}\left(\tau_\lambda^G, \mu_G^{\mathrm{can}}, \mu_{I_\gamma}^{\mathrm{can}}\right) \leqslant O_{\overline{\gamma}}^{\overline{G}(F)}\left(\tau_{\overline{\lambda}}^{\overline{G}}, \mu_{\overline{G}}^{\mathrm{can}}, \mu_{\overline{I}_{\overline{\gamma}}}^{\mathrm{can}}\right).$$

Indeed, this follows from the fact that $I_\gamma(F) \backslash G(F) \simeq \overline{I}_{\overline{\gamma}}(F) \backslash \overline{G}(F)$ carries $\frac{\mu_G^{\mathrm{can}}}{\mu_{I_\gamma}^{\mathrm{can}}}$ to $\frac{\mu_{\overline{G}}^{\mathrm{can}}}{\mu_{\overline{I}_{\overline{\gamma}}}^{\mathrm{can}}}$. As the proposition is assumed to hold for $\overline{G}$, the right hand side is bounded by $q_v^{a_{\overline{G}}+b_{\overline{G}}\kappa} \cdot D^{\overline{G}}(\gamma)^{-e_{\overline{G}}/2} = q_v^{a_{\overline{G}}+b_{\overline{G}}\kappa} \cdot D^G(\gamma)^{-e_{\overline{G}}/2}$. Hence the proposition holds for G if we set $a_G = a_{\overline{G}}$, $b_G = b_{\overline{G}}$ and $e_G = e_{\overline{G}}$. This finishes Step 1.

Step 2. When $Z(G)$ is anisotropic.
The problem will be divided into three cases depending on γ. In each case we find a sufficient condition on a_G, b_G and e_G for (7.1) to be true.

Step 2-1. When $\gamma \in Z(G)(F)$.
In this case the proposition holds for any $a_G, b_G, e_G \geqslant 0$ since $O_\gamma(\tau_\lambda^G) = 0$ or 1 and $D^G(\gamma) = 1$.

Step 2-2. When γ is non-central and non-elliptic.
Then there exists a nontrivial split torus $S \subset Z(I_\gamma)$. Set $M := Z_G(S)$, which is an F-rational Levi subgroup of G. Then $I_\gamma \subset M \subsetneq G$. Note that γ is (G, M)-regular. Lemma 6.1 reads

$$O_\gamma^{G(F)}(\mathbf{1}_{K\lambda(\varpi)K}) = D_M^G(\gamma)^{-1/2} O_\gamma^{M(F)}((\mathbf{1}_{K\lambda(\varpi)K})_M). \qquad (7.2)$$

By conjugation we may assume without loss of generality that $\lambda(\varpi) \in M(F)$. (To justify, find $x \in G(F)$ such that xMx^{-1} contains A. Then $\lambda(\varpi) \in$

$xM(F)x^{-1}$ and $O_\gamma^M = O_{x\gamma x^{-1}}^{xMx^{-1}}$.) Moreover by conjugating λ we may assume that λ is $B \cap M$-dominant. We can write

$$(\mathbf{1}_{K\lambda(\varpi)K})_M = \sum_{\mu \leqslant_{\mathbb{R}} \lambda} c_{\lambda,\mu} \mathbf{1}_{K_M \mu(\varpi) K_M}. \tag{7.3}$$

The ordering in the sum is relative to $B \cap M$. For any $m = \mu(\varpi)$, $c_{\lambda,\mu}$ is equal to

$$(\mathbf{1}_{K\lambda(\varpi)K})_M(m) = \delta_P(m)^{1/2} \int_{N(F)} \mathbf{1}_{K\lambda(\varpi)K}(mn)dn$$
$$= q_v^{\langle \rho_P, \mu \rangle} \mu_G^{\mathrm{can}}(mN(F)K \cap K\lambda(\varpi)K).$$

Lemma 2.13 and the easy inequality $\langle \rho_P, \mu \rangle \leqslant \langle \rho, \lambda \rangle$ allow us to deduce that

$$0 \leqslant c_{\lambda,\mu} \leqslant q_v^{\langle \rho_P, \mu \rangle} \mu_G^{\mathrm{can}}(K\lambda(\varpi)K) \leqslant q_v^{d_G + r_G + 2\langle \rho, \lambda \rangle}.$$

The sum in (7.3) runs over the set of

$$\mu = \lambda - \sum_{\alpha \in \Delta^+} a_\alpha \cdot \alpha \quad \text{with } a_\alpha \in \frac{1}{\delta_G}\mathbb{Z}, \ a_\alpha \geqslant 0$$

such that $\mu \in (X^*(T)_{\mathbb{R}})^+$. Here we need to explain δ_G: If $\mu \leqslant_{\mathbb{R}} \lambda$ then $\lambda - \mu$ is a linear combination of positive coroots with nonnegative rational coefficients. The denominators of such coefficients under the constraint $c_{\lambda,\mu} \neq 0$ are uniformly bounded, where the bound depends on the coroot datum. We write δ_G for this bound.

The above condition on μ and $\|\lambda\| \leqslant \kappa$ imply that $a_\alpha \leqslant \kappa$. We get, by using the induction hypothesis for O_γ^M,

$$0 \leqslant O_\gamma^{M(F)}((\mathbf{1}_{K\lambda(\varpi)K})_M) \leqslant \sum_{\mu \leqslant_{\mathbb{R}} \lambda} c_{\lambda,\mu} O_\gamma^M(\mathbf{1}_{K_M\mu(\varpi)K_M})$$
$$\leqslant \sum_{\mu \leqslant_{\mathbb{R}} \lambda} c_{\lambda,\mu} q_v^{a_M + b_M \kappa} \cdot D^M(\gamma)^{-e_M/2}$$
$$\leqslant (\delta_G(\kappa+1))^{|\Delta^+|} q_v^{d_G + r_G + 2\langle \rho, \lambda \rangle} q_v^{a_M + b_M \kappa} \cdot D^M(\gamma)^{-e_M/2}$$
$$\leqslant q_v^{d_G + r_G(\delta_G \kappa + \delta_G + 1) + 2\langle \rho, \lambda \rangle + a_M + b_M \kappa} D^M(\gamma)^{-e_M/2}.$$

Set

$$c_G := d_G + r_G(\delta_G + 1) + 2\langle \rho, \lambda \rangle \leqslant d_G + r_G(\delta_G + 1) + |\Phi^+|\kappa.$$

In view of (7.2) it suffices to find $a_G, b_G, e_G \geqslant 0$ such that

$$D_M^G(\gamma)^{-1/2} D^M(\gamma)^{-e_M/2} q_v^{a_M + c_G + (b_M + r_G \delta_G)\kappa} \leqslant D^G(\gamma)^{-e_G/2} q_v^{a_G + b_G \kappa}$$

or equivalently

$$D_M^G(\gamma)^{\frac{e_G - 1}{2}} D^M(\gamma)^{\frac{e_G - e_M}{2}} \leqslant q_v^{a_G - a_M - c_G + (b_G - b_M - r_G \delta_G)\kappa} \tag{7.4}$$

whenever a conjugate of γ lies in $K\lambda(\varpi)K$. For each $\alpha \in \Phi$,

$$\begin{array}{ll} v(1 - \alpha(\gamma)) \geqslant 0 & \text{if } v(\alpha(\gamma)) \geqslant 0, \\ v(1 - \alpha(\gamma)) = v(\alpha(\gamma)) \geqslant -b_\Xi \kappa & \text{if } v(\alpha(\gamma)) < 0 \end{array} \tag{7.5}$$

where b_Ξ is the constant B_5 (depending only on G and Ξ and not on v) of Lemma 2.18. Hence

$$D_M^G(\gamma) = \sum_{\substack{\alpha \in \Phi \setminus \Phi_M \\ \alpha(\gamma) \neq 1}} 1 - \alpha(\gamma)|_v \leqslant q_v^{|\Phi \setminus \Phi_M| b_\Xi \kappa / 2}$$

and likewise $D^M(\gamma) \leqslant q_v^{|\Phi_M| b_\Xi \kappa / 2}$. (We divide the exponents by 2 because it cannot happen simultaneously that $v(\alpha(\gamma)) < 0$ and $v(\alpha^{-1}(\gamma)) < 0$.) Therefore condition (7.4) on a_G, b_G, e_G is implied by the two conditions

$$e_G \geqslant \max(1, e_M), \tag{7.6}$$

$$\begin{aligned} &\frac{e_G - 1}{2} \frac{|\Phi \setminus \Phi_M| b_\Xi \kappa}{2} + \frac{e_G - e_M}{2} \frac{|\Phi_M| b_\Xi \kappa}{2} \\ &\leqslant a_G - a_M - (d_G + r_G(\delta_G + 1) + |\Phi^+|\kappa) + (b_G - b_M - r_G \delta_G)\kappa. \end{aligned} \tag{7.7}$$

There are only finitely many Levi subgroups M (up to conjugation) giving rise to the triples (a_M, b_M, e_M). It is elementary to observe that (7.7) holds as long as a_G and b_G are sufficiently large while e_G has any fixed value such that (7.6) holds. We will impose another condition on a_G, b_G, e_G in Step 2-3.

Step 2-3. When γ is noncentral and elliptic in G.
This case is essentially going to be worked out in Sect. 7.3. Let $Z_1, Z_2 \geqslant 0$ be as in Lemma 7.9 below. By (7.11) and Corollary 7.11 below, (7.1) will hold if

$$q_v^{r_G(d_G + 1)} q_v^{1 + Z_1 \kappa} D^G(\gamma)^{-Z_2} \leqslant q_v^{a_G + b_G \kappa} D^G(\gamma)^{-e_G/2}. \tag{7.8}$$

We have $D^G(\gamma) \leqslant q_v^{|\Phi||b_\Xi\kappa/2}$ thanks to (7.5) (cf. Step 2-2). So (7.8) (is not equivalent to but) is implied by the combination of the following two inequalities:

$$-Z_2 + \frac{e_G}{2} \geqslant 0. \tag{7.9}$$

$$r_G(d_G + 1) + 1 + Z_1\kappa + |\Phi||b_\Xi \frac{\kappa}{2}(-Z_2 + \frac{e_G}{2}) \leqslant a_G + b_G\kappa. \tag{7.10}$$

The latter two will hold true, for instance, if e_G has any fixed value greater than or equal to $2Z_2$ and if a_G and b_G are sufficiently large. (We will see in Sect. 7.3 below that Z_1 and Z_2 are independent of λ, γ and κ.)

Now that we are done with analyzing three different cases, we finish Step 2. For this we use the induction on semisimple ranks (to ensure the existence of a_M, b_M and e_M in Step 2-2) to find $a_G, b_G, e_G \geqslant 0$ which satisfy the conditions described at the ends of Step 2-2 and Step 2-3. We are done with the proof of Proposition 7.1.

7.2 A global consequence

Here we switch to a global setup. Let $\mathbf{F}$ be a number field. For a finite place v of $\mathbf{F}$, let $k(v)$ denote the residue field and put $q_v := |k(v)|$.

- G is a connected reductive group over $\mathbf{F}$.
- $\mathrm{Ram}(G)$ is the set of finite places v of $\mathbf{F}$ such that G is ramified at $\mathbf{F}_v$.
- $\mathbf{G}$ is the Chevalley group for $G \times_{\mathbf{F}} \overline{\mathbf{F}}$, and $\mathbf{B}$, $\mathbf{T}$ are as in Sect. 7.1.
- $\Xi^{\mathrm{spl}} \colon \mathbf{G} \hookrightarrow \mathrm{GL}_m$, fixed once and for all, is a closed embedding defined over $\mathbb{Z}[1/R]$ for a large enough integer R such that $\Xi^{\mathrm{spl}}(\mathbf{T})$ [resp. $\Xi^{\mathrm{spl}}(\mathbf{B})$] lies in the group of diagonal (resp. upper triangular) matrices of GL_m. The choice of R depends only on $\mathbf{G}$ and Ξ^{spl}. (We defer to Sect. 7.4 more details and the explanation that there exists such a Ξ^{spl}.)
- S_{bad} is the set of finite places v such that either $v \in \mathrm{Ram}(G)$, $\mathrm{char}\, k(v) \leqslant w_G s_G$, $\mathrm{char}\, k(v)$ divides R, or $\mathrm{char}\, k(v)$ divides at least one of the constants for the Chevalley commutator relations for $\mathbf{G}$, cf. (7.34) below.

Examining the dependence of various constants in Proposition 7.1 leads to the following main result of this section. For each finite place $v \notin S_{\mathrm{bad}}$, denote by A_v a maximal $\mathbf{F}_v$-split torus of $G \times_{\mathbf{F}} \mathbf{F}_v$.

Theorem 7.3 *There exist* $a_G, b_G \geqslant 0$ *and* $e_G \geqslant 1$ (*depending on* $\mathbf{F}$, G *and* Ξ^{spl}) *such that*

- *for every finite* $v \notin S_{\mathrm{bad}}$,
- *for every semisimple* $\gamma \in G(\mathbf{F}_v)$,

- *for every $\lambda \in X_*(A_v)$ and $\kappa \in \mathbb{Z}_{\geqslant 0}$ such that $\|\lambda\| \leqslant \kappa$,*

$$0 \leqslant O_\gamma^{G(\mathbf{F}_v)}\left(\tau_\lambda^G, \mu_{G,v}^{\mathrm{can}}, \mu_{I_\gamma,v}^{\mathrm{can}}\right) \leqslant q_v^{a_G+b_G\kappa} \cdot D_v^G(\gamma)^{-e_G/2}.$$

Remark 7.4 It is worth drawing a comparison between the above theorem and Theorem 13.1 proved by Kottwitz. In the latter the test function (in the full Hecke algebra) and the base p-adic field are fixed whereas the main point of the former is to allow the test function (in the unramified Hecke algebra) and the place v to vary. The two theorems are complementary to each other and will play a crucial role in the proof of Theorem 9.19.

Remark 7.5 In an informal communication Kottwitz and Ngô pointed out that there might be yet another approach based on a geometric argument involving affine Springer fibers, as in [46, §15], which might lead to a streamlined and conceptual proof, as well as optimized values of the constants a_G and b_G. Appendix B provides an important step in that direction, see Theorem 14.7 which implies that the constants are transferable from finite characteristic to characteristic zero.

Proof Since the case of tori is clear, we may assume that $r_G^{\mathrm{ss}} \geqslant 1$. Let $\theta \in \mathscr{C}(\Gamma_1)$. (Recall the definition of Γ_1 and $\mathscr{C}(\Gamma_1)$ from Sects. 5.1 and 5.2.) Our strategy is to find $a_{G,\theta}, b_{G,\theta}, e_{G,\theta} \geqslant 0$ which satisfy the requirements (7.7), (7.9), and (7.10) on $a_{G,v}, b_{G,v}, e_{G,v}$ at all $v \in \mathcal{V}_{\mathbf{F}}(\theta)\backslash S_{\mathrm{bad}}$. As for (7.7), we inductively find $a_{M,\theta}, b_{M,\theta}, e_{M,\theta} \geqslant 0$ for all local Levi subgroups M of G as will be explained below.

We would like to explain an inductive choice of $a_{M,\theta}, b_{M,\theta}, e_{M,\theta} \geqslant 0$ for a fixed θ. To do so we ought to clarify what Levi subgroups M of G we consider. Let Δ denote the set of $\mathbf{B}$-positive simple roots for $(\mathbf{G}, \mathbf{T})$. Via an identification $G \times_{\mathbf{F}} \overline{\mathbf{F}} \simeq \mathbf{G} \times_{\mathbb{Z}} \overline{\mathbf{F}}$ we may view Δ as the set of simple roots for G equipped with an action by Γ_1, cf. [9, §1.3]. Note that Frob_v acts as $\theta \in \Gamma_1$ on Δ for all $v \in \mathcal{V}_{\mathbf{F}}(\theta)\backslash S_{\mathrm{bad}}$. According to [9, §3.2], the θ-stable subsets of Δ are in bijection with $G(\mathbf{F}_v)$-conjugacy classes of $\mathbf{F}_v$-parabolic subgroups of G. For each $v \in \mathcal{V}_{\mathbf{F}}(\theta)\backslash S_{\mathrm{bad}}$, fix a Borel subgroup B_v of G over $\mathbf{F}_v$ containing the centralizer T_v of A_v in G so that the following are in a canonical bijection with one another.

- θ-stable subsets Υ of Δ
- parabolic subgroups P_v of G containing B_v

Denote by $P_{\Upsilon,v}$ the parabolic subgroup corresponding to Υ and by $M_{\Upsilon,v}$ its Levi subgroup containing T_v. Here is an important observation. The constants Z_1, Z_2 (see Remark 7.10 below) and the inequalities (7.7), (7.9), and (7.10) to be satisfied by $a_{M_\Upsilon,v}, b_{M_\Upsilon,v}, e_{M_\Upsilon,v}$ depend only on θ and not on

$v \in \mathcal{V}_{\mathbf{F}}(\theta) \backslash S_{\mathrm{bad}}$. (We consider the case where G and M of those inequalities are M_{Υ} and a $\mathbf{F}_v$-Levi subgroup of M_{Υ}, respectively.) Hence we will write $a_{M_{\Upsilon},\theta}, b_{M_{\Upsilon},\theta}, e_{M_{\Upsilon},\theta} \geqslant 0$ for these constants. What we need to do is to define them inductively according to the semisimple rank of M such that (7.7), (7.9), and (7.10) hold true. In particular the desired $a_{G,\theta}, b_{G,\theta}, e_{G,\theta}$ will be obtained and the proof will be finished (by returning to the first paragraph in the current proof).

Now the inductive choice of $a_{M_{\Upsilon},\theta}, b_{M_{\Upsilon},\theta}, e_{M_{\Upsilon},\theta}$ is easy to make once the choice of $a_{M_{\Omega},\theta}, b_{M_{\Omega},\theta}, e_{M_{\Omega},\theta}$ has been made for all $\Omega \subsetneq \Upsilon$. Indeed, we may choose $e_{M_{\Omega},\theta} \in \mathbb{Z}_{\geqslant 1}$ to fulfill (7.9) and then choose $a_{M_{\Omega},\theta}, b_{M_{\Omega},\theta}$ to be large enough to verify (7.7) and (7.10). Notice that Z_1, Z_2, Z_3 of (7.10) (which are constructed in Lemma 7.9 below) depend only on the group-theoretic information of M_{Υ} (such as the dimension, rank, affine root data, $\delta_{M_{\Upsilon}}$ of M_{Υ} as well as an embedding of the Chevalley form of M_{Υ} into GL_d coming from Ξ^{spl}) but not on v, cf. Remark 7.10.

In view of Theorem 13.1 and other observations in harmonic analysis, a natural question is whether it is possible to achieve $e_G = 1$. This is a deep and difficult question which is of independent interest. It was a pleasant surprise to the authors that the theory of arithmetic motivic integration provides a solution. A precise theorem due to Cluckers, Gordon, and Halupczok is stated in Theorem 14.1 below. It is worth remarking that their method of proof is significantly different from that of this section and also that they make use of Theorem 13.1, the local boundedness theorem. Finally it would be interesting to ask about the analogue in the case of twisted or weighted orbital integrals. Such a result would be useful in the more general situation than the one considered in this paper.

7.3 The noncentral elliptic case

The objective of this subsection is to establish Corollary 7.11, which was used in Step 2-3 of the proof of Proposition 7.1 above. Since the proof is quite complicated let us guide the reader. The basic idea, going back to Langlands, is to interpret the orbital integral $O_{\gamma}^{G(F)}(\tau_{\lambda}^G)$ in question as the number of points in the building fixed "up to λ" under the action of γ. The set of such points, denoted $X_F(\gamma, \lambda)$ below, is finite since γ is elliptic. Then it is shown that every point of $X_F(\gamma, \lambda)$ is within a certain distance from a certain apartment, after enlarging the ground field F to a finite extension. We exploit this to bound $X_F(\gamma, \lambda)$ by a ball of an explicit radius in the building. By counting the number of points in the ball (which is of course much more tractable than counting $|X_F(\gamma, \lambda)|$) we arrive at the desired bound on the orbital integral. The proof presented here is inspired by the beautiful exposition of [66, §§3–5] but

uses brute force and crude bounds at several places. We defer some technical lemmas and their proofs to Sect. 7.4 below and refer to them in this subsection but there is no circular logic since no results of this subsection are used in Sect. 7.4.

Throughout this subsection the notation of Sect. 7.1 is adopted and γ is assumed to be noncentral and elliptic in $G(F)$. (However γ need not be regular.) We assume $Z(G)$ to be anisotropic over F as we did in Step 2 of the proof of Proposition 7.1. Then $I_\gamma(F)$ is a compact group, on which the Euler-Poincare measure $\mu_{I_\gamma}^{\mathrm{EP}}$ assigns total volume 1. Our aim is to bound $O_\gamma^{G(F)}(\mathbf{1}_{K\mu(\varpi)K}, \mu_G^{\mathrm{can}}, \mu_{I_\gamma}^{\mathrm{can}})$. It follows from [47, Thm 5.5] (for the equality) and Proposition 6.3 that

$$
\begin{aligned}
\left| \frac{\mu_{I_\gamma}^{\mathrm{EP}}}{\mu_{I_\gamma}^{\mathrm{can}}} \right| &= \frac{\prod_{d \geqslant 1} \det\left(1 - \mathrm{Frob}_v \, q_v^{d-1} \,\big|(\mathrm{Mot}_{I_\gamma, d})^{I_v}\right)}{|H^1(F, I_\gamma)|} \\
&\leqslant \prod_{d \geqslant 1} \left(1 + q_v^{d-1}\right)^{\dim \mathrm{Mot}_{I_\gamma, d}} \\
&\leqslant \left(1 + q_v^{(\dim I_\gamma + 1)/2}\right)^{\mathrm{rk} I_\gamma} \leqslant \left(1 + q_v^{d_G}\right)^{r_G} \leqslant q_v^{r_G(d_G+1)}. \quad (7.11)
\end{aligned}
$$

Thus we may as well bound $O_\gamma^{G(F)}(\mathbf{1}_{K\mu(\varpi)K}, \mu_G^{\mathrm{can}}, \mu_{I_\gamma}^{\mathrm{EP}})$.

Let T_γ be an elliptic maximal torus of I_γ defined over F containing γ. By Lemma 6.5, there exists a Galois extension F'/F with

$$
[F' : F] \leqslant w_G s_G \quad (7.12)
$$

such that T_γ is a split torus over F'. Hence I_γ and G are split groups over F'. Note that F' is a tame extension of F under the assumption that char $k_F > w_G s_G$. Let A' be a split maximal torus of G over F' such that $A \times_F F' \subset A'$. Since F'-split maximal tori are conjugate over F', we find

$$
y \in G(F') \quad \text{such that } A' = y T_\gamma y^{-1}
$$

and fix such a y. Write $\mathcal{O}'$, ϖ' and v' for the integer ring of F', a uniformizer and the valuation on F' such that $v'(\varpi') = 1$. With respect to the integral model of G over $\mathcal{O}$ at the beginning of Sect. 7.1, we put $K' := G(\mathcal{O}')$. A point of $G(F)/K$ will be denoted $\overline{x}$ and any of its lift in $G(F)$ will be denoted x. Let $\overline{x}_0 \in G(F)/K$ [resp. $\overline{x}_0' \in G(F')/K'$] denote the element represented by the trivial coset of K (resp. K'). Then $\overline{x}_0$ (resp. $\overline{x}_0'$) may be thought of as a base point of the building $\mathcal{B}(G(F), K)$ [resp. $\mathcal{B}(G(F'), K')$] and its stabilizer is identified with K (resp. K'). There exists an injection

$$\mathcal{B}(G(F), K) \hookrightarrow \mathcal{B}(G(F'), K') \tag{7.13}$$

such that $\mathcal{B}(G(F), K)$ is the $\mathrm{Gal}(F'/F)$-fixed points of $\mathcal{B}(G(F'), K')$. (This is the case because F' is tame over F.) The natural injection $G(F)/K \hookrightarrow G(F')/K'$ coincides with the injection induced by (7.13) on the set of vertices.

Define $\lambda' \in X_*(A')$ by $\lambda' := e_{F'/F}\lambda$ (where $e_{F'/F}$ is the ramification index of F' over F) so that $\lambda'(\varpi') = \lambda(\varpi)$ and

$$\|\lambda'\| = e_{F'/F}\|\lambda\| \leqslant e_{F'/F}\kappa. \tag{7.14}$$

For (the fixed γ and) a semisimple element $\delta \in G(F')$, set

$$X_F(\gamma, \lambda) := \{\overline{x} \in G(F)/K : \overline{x}^{-1}\gamma\overline{x} \in K\lambda(\varpi)K\}$$
$$X_{F'}(\delta, \lambda') := \{\overline{x}' \in G(F')/K' : (\overline{x}')^{-1}\delta\overline{x}' \in K'\lambda'(\varpi')K'\}.$$

By abuse of notation we write $\overline{x}^{-1}\gamma\overline{x} \in K\lambda(\varpi)K$ for the condition that $x^{-1}\gamma x \in K\lambda(\varpi)K$ for some (thus every) lift $x \in G(F)$ of $\overline{x}$ and similarly for the condition on $\overline{x}'$. It is clear that $X_F(\gamma, \lambda) \subset X_{F'}(\gamma, \lambda') \cap (G(F)/K)$. By (3.4.2) of [66],

$$O_\gamma^{G(F)}\left(\mathbf{1}_{K\lambda(\varpi)K}, \lambda_G, \lambda_{I_\gamma}^{\mathrm{EP}}\right) = |X_F(\gamma, \lambda)|. \tag{7.15}$$

Our goal of bounding the orbital integrals on the left hand side can be translated into a problem of bounding $|X_F(\gamma, \lambda)|$.

Let $\mathrm{Apt}(A'(F'))$ denote the apartment for $A'(F')$. Likewise $\mathrm{Apt}(T_\gamma(F))$ and $\mathrm{Apt}(T_\gamma(F'))$ are given the obvious meanings. We have $\overline{x}_0' \in \mathrm{Apt}(A'(F'))$. The metrics on $\mathcal{B}(G(F), K)$ and $\mathcal{B}(G(F'), K')$ are chosen such that (7.13) is an isometry. The metric on $\mathcal{B}(G(F'), K')$ is determined by its restriction to $\mathrm{Apt}(A'(F'))$, which is in turn pinned down by a (non-canonical choice of) a Weyl-group invariant scalar product on $X_*(A')$, cf. [103, §2.3]. Henceforth we fix the scalar product once and for all. Scaling the scalar product does not change our main results of this subsection.

Remark 7.6 For any other tame extension F'' of F and a split maximal torus A'' of G over F'', we can find an isomorphism $X_*(A')$ and $X_*(A'')$ over the composite field of F' and F'', well defined up to the Weyl group action. So the scalar product on $X_*(A'')$ is uniquely determined by that on $X_*(A')$. So we need not choose a scalar product again when considering a different $\gamma \in G(F)$.

We define certain length functions. Consider an F'-split maximal torus A'' of G (for instance $A'' = T_\gamma$ or $A'' = A'$) and the associated set of roots $\Phi = \Phi(G, A'')$ and the set of coroots $\Phi^\vee = \Phi^\vee(G, A'')$. Let $l_{\max}(\Phi)$ denote the largest length of a positive coroot in $\Phi^\vee$. Note that these are independent

of the choice of A'' and completely determined by the previous choice of a Weyl group invariant scalar product on $X_*(A')$. It is harmless to assume that we have chosen the scalar product such that the longest positive coroot in each irreducible system of $X_*(A')$ has length $l_{\max}(\Phi)$.

Fix a Borel subgroup B' of G over F' containing A' so that $y^{-1}B'y$ is a Borel subgroup containing T_γ. Relative to these Borel subgroups we define the subset of positive roots $\Phi^+(G, A')$ and $\Phi^+(G, T_\gamma)$. Let $m_{\Xi^{\mathrm{spl}}}$ be as in Lemma 7.12 below. In order to bound $|X_F(\gamma, \lambda)|$ in (7.15), we control the larger set $X_{F'}(\delta, \lambda')$ by bounding the distance from its points to the apartment for A'.

Lemma 7.7 *Let $\delta \in A'(F')$ and $\overline{x}' \in G(F')/K'$. Then there exist constants $C = C(\mathbf{G}, \Xi) > 0$, $c_{\mathbf{G}} > 0$, and $Y = Y(\mathbf{G}) \in \mathbb{Z}_{\geqslant 1}$ such that whenever $(\overline{x}')^{-1}\delta\overline{x}' \in K'\lambda'(\varpi')K'$ [i.e. whenever $\overline{x}' \in X_{F'}(\delta, \lambda')$],*

$$
\begin{aligned}
d(\overline{x}', \mathrm{Apt}(A'(F'))) &\leqslant l_{\max}(\Phi) \cdot C|\Delta^+| \cdot Y^{|\Phi^+|} w_G s_G \\
&\times \sum_{\alpha \in \Phi^+(G, A')} \left(|v(1 - \alpha^{-1}(\delta))| + Y(m_{\mathbf{G}}m_{\Xi^{\mathrm{spl}}} + m_{\mathbf{G}}c_{\mathbf{G}} + m_{\Xi^{\mathrm{spl}}})\kappa \right),
\end{aligned}
$$

where the left hand side denotes the shortest distance from $\overline{x}'$ to $\mathrm{Apt}(A'(F'))$.

Proof of Lemma 7.7 Write $\overline{x}' = an\overline{x}'_0$ for some $a \in A'(F')$ and $n \in N(F')$ using the Iwahori decomposition. As both sides of the above inequality are invariant under multiplication by a, we may assume that $a = 1$. Let $\lambda_\delta \in X_*(A')$ be such that $\delta \in \lambda_\delta(\varpi')A'(\mathcal{O}')$. For each $\lambda_0 \in X_*(A')^+$ recall the definition of $n_{\mathbf{G}}(\lambda_0)$ from (2.6). Let $c_{\mathbf{G}} > 0$ be a constant depending only on $\mathbf{G}$ such that every $\lambda_0 \in X_*(A')$ satisfies the inequality $\langle \alpha, \lambda_0 \rangle \leqslant c_{\mathbf{G}}\|\lambda_0\|$ for all $\alpha \in \Phi^+(G, A')$.

Step 1. Show that $\delta^{-1}n^{-1}\delta n \in K'\lambda_0(\varpi')K'$ for some $\lambda_0 \in X_*(A')^+$ such that $n_{\mathbf{G}}(\lambda_0) \leqslant (m_{\Xi^{\mathrm{spl}}} + c_{\mathbf{G}})e_{F'/F}\kappa$.

By Cartan decomposition there exists a B'-dominant $\lambda_0 \in X_*(A')$ such that $\delta^{-1}n^{-1}\delta n \in K'\lambda_0(\varpi')K'$. The condition on δ in the lemma is unraveled as $(x'_0)^{-1}n^{-1}\delta nx'_0 \in K'\lambda'(\varpi')K'$. So

$$
\delta^{-1}n^{-1}\delta n \in \delta^{-1}K'\lambda'(\varpi')K' \subset (K'\lambda_\delta^{-1}(\varpi')K')(K'\lambda'(\varpi')K').
$$

Let w be a Weyl group element for A' in G such that $w\lambda_\delta^{-1}$ is B'-dominant. The fact that $K'\lambda_0(\varpi')K'$ intersects $(K'\lambda_\delta^{-1}(\varpi')K')(K'\lambda'(\varpi')K')$ implies [16, Prop4.4.4.(iii)] that

$$
\langle \alpha, \lambda_0 \rangle \leqslant \left\langle \alpha, w\lambda_\delta^{-1} + \lambda' \right\rangle, \quad \alpha \in \Phi^+(G, A').
$$

We have $\langle \alpha, \lambda' \rangle \leqslant c_{\mathbf{G}} \|\lambda'\|$. Note also that

$$v'(\alpha(\delta)) \in [-m_{\Xi^{\mathrm{spl}}} \|\lambda'\|, m_{\Xi^{\mathrm{spl}}} \|\lambda'\|] \tag{7.16}$$

by Lemma 7.12 since a conjugate of δ belongs to $K'\lambda'(\varpi')K'$. This implies that

$$\left\langle \alpha, w\lambda_\delta^{-1} \right\rangle = v'(w\alpha^{-1}(\delta)) \leqslant m_{\Xi^{\mathrm{spl}}} \|\lambda'\|.$$

On the other hand $\|\lambda'\| \leqslant e_{F'/F}\kappa$ according to (7.14). These inequalities imply the desired bound on $n_{\mathbf{G}}(\lambda_0)$, which is the maximum of $\langle \alpha, \lambda_0 \rangle$ over $\alpha \in \Phi^+(G, A')$.

Before entering Step 2, we notify the reader that we are going to use the convention and notation for the Chevalley basis as recalled in Sect. 7.4 below. In particular $n \in N(F')$ can be written as [cf. (7.33)]

$$n = x_{\alpha_1}(X_{\alpha_1}) \dots x_{\alpha_{|\Phi^+|}}(X_{\alpha_{|\Phi^+|}}) \tag{7.17}$$

for unique $X_{\alpha_1}, \dots, X_{\alpha_{|\Phi^+|}} \in F'$.

Step 2. Show that there exists a constant $\mathcal{M}_{|\Phi^+|} \geqslant 0$ [explicitly defined in (7.20) below] such that $v'(X_{\alpha_i}) \geqslant -\mathcal{M}_{|\Phi^+|}$ for all $1 \leqslant i \leqslant |\Phi^+|$. In our setting we compute

$$\delta^{-1}n^{-1}\delta n = \delta^{-1} \left(\prod_{i=|\Phi^+|}^{1} x_{\alpha_i}(-X_{\alpha_i}) \right) \delta \prod_{i=1}^{|\Phi^+|} x_{\alpha_i}(X_{\alpha_i})$$

$$= \left(\prod_{i=|\Phi^+|}^{1} x_{\alpha_i}(-\alpha_i^{-1}(\delta)X_{\alpha_i}) \right) \prod_{i=1}^{|\Phi^+|} x_{\alpha_i}(X_{\alpha_i})$$

$$= \prod_{i=1}^{|\Phi^+|} x_{\alpha_i} \left((1 - \alpha_i^{-1}(\delta))X_{\alpha_i} + P_{\alpha_i} \right) \tag{7.18}$$

where the last equality follows from the repeated use of (7.34) to rearrange the terms. Here P_{α_i} is a polynomial (which could be zero) in $\alpha_j^{-1}(\delta)$ and X_{α_j} with integer coefficients for $j < i$. It is not hard to observe from (7.34) that P_{α_i} has no constant term. As i varies in $[1, |\Phi^+|]$, let Y denote the highest degree for the nonzero monomial term appearing in P_{α_i} viewed as a polynomial in either $\alpha_i^{-1}(\delta)$ or X_{α_i} (but not both).[7] Set $Y = 1$ if $P_{\alpha_i} = 0$. As mentioned above, the

---

[7] For instance if $P_{\alpha_i} = \alpha_i^{-1}(\delta)^2 X_{\alpha_i}^4 + \alpha_i^{-1}(\delta)^3 X_{\alpha_i}^3$ then $Y = 4$.

positive roots for a given $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ are ordered once and for all so that Y depends only on $\mathbf{G}$ in the sense that for any G having $\mathbf{G}$ as its Chevalley form, Y is independent of the local field F over which G is defined.

Applying Corollary 7.14 below, we obtain from (7.18) and the condition $\delta^{-1}n^{-1}\delta n \in K'\lambda_0(\varpi')K'$ that

$$v'\left((1 - \alpha_i^{-1}(\delta))X_{\alpha_i} + P_{\alpha_i}\right) \geqslant -m_{\mathbf{G}}n_{\mathbf{G}}(\lambda_0). \tag{7.19}$$

For $1 \leqslant i \leqslant |\Phi^+|$, put

$$\mathcal{M}_i := \sum_{j=1}^{i}\left(Y^{i-j}(|v'(1 - \alpha_j^{-1}(\delta))| + m_{\mathbf{G}}n_{\mathbf{G}}(\lambda_0))\right)$$
$$+ \sum_{j=1}^{i-1} Y^j m_{\Xi^{\mathrm{spl}}} e_{F'/F}\kappa. \tag{7.20}$$

Obviously $0 \leqslant \mathcal{M}_1 \leqslant \mathcal{M}_2 \leqslant \cdots \leqslant \mathcal{M}_{|\Phi^+|}$. We claim that for every $i \geqslant 1$,

$$v'(X_{\alpha_i}) \geqslant -\mathcal{M}_i. \tag{7.21}$$

When $i = 1$, this follows from (7.19) as $P_{\alpha_1} = 0$. (Use the fact that $x_{\alpha_1}(a_1 X_{\alpha_1})$ commutes with any other $x_{\alpha_j}(a_j X_{\alpha_j})$ in view of (7.34) since α_1 is a simple root.) Now by induction, suppose that (7.21) is verified for all $j < i$. By (7.19),

$$v'(X_{\alpha_i}) + v'\left(1 - \alpha_i^{-1}(\delta)\right) \geqslant \min\left(-m_{\mathbf{G}}n_{\mathbf{G}}(\lambda_0), v'(P_{\alpha_i})\right).$$

Note that P_{α_i} is the sum of monomials of the form $\alpha_j^{-1}(\delta)^{k_1}X_{\alpha_j}^{k_2}$ with $j, k_1, k_2 \in \mathbb{Z}$ such that $1 \leqslant j < i$ and $0 \leqslant k_1, k_2 \leqslant Y$. Each monomial satisfies

$$v'\left(\alpha_j^{-1}(\delta)^{k_1}X_{\alpha_j}^{k_2}\right) = k_1 v'\left(\alpha_j^{-1}(\delta)\right) + k_2 v'(X_{\alpha_j})$$
$$\geqslant -Y m_{\Xi^{\mathrm{spl}}} e_{F'/F}\kappa - Y\mathcal{M}_{i-1},$$

where the inequality follows from (7.16), (7.14), the induction hypothesis, and the fact that $0 \leqslant \mathcal{M}_j \leqslant \mathcal{M}_{i-1}$. Hence

$$v'(P_{\alpha_i}) \geqslant -Y m_{\Xi^{\mathrm{spl}}} e_{F'/F}\kappa - Y\mathcal{M}_{i-1}.$$

Now

$$v'(X_{\alpha_i}) \geqslant \min\left(-m_{\mathbf{G}}n_{\mathbf{G}}(\lambda_0), v'(P_{\alpha_i})\right) - v'\left(1 - \alpha_i^{-1}(\delta)\right)$$
$$\geqslant -m_{\mathbf{G}}n_{\mathbf{G}}(\lambda_0) - Ym_{\Xi^{\mathrm{spl}}}e_{F'/F}\kappa - Y\mathcal{M}_{i-1}$$
$$- \left|v'\left(1 - \alpha_i^{-1}(\delta)\right)\right| = -\mathcal{M}_i,$$

as desired. Now that the claim is verified, we have a fortiori

$$v'(X_{\alpha_i}) \geqslant -\mathcal{M}_{|\Phi^+|}, \quad \forall 1 \leqslant i \leqslant |\Phi^+|. \tag{7.22}$$

For our purpose it suffices to use the following upper bound, which is simpler than $\mathcal{M}_{|\Phi^+|}$. Note that we used the upper bound on $n_{\mathbf{G}}(\lambda_0)$ from Step 1.

$$\mathcal{M}_{|\Phi^+|} \leqslant Y^{|\Phi^+|} \sum_{\alpha \in \Phi^+} \left(|v'(1 - \alpha^{-1}(\delta)| + (m_{\mathbf{G}}m_{\Xi^{\mathrm{spl}}} + m_{\mathbf{G}}c_{\mathbf{G}}\right.$$
$$\left. + m_{\Xi^{\mathrm{spl}}})e_{F'/F}\kappa\right). \tag{7.23}$$

Step 3. Find $a \in A'(F')$ such that $a^{-1}na \in K'$.
We can choose a sufficiently large $C = C(\mathbf{G}, \Xi) > 0$, depending only on the Chevalley group $\mathbf{G}$ and Ξ, and integers $a_\alpha^0 \in [-C, 0]$ for $\alpha \in \Delta^+$ such that

$$1 \leqslant \sum_{\alpha \in \Delta^+} (-a_\alpha^0)\langle \beta, \alpha^\vee \rangle \leqslant C, \quad \forall \beta \in \Delta^+.$$

[This is possible because the matrix $(\langle \beta, \alpha^\vee \rangle)_{\beta, \alpha \in \Delta^+}$ is nonsingular. For instance one finds $a_\alpha^0 \in \mathbb{Q}$ satisfying the above inequalities for $C = 1$ and then eliminate denominators in a_α^0 by multiplying a large positive integer.] Now put $a_\alpha := \mathcal{M}_{|\Phi^+|}a_\alpha^0 \in [-C\mathcal{M}_{|\Phi^+|}, 0]$ and $a := \sum_{\alpha \in \Delta^+} a_\alpha \alpha^\vee(\varpi') \in A'(F')$ so that

$$\mathcal{M}_{|\Phi^+|} \leqslant -v(\beta(a)) \leqslant C \cdot \mathcal{M}_{|\Phi^+|}, \quad \forall \beta \in \Delta^+. \tag{7.24}$$

In fact (7.24) implies that the left inequality holds for all $\beta \in \Phi^+$. Hence

$$a^{-1}na = \prod_{i=1}^{|\Phi^+|} x_{\alpha_i}\left(\alpha_i(a)^{-1}X_{\alpha_i}\right)$$

$$\in \prod_{i=1}^{|\Phi^+|} U_{\alpha_i, v(X_{\alpha_i}) - v(\alpha_i(a))} \subset \prod_{i=1}^{|\Phi^+|} U_{\alpha_i, \mathcal{M}_{|\Phi^+|} + v(X_{\alpha_i})}.$$

Here we have written $U_{\alpha, m}$ with $m \in \mathbb{R}$ for the image under the isomorphism $x_\alpha \colon F \simeq U_\alpha(F)$ of the set $\{a \in F : v(a) \geqslant m\}$. In light of (7.21), $\mathcal{M}_{|\Phi^+|} + v(X_{\alpha_i}) \geqslant 0$. Hence $a^{-1} n a \in K'$.

Step 4. Conclude the proof.

Step 3 shows that $a \overline{x}'_0 \in \mathrm{Apt}(A'(F'))$ is invariant under the left multiplication action by n on $\mathcal{B}(G(F'), K')$, which acts as an isometry. Recalling that $\overline{x}' = n \overline{x}'_0$ we have

$$d(\overline{x}', \mathrm{Apt}(A'(F'))) \leqslant d(n \overline{x}'_0, a \overline{x}'_0) = d(n \overline{x}'_0, n a \overline{x}'_0)$$
$$= d(\overline{x}'_0, a \overline{x}'_0). \tag{7.25}$$

On the other hand, for any $\overline{x}' \in \mathrm{Apt}(A'(F'))$ and any positive simple coroot $\alpha^\vee$, we have

$$d(\overline{x}', \alpha^\vee(\varpi')^{-1} \overline{x}') \leqslant l_{\max}(\Phi). \tag{7.26}$$

Indeed this holds by the definition of $l_{\max}(\Phi)$ as the left hand side is the length of $\alpha^\vee$. Since $a = \prod_{\alpha \in \Delta^+} (\alpha^\vee(\varpi'))^{a_\alpha}$ with $a_\alpha \in [-C\mathcal{M}_{|\Phi^+|}, 0]$, a repeated use of (7.26), together with a triangle inequality, shows that

$$d(\overline{x}'_0, a \overline{x}'_0) \leqslant l_{\max}(\Phi) \cdot C \cdot \mathcal{M}_{|\Phi^+|} \cdot |\Delta^+|. \tag{7.27}$$

Lemma 7.7 follows from (7.25), (7.27), (7.22), (7.23), and $e_{F'/F} \leqslant [F' : F] \leqslant w_G s_G$ as we saw in (7.12). $\qquad \square$

Since γ is elliptic and G is anisotropic over F, $\mathrm{Apt}(T_\gamma(F))$ is a singleton. Let $\overline{x}_1$ denote its only point. Then the $\mathrm{Gal}(F'/F)$-action on $\mathrm{Apt}(T_\gamma(F'))$ has $\overline{x}_1$ as the unique fixed point. Motivated by Lemma 7.7 we set $\mathcal{M}(\gamma, \kappa)$ to be

$$l_{\max}(\Phi) \cdot C|\Delta^+| \cdot Y^{|\Phi^+|} w_G s_G$$
$$\times \sum_{\alpha \in \Phi(G, T_\gamma)} \left(|v(1 - \alpha^{-1}(\gamma))| + Y(m_{\mathbf{G}} m_{\Xi^{\mathrm{spl}}} + m_{\mathbf{G}} c_{\mathbf{G}} + m_{\Xi^{\mathrm{spl}}}) \kappa \right)$$

and similarly $\mathcal{M}(\delta, \kappa)$ using $\alpha \in \Phi(G, A')$ in the sum instead. Note that we are summing over all roots, not just positive roots as in the lemma. This is okay since it will only improve the inequality of the lemma. We do this such that $\mathcal{M}(\gamma, \kappa) = \mathcal{M}(\delta, \kappa)$. Indeed the equality is induced by a bijection $\Phi(G, T_\gamma) \simeq \Phi(G, A')$ coming from any element $y' \in G(F')$ such that $A' =$

$y'T_\gamma(y')^{-1}$ (for example one can take $y' = y$). Define a closed ball in $G(F)/K$: for $\overline{z} \in G(F)/K$ and $R \geqslant 0$,

$$\text{Ball}(\overline{z}, R) := \{\overline{x} \in G(F)/K : d(\overline{x}, \overline{z}) \leqslant R\}.$$

Lemma 7.8 $X_F(\gamma, \lambda) \subset \text{Ball}(\overline{x}_1, \mathcal{M}(\gamma, \kappa))$.

Proof As we noted above, $X_F(\gamma, \lambda) \subset X_{F'}(\gamma, \lambda') = X_{F'}(y^{-1}\delta y, \lambda')$. Lemma 7.7 tells us that

$$\overline{x} \in X_F(\gamma, \lambda) \implies d(y\overline{x}, \text{Apt}(A'(F'))) \leqslant \mathcal{M}(\delta, \kappa) \implies d(\overline{x}, \text{Apt}(T_\gamma(F')))$$
$$\leqslant \mathcal{M}(\delta, \kappa).$$

The last implication uses $\text{Apt}(A'(F')) = y\text{Apt}(T_\gamma(F'))$ (recall $A' = yT_\gamma y^{-1}$). We have viewed $\overline{x}$ as a point of $\mathcal{B}(G(F'), K')$ via the isometric embedding $\mathcal{B}(G(F), K) \hookrightarrow \mathcal{B}(G(F'), K')$. In order to prove the lemma, it is enough to check that $d(\overline{x}, \overline{x}_1) \leqslant d(\overline{x}, \overline{x}_2)$ for every $\overline{x}_2 \in \text{Apt}(T_\gamma(F'))$. To this end, we suppose that there exists an $\overline{x}_2$ with

$$d(\overline{x}, \overline{x}_1) > d(\overline{x}, \overline{x}_2) \tag{7.28}$$

and will draw a contradiction.

As $\sigma \in \text{Gal}(F'/F)$ acts on $\mathcal{B}(G(F'), K')$ by isometry, $d(\overline{x}, \sigma\overline{x}_2) = d(\overline{x}, \overline{x}_2)$. As $\text{Apt}(T_\gamma(F'))$ is preserved under the Galois action, $\sigma\overline{x}_2 \in \text{Apt}(T_\gamma(F'))$. According to the inequality of [103, 2.3], for any $x, y, z \in \mathcal{B}(G(F'), K')$ and for the unique mid point $m = m(x, y) \in \mathcal{B}(G(F'), K')$ such that $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$,

$$d(x, z)^2 + d(y, z)^2 \geqslant 2d(m, z)^2 + \frac{1}{2}d(x, y)^2. \tag{7.29}$$

Consider the convex hull $\mathscr{C}$ of $\mathscr{C}_0 := \{\sigma\overline{x}_2\}_{\sigma \in \text{Gal}(F'/F)}$. Since $\mathscr{C}_0$ is contained in $\text{Apt}(T_\gamma(F'))$, so is $\mathscr{C}$. Moreover $\mathscr{C}_0$ is fixed under $\text{Gal}(F'/F)$, from which it follows that $\mathscr{C}$ is also preserved under the same action. [One may argue as follows. Inductively define $\mathscr{C}_{i+1}$ to be the set consisting of the mid points $m(x, y)$ for all $x, y \in \mathscr{C}_i$. Then it is not hard to see that $\mathscr{C}_i$ must be preserved under $\text{Gal}(F'/F)$ and that $\cup_{i \geqslant 0}\mathscr{C}_i$ is a dense subset of $\mathscr{C}$.] As $\mathscr{C}$ is a compact set, one may choose $\overline{x}_3 \in \mathscr{C}$ which has the minimal distance to $\overline{x}$ among the points of $\mathscr{C}$. By construction

$$d(\overline{x}_3, \overline{x}) \leqslant d(\overline{x}_2, \overline{x}). \tag{7.30}$$

Applying (7.29) to $(x, y, z) = (\overline{x}_3, \sigma\overline{x}_3, \overline{x})$, where $\sigma \in \text{Gal}(F'/F)$,

$$2d(\overline{x}_3, \overline{x})^2 = d(\overline{x}_3, \overline{x})^2 + d(\sigma\overline{x}_3, \overline{x})^2 \geqslant 2d(m(\overline{x}_3, \sigma\overline{x}_3), \overline{x})^2 + \frac{1}{2}d(\overline{x}_3, \sigma\overline{x}_3)^2.$$

As $\overline{x}_3, \sigma\overline{x}_3 \in \mathscr{C}$, we also have $m(\overline{x}_3, \sigma\overline{x}_3) \in \mathscr{C}$ by the convexity of $\mathscr{C}$. The choice of $\overline{x}_3$ ensures that $d(\overline{x}_3, \overline{x}) \leqslant d(m(\overline{x}_3, \sigma\overline{x}_3), \overline{x})$, therefore $d(\overline{x}_3, \sigma\overline{x}_3) = 0$, i.e. $\overline{x}_3 = \sigma\overline{x}_3$. Hence $\overline{x}_3$ is a $\mathrm{Gal}(F'/F)$-fixed point of $\mathrm{Apt}(T_\gamma(F'))$. This implies that $\overline{x}_3 = \overline{x}_1$, but then (7.30) contradicts (7.28). □

Lemma 7.9 *There exist constants $Z_1, Z_2 \geqslant 0$, independent of γ and λ, such that*

$$|\mathrm{Ball}(\overline{x}_1, \mathcal{M}(\gamma, \kappa))| \leqslant q_v^{1+Z_1\kappa} D^G(\gamma)^{-Z_2}.$$

Remark 7.10 A scrutiny into the defining formulas for Z_1 and Z_2 (as well as Z_1' and Z_2') at the end of the proof reveals that Z_1 and Z_2 depend only on the affine root data, the group-theoretic constants for G (and its Chevalley form), and Ξ. An important point is that, in the situation where local data arise from some global reductive group over a number field by localization, the constants Z_1 and Z_2 do not depend on the residue characteristic p or the p-adic field F as long as the affine root data remain unchanged. This observation is used in the proof of Theorem 7.3 to establish a kind of uniformity when traveling between places in $\mathcal{V}(\theta)\backslash S_{\mathrm{bad}}$ for a fixed $\theta \in \mathscr{C}(\Gamma_1)$ in the notation there.

Proof To ease notation we write $\mathcal{M}$ for $\mathcal{M}(\gamma, \kappa)$ in the proof. Let us introduce some quantities and objects of geometric nature for the building $\mathcal{B}(G(F), K)$. Write $e_{\max} > 0$ for the maximum length of the edges of $\mathcal{B}(G(F), K)$. For a subset S of $\mathcal{B}(G(F), K)$, define $\mathrm{Ch}^+(S)$ to be the set of chambers $\mathscr{C}$ of the building such that $\mathscr{C} \cap S$ contains a vertex. Let $v \in \mathcal{B}(G(F), K)$ be a vertex. (We are most interested in the case $v = \overline{x}_1$.) We put $\mathscr{C}_1(v)$ to be the union of chambers in $\mathrm{Ch}^+(\{v\})$ and define $\mathscr{C}_{i+1}(v)$ to be the union of chambers in $\mathrm{Ch}^+(\mathscr{C}_i(v))$ for all $i \in \mathbb{Z}_{\geqslant 1}$ so as to obtain a strictly increasing chain $\{v\} \subsetneq \mathscr{C}_1(v) \subsetneq \mathscr{C}_2(v) \subsetneq \mathscr{C}_3(v) \subsetneq \cdots$. Denote by $\mathrm{V}_i(v)$ (resp. $\mathrm{Ch}_i(v)$) the set of vertices (resp. chambers) contained in $\mathscr{C}_i(v)$ for $i \in \mathbb{Z}_{\geqslant 1}$.

Choose any chamber $\mathscr{C}$ in $\mathcal{B}(G(F), K)$. Define $\mathscr{C}^+$ to be the union of all chambers in $\mathrm{Ch}^+(\mathscr{C})$. Clearly $\mathscr{C}^+$ is compact and its interior contains the compact subset $\mathscr{C}$. Hence there exists a maximal $R_G > 0$ such that for every point $y \in \mathscr{C}$ (which may not be a vertex), the ball centered at y of radius R_G is contained in $\mathscr{C}^+$. Since the isometric action of $G(F)$ is transitive on the set of chambers, R_G does not depend on the choice of $\mathscr{C}$. Moreover the ratio $l_{\max}(\Phi)/R_G$ does not depend on the choice of metric on the building.

From the definitions we have $\mathrm{Ball}(\overline{x}_1, R_G) \subset \mathscr{C}_1(\overline{x}_1)$ and deduce recursively that

$$\mathrm{Ball}(\overline{x}_1, i R_G) \subset \mathrm{V}_i(\overline{x}_1) \subset \mathscr{C}_i(\overline{x}_1), \quad \forall i \in \mathbb{Z}_{\geqslant 1}.$$

Take $\mathcal{M}'$ to be the integer such that $\frac{\mathcal{M}}{R_G} \leqslant \mathcal{M}' < \frac{\mathcal{M}}{R_G} + 1$ so that in particular

$$\text{Ball}(\overline{x}_1, \mathcal{M}) \subset V_{\mathcal{M}'}(\overline{x}_1). \tag{7.31}$$

Let us bound $|\text{Ch}_1(v)|$ for every vertex $v \in \mathcal{B}(G(F), K)$. The stabilizer of v, denoted by $\text{Stab}(v)$, acts transitively on $\text{Ch}_1(v)$. Let $\mathscr{C} \in \text{Ch}_1(v)$. Then

$$|\text{Ch}_1(v)| = |\text{Stab}(v)/\text{Stab}(\mathscr{C})| \leqslant |G(\mathcal{O})/\text{Iw}| \leqslant |G(k_F)| \leqslant q_v^{d_G+r_G}$$

where Iw denotes an Iwahori subgroup of $G(\mathcal{O})$, which is conjugate to $\text{Stab}(\mathscr{C})$. The group $\text{Stab}(v)$ may not be hyperspecial, but the first inequality follows from the fact that the hyperspecial has the largest volume among all maximal compact subgroups [103, 3.8.2]. See the proof of Lemma 2.13 for the last inequality.

Each chamber contains $\dim A + 1$ vertices as a $\dim A$-dimensional simplex. Hence for each $i \geqslant 1$,

$$|V_i(\overline{x}_1)| \leqslant (\dim A + 1) \cdot |\text{Ch}_i(\overline{x}_1)|.$$

On the other hand,

$$\begin{aligned}
|\text{Ch}_{i+1}(\overline{x}_1)| &\leqslant \sum_{v \in V_i(\overline{x}_1)} |\text{Ch}_1(v)| \leqslant q_v^{d_G+r_G} |V_i(\overline{x}_1)| \\
&\leqslant q_v^{d_G+r_G} (\dim A + 1) \cdot |\text{Ch}_i(\overline{x}_1)|.
\end{aligned}$$

We see that $\text{Ch}_i(\overline{x}_1)| \leqslant q_v^{i(d_G+r_G)} (\dim A + 1)^{i-1}$ and thus

$$|V_{\mathcal{M}'}(\overline{x}_1)| \leqslant (\dim A + 1)^{\mathcal{M}'} q_v^{\mathcal{M}'(d_G+r_G)} \leqslant (r_G + 1)^{\mathcal{M}'} q_v^{\mathcal{M}'(d_G+r_G)}. \tag{7.32}$$

Note that

$$\begin{aligned}
\mathcal{M}' &\leqslant 1 + \frac{\mathcal{M}}{R_G} \leqslant 1 + \frac{l_{\max}(\Phi)}{R_G} C |\Delta^+| \cdot Y^{|\Phi^+|} w_G s_G \\
&\times \left(\sum_{\alpha \in \Phi} |v(1 - \alpha^{-1}(\gamma))| + Y(m_{\mathbf{G}} m_{\Xi^{\text{spl}}} + m_{\mathbf{G}} c_{\mathbf{G}} + m_{\Xi^{\text{spl}}}) \kappa \right),
\end{aligned}$$

which can be rewritten in the form

$$\mathcal{M}' \leqslant 1 + + Z_1' \kappa + Z_2' \sum_{\alpha \in \Phi} |v(1 - \alpha^{-1}(\gamma))|.$$

Since $|v(1-\alpha(\gamma))|+|v(1-\alpha^{-1}(\gamma))| \leqslant v(1-\alpha(\gamma))+v(1-\alpha^{-1}(\gamma))+2b_\Xi\kappa$ in view of (7.5), we have

$$q^{\mathcal{M}'} \leqslant q^{1+(Z_1'+b_\Xi Z_2')\kappa} D^G(\gamma)^{-Z_2'}.$$

Returning to (7.31) and (7.32),

$$|\mathrm{Ball}(\overline{x}_1, \mathcal{M})| \leqslant |V_{\mathcal{M}'}(\overline{x}_1)| \leqslant q_v^{(r_G+1)\mathcal{M}'} q_v^{\mathcal{M}'(d_G+r_G)}$$
$$\leqslant \left(q_v^{1+(Z_1'+2b_\Xi Z_2')\kappa} D^G(\gamma)^{-Z_2'} \right)^{d_G+2r_G+1}.$$

The proof of Lemma 7.9 is complete once we set Z_1 and Z_2 as follows, the point being that they

- $Z_1 := (Z_1' + 2b_\Xi Z_2')(d_G + 2r_G + 1)$,
- $Z_2 := Z_2'(d_G + 2r_G + 1)$.

Corollary 7.11 $|O_\gamma^{G(F)}(\mathbf{1}_{K\lambda(\varpi)K}, \mu_G, \mu_{I_\gamma}^{\mathrm{EP}})| \leqslant q_v^{r_G(d_G+1)} q_v^{1+Z_1\kappa} D^G(\gamma)^{-Z_2}.$

Proof Follows from (7.15), Lemmas 7.8 and 7.9.

7.4 Lemmas in the split case

This subsection plays a supporting role for the previous subsections, especially Sect. 7.3. As in Sect. 7.2 let $\mathbf{G}$ be a Chevalley group with a Borel subgroup $\mathbf{B}$ containing a split maximal torus $\mathbf{T}$, all over $\mathbb{Z}$. Let $\Xi_{\mathbb{Q}}^{\mathrm{spl}} \colon \mathbf{G} \hookrightarrow \mathrm{GL}_m$ be a closed embedding of algebraic groups over $\mathbb{Q}$. Let $\mathbb{T}$ denote the diagonal maximal torus of GL_m, $\mathbb{B}$ the upper triangular Borel subgroup of GL_m, and $\mathbb{N}$ the unipotent radical of $\mathbb{B}$.

Extend $\Xi_{\mathbb{Q}}^{\mathrm{spl}}$ to a closed embedding $\Xi^{\mathrm{spl}} \colon \mathbf{G} \hookrightarrow \mathrm{GL}_m$ defined over $\mathbb{Z}[1/R]$ for some integer R such that $\Xi^{\mathrm{spl}}(\mathbf{T})$ [resp. $\Xi^{\mathrm{spl}}(\mathbf{B})$] lies in the group of diagonal (resp. upper triangular) matrices of GL_m. To see that this is possible, find a maximal $\mathbb{Q}$-split torus $\mathbb{T}'$ of GL_m containing $\Xi_{\mathbb{Q}}^{\mathrm{spl}}(\mathbf{T})$. Choose any Borel subgroup $\mathbb{B}'$ over $\mathbb{Q}$ containing $\mathbb{T}$. Then there exists $g \in \mathrm{GL}_m(\mathbb{Q})$ such that the inner automorphism $\mathrm{Int}(g) \colon \mathrm{GL}_m \to \mathrm{GL}_m$ by $\gamma \mapsto g\gamma g^{-1}$ carries $(\mathbb{B}', \mathbb{T}')$ to $(\mathbb{B}, \mathbb{T})$. Then $\Xi_{\mathbb{Q}}^{\mathrm{spl}}$ and $\mathrm{Int}(g)$ extend over $\mathbb{Q}$ to over $\mathbb{Z}[1/R]$ for some $R \in \mathbb{Z}$, namely at the expense of inverting finitely many primes [basically those in the denominators of the functions defining $\Xi_{\mathbb{Q}}^{\mathrm{spl}}$ and $\mathrm{Int}(g)$].

Now suppose that p is a prime not diving R. Let F be a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}$ and a uniformizer ϖ. The field F is equipped with a unique discrete valuation v_F such that $v_F(\varpi) = 1$. Let $\lambda \in X_*(\mathbf{T})$. We are interested in assertions which work for F as the residue characteristic p varies.

Lemma 7.12 (resp. Corollary 7.14) below is used in Step 1 (resp. Step 2) of the proof of Lemma 7.7.

Lemma 7.12 *There exists $m_{\Xi^{\mathrm{spl}}} \in \mathbb{Z}_{>0}$ such that for every p, F and λ as above and for every semisimple $\delta \in \mathbf{G}(\mathcal{O})\lambda(\varpi)\mathbf{G}(\mathcal{O})$ (and for any choice of T_δ containing δ),*

$$\forall \alpha \in \Phi_\delta, \quad v_F(\alpha(\delta)) \in [-m_{\Xi^{\mathrm{spl}}}\|\lambda\|, m_{\Xi^{\mathrm{spl}}}\|\lambda\|].$$

Proof The argument is the same as in the proof of Lemma 2.18. The constant $m_{\Xi^{\mathrm{spl}}}$ corresponds to the constant B_5 in that lemma. To see that it is independent of p, F and λ, it suffices to examine the argument and see that the constant depends only on $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ (and the auxiliary choice of $\tilde{\alpha}$'s as in the proof of Lemma 2.17, which is fixed once and for all).

The unipotent radical of $\mathbf{B}$ is denoted $\mathbf{N}$. For F as above, let x_0 be the hyperspecial vertex on the building of $\mathbf{G}(F)$ corresponding to $\mathbf{G}(\mathcal{O})$. As usual put $\Phi^+ := \Phi^+(\mathbf{G}, \mathbf{T})$ be the set of positive roots with respect to $(\mathbf{B}, \mathbf{T})$.

Let us recall some facts about the Chevalley basis. For each $\alpha \in \Phi^+$, let U_α denote the corresponding unipotent subgroup equipped with $x_\alpha \colon \mathbb{G}_a \simeq U_\alpha$. Order the elements of Φ^+ as $\alpha_1, \ldots, \alpha_{|\Phi^+|}$ once and for all such that simple roots appear at the beginning. The multiplication map

$$\mathrm{mult} \colon U_{\alpha_1} \times \cdots \times U_{\alpha_{|\Phi^+|}} \to \mathbf{N}, \quad (u_1, \ldots, u_{|\Phi^+|}) \mapsto u_1 \ldots u_{|\Phi^+|}$$

is an isomorphism of schemes (but not as group schemes) over $\mathbb{Z}$. This can be deduced from [5, Exp XXII, 5.5.1], which deals with a Borel subgroup of a Chevalley group. In particular (since the ordering on Φ^+ is fixed) any $n \in \mathbf{N}(F)$ can be uniquely written as

$$y = x_{\alpha_1}(Y_{\alpha_1}) \ldots x_{\alpha_{|\Phi^+|}}(Y_{\alpha_{|\Phi^+|}}) \tag{7.33}$$

for unique $Y_{\alpha_i} \in \mathbb{G}_a(F) \simeq F$'s. The Chevalley commutation relation ([20, §III]) has the following form: for all $1 \leqslant i < j \leqslant |\Phi^+|$ and all $Y_{\alpha_i} \in F$'s,

$$x_{\alpha_i}(Y_{\alpha_i})x_{\alpha_j}(Y_{\alpha_j}) = x_{\alpha_j}(Y_{\alpha_j})x_{\alpha_i}(Y_{\alpha_i}) \prod_{\substack{c,d \geqslant 1 \\ \alpha_k = c\alpha_i + d\alpha_j}} x_{\alpha_k}(C_{ij}(Y_{\alpha_i})^c(Y_{\alpha_j})^d)$$

$$\tag{7.34}$$

where C_{ij} are certain integers (depending on $\mathbf{G}$) which we need not know explicitly. It suffices to know that, in the cases of F we are interested in, the

constants C_{ij} are units in $\mathcal{O}$ (cf. the assumption in the paragraph preceding Proposition 7.1).

We thank Kottwitz for explaining the proof of the following lemma.

Lemma 7.13 *Suppose that the Chevalley group* **G** *is semisimple and simply connected. Let* $\Omega \subset X^*(\mathbf{T})$ *denote the set of fundamental weights and* $\rho^\vee \in X_*(\mathbf{T})$ *the half sum of all positive coroots. Let* $\lambda \in X^*(\mathbf{T})$ *and define* $n_0(\lambda) := \max_{\omega \in \Omega} \langle \omega, \lambda \rangle$. *For every prime p, every p-adic field F, and every cocharacter* $\lambda \in X_*(\mathbf{T})$ *as above, the following is true: in terms of the decomposition* (7.33), *each* $y \in \mathbf{G}(\mathcal{O})\lambda(\varpi)\mathbf{G}(\mathcal{O}) \cap \mathbf{N}(F)$ *satisfies the inequality*

$$v_F(Y_i) \geqslant -2n_0(\lambda)\langle \alpha_i, \rho^\vee \rangle, \quad 1 \leqslant i \leqslant |\Phi^+|.$$

Proof It suffices to check that

$$\varpi^{2n_0(\lambda)\rho^\vee} y \varpi^{-2n_0(\lambda)\rho^\vee} \in \mathbf{N}(\mathcal{O}). \tag{7.35}$$

[Here we write $\varpi^{2n_0(\lambda)\rho^\vee}$ for $(\rho^\vee(\varpi))^{2n_0(\lambda)}$.] Indeed, this implies the desired inequality in the lemma since the decomposition (7.33) is defined over $\mathcal{O}$.

Let us introduce some notation. For each $\omega \in \Omega$ let V_ω denote the irreducible representation of $\mathbf{G}(F)$ of highest weight ω on an F-vector space. Write $V_\omega = \oplus_{\mu \in X^*(\mathbf{T})} V_{\omega,\mu}$ for the weight decomposition. The geometric construction of V_ω and its weight decomposition by using flag varieties gives us a natural $\mathcal{O}$-integral structures $V_\omega(\mathcal{O})$ in V_ω such that $V_\omega(\mathcal{O}) = \oplus_{\mu \in X^*(\mathbf{T})} V_{\omega,\mu}(\mathcal{O})$, where $V_{\omega,\mu}(\mathcal{O}) := V_\omega(\mathcal{O}) \cap V_{\omega,\mu}$. Note that each V_ω receives an action of $\mathbb{G}_m$ via $\mathbb{G}_m \xrightarrow{\rho^\vee} \mathbf{T} \hookrightarrow \mathbf{G}$. We may consider a coarser decomposition $V_\omega = \oplus_{i \in \mathbb{Z}} V_{\omega,i}$, where $V_{\omega,i} := \oplus_{\langle \mu, 2\rho^\vee \rangle = i} V_{\omega,\mu}$. For any $\omega \in \Omega$ and $V = V_\omega$, set $V_{\geqslant i} := \oplus_{j \geqslant i} V_j$, $V_{\geqslant i}(\mathcal{O}) := V_{\geqslant i} \cap V(\mathcal{O})$, and $V_i(\mathcal{O}) := V_i \cap V(\mathcal{O})$. Observe that $\mathbf{B}(F)$ preserves the filtration $\{V_{\geqslant i}\}_{i \in \mathbb{Z}}$ and that $\mathbf{N}(F)$ acts trivially on $V_{\geqslant i}/V_{\geqslant i+1}$.

As a preparation, suppose that $g \in \mathbf{G}(\mathcal{O})\lambda(\varpi)\mathbf{G}(\mathcal{O})$ and let us prove that $gV_\omega(\mathcal{O}) \subset \varpi^{-n_0(\lambda)} V_\omega(\mathcal{O})$ for all $\omega \in \Omega$. Since $\mathbf{G}(\mathcal{O})$ stabilizes $V_\omega(\mathcal{O})$, the latter condition is true if and only if $\lambda(\varpi)V_\omega(\mathcal{O}) \subset \varpi^{-n_0(\lambda)} V_\omega(\mathcal{O})$, which holds if and only if

$$\langle \mu, \lambda \rangle \geqslant -n_0(\lambda)$$

for all weights μ for V_ω by considering the weight decomposition. The above inequality for all weights μ is equivalent to that for the lowest weight μ for V_ω. Since $\mu = w_0\omega_\omega$ for the longest Weyl element w_0, the condition is that $\langle -w_0\omega, \lambda \rangle \leqslant n_0(\lambda)$ for all ω. This is verified by the definition of $n_0(\lambda)$ since $-w_0$ preserves the set Ω.

Now consider $\varpi^{2n_0(\lambda)\rho^\vee}(y-1)\varpi^{-2n_0(\lambda)\rho^\vee}$, where y is as in the lemma. Since $\varpi^{2\rho^\vee}$ acts on V_j as ϖ^j, we see from this and the last paragraph that for all $\omega \in \Omega$ and $i \in \mathbb{Z}$,

$$
\begin{aligned}
&(\varpi^{2n_0(\lambda)\rho^\vee}(y-1)\varpi^{-2n_0(\lambda)\rho^\vee})(V_{\omega,i}(\mathcal{O})) \\
&= (\varpi^{2n_0(\lambda)\rho^\vee}(y-1))(\varpi^{-in_0(\lambda)}V_{\omega,i}(\mathcal{O})) \\
&\subset \varpi^{2n_0(\lambda)\rho^\vee}(\varpi^{-(i+1)n_0(\lambda)}V_{\omega,\geqslant i+1}(\mathcal{O})) \subset V_{\omega,i}(\mathcal{O}).
\end{aligned}
$$

It follows that $\varpi^{2n_0(\lambda)\rho^\vee}y\varpi^{-2n_0(\lambda)\rho^\vee}$ also preserves $V_{\omega,i}(\mathcal{O})$, hence $V_\omega(\mathcal{O})$. Therefore the element belongs to $\mathcal{N}(\mathcal{O}) = \mathcal{N}(F) \cap \mathbf{G}(\mathcal{O})$, concluding the proof of (7.35). $\qquad\square$

For an arbitrary Chevalley group $\mathbf{G}$ and $\lambda \in X_*(\mathbf{T})^+$, define a nonnegative integer

$$
n_\mathbf{G}(\lambda) := \max_{\alpha \in \Phi^+} \langle \alpha, \lambda \rangle. \tag{7.36}
$$

Corollary 7.14 *Let $\mathbf{G}$ be an arbitrary Chevalley group. For every prime p, every p-adic field F, and every cocharacter $\lambda \in X_*(\mathbf{T})$, there exists a constant $m_\mathbf{G} > 0$ such that the following is true: each $y \in \mathbf{G}(\mathcal{O})\lambda(\varpi)\mathbf{G}(\mathcal{O}) \cap \mathbf{N}(F)$, uniquely decomposed as in (7.33), satisfies the inequality*

$$
v_F(Y_i) \geqslant -2m_\mathbf{G}n_\mathbf{G}(\lambda), \quad 1 \leqslant i \leqslant |\Phi^+|.
$$

Proof The corollary is immediate from the lemma if $\mathbf{G}$ is semisimple and simply connected. Indeed, define $n_1(\lambda)$ to be the maximum of $\langle \alpha, \lambda \rangle$ as α runs over Δ^+, the set of simple roots. Observe that both the sets Ω and Δ^+ are bases for $X^*(\mathbf{T})_\mathbb{Q}$. By using the change of basis matrix, it is easy to deduce from Lemma 7.13 that for some constant $c > 0$ depending only on $\mathbf{G}$, we have that

$$
v_F(Y_i) \geqslant -2cn_1(\lambda)\langle \alpha_i, \rho^\vee \rangle
$$

for all p, F, λ, and i. A fortiori the same holds with $n_\mathbf{G}(\lambda)$ in place of $n_1(\lambda)$. The proof is completed by setting $m_\mathbf{G} := c \max_{\alpha \in \Phi^+}\langle \alpha, \rho^\vee \rangle$.

It remains to extend from the simply connected case to the general case. As usual write $\mathbf{G}_{\mathrm{ad}}$ for the adjoint group of $\mathbf{G}$ and $\mathbf{G}_{\mathrm{sc}}$ for the simply connected cover of $\mathbf{G}_{\mathrm{ad}}$. The pair $(\mathbf{B}, \mathbf{T})$ induces the Borel pairs $(\mathbf{B}_{\mathrm{ad}}, \mathbf{T}_{\mathrm{ad}})$ for $\mathbf{G}_{\mathrm{ad}}$ and $(\mathbf{B}_{\mathrm{sc}}, \mathbf{T}_{\mathrm{sc}})$ for $\mathbf{G}_{\mathrm{sc}}$. Write Φ^+_{ad} and Φ^+_{sc} for the associated sets of roots. Let $\mathbf{N}_{\mathrm{ad}}$ and $\mathbf{N}_{\mathrm{sc}}$ denote the unipotent radicals of $\mathbf{B}_{\mathrm{ad}}$ and $\mathbf{B}_{\mathrm{sc}}$, respectively. Then the natural maps $\mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{ad}}$ induce isomorphisms $\mathbf{N} \simeq \mathbf{N}_{\mathrm{ad}}$ and $\mathbf{N}_{\mathrm{sc}} \simeq \mathbf{N}_{\mathrm{ad}}$ as well as set-theoretic bijections $\Phi^+ \to \Phi^+_{\mathrm{ad}}$ and $\Phi^+_{\mathrm{sc}} \to \Phi^+_{\mathrm{ad}}$.

In particular the ordering on Φ^+ induces unique orderings on Φ_{ad}^+ and Φ_{sc}^+. With respect to these orderings, the decomposition (7.33) is compatible with the maps $\mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ and $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{ad}}$. From all this it follows that the corollary for $\mathbf{G}_{\mathrm{sc}}$ implies that for $\mathbf{G}_{\mathrm{ad}}$, and then for $\mathbf{G}$. $\qquad\square$

8 Lemmas on conjugacy classes and level subgroups

This section contains several results which are useful for estimating the geometric side of Arthur's invariant trace formula in the next section.

8.1 Notation and basic setup

Let us introduce some global notation in addition to that at the start of Sect. 4.

- M_0 is a minimal F-rational Levi subgroup of G.
- A_{M_0} is the maximal split F-torus in the center of M_0.
- $\mathrm{Ram}(G) := \{v \in \mathcal{V}_F^\infty : G$ is ramified at $v\}$.
- $S \subset \mathcal{V}_F^\infty$ is a finite subset, often with a partition $S = S_0 \coprod S_1$.
- $r : {}^L G \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible continuous representation such that $r|_{\widehat{G}}$ is algebraic.
- $\Xi : G \to \mathrm{GL}_m$ is a faithful algebraic representation defined over F (or over $\mathcal{O}_F$ as explained below)
- For any $\mathbb{C}$-subspace $\mathcal{H}' \subset C_c^\infty(G(F_S))$, define

$$\mathrm{supp}\,\mathcal{H}' = \cup\,\mathrm{supp}\,\phi_S$$

  where the union is taken over $\phi_S \in \mathcal{H}'$.
- $q_S := \prod_{v \in S} q_v$ where q_v is the cardinality of the residue field at v. (Convention: $q_S = 1$ if $S = \emptyset$.)

For each finite place $v \in \mathrm{Ram}(G)$ of F, fix a special point x_v on the building of G once and for all, where x_v is required to belong to an apartment corresponding to a maximal F_v-split torus A_v containing A_{M_0}. The stabilizer K_v of x_v is a good special maximal compact subgroup of $G(F_v)$ (good in the sense of [16]). Set $K_{M,v} := K_v \cap M(F_v)$ for each F_v-rational Levi subgroup M of G containing A_v. Then $K_{M,v}$ is a good special maximal compact subgroup of $M(F_v)$.

It is worth stressing that this article treats a reductive group G without any hypothesis on G being split (or quasi-split). To do so, we would like to carefully choose an integral model of G over $\mathcal{O}_F$ for convenience and also for clarifying a notion like "level $\mathfrak{n}$ subgroups". We thank Brian Conrad for explaining us crucial steps in the proof below (especially how to proceed by using the facts from [12]).

Proposition 8.1 *The F-group G extends to a group scheme $\mathfrak{G}$ over $\mathcal{O}_F$ (thus equipped with an isomorphism $\mathfrak{G} \times_{\mathcal{O}_F} F \simeq G$) such that*

- *$\mathfrak{G} \times_{\mathcal{O}_F} \mathcal{O}_F[\frac{1}{\mathrm{Ram}(G)}]$ is a reductive group scheme (cf. [32]),*
- *$\mathfrak{G}(\mathcal{O}_v) = K_v$ for all $v \in \mathrm{Ram}(G)$ (where K_v are chosen above),*
- *there exists a faithful embedding of algebraic groups $\Xi: \mathfrak{G} \hookrightarrow \mathrm{GL}_m$ over $\mathcal{O}_F$ for some $m \geqslant 1$.*

Remark 8.2 If G is split then $\mathrm{Ram}(G)$ is empty and the above proposition is standard in the theory of Chevalley groups.

Proof For any finite place v of F, we will write $\mathcal{O}_{(v)}$ for the localization of $\mathcal{O}_F$ at v (to be distinguished from the completion $\mathcal{O}_v$). As a first step there exists an injective morphism of group schemes $\Xi_F: G \hookrightarrow GL_m$ defined over F for some $m \geqslant 1$ ([33, Prop A.2.3]. The scheme-theoretic closure $\mathfrak{G}'$ of G in $GL_{m'}$ is a smooth affine scheme over $\mathrm{Spec}\, \mathcal{O}_F[1/S]$ for a finite set S of primes of $\mathcal{O}_F$ by arguing as in the first paragraph of [32, §2]. We may assume that $S \supset \mathrm{Ram}(G)$. By [32, Prop 3.1.9.(1)], by enlarging S if necessary, we can arrange that $\mathfrak{G}'$ is reductive. For $v \in \mathrm{Ram}(G)$ we have fixed special points x_v, which give rise to the Bruhat-Tits group schemes $\widehat{\mathfrak{G}}(v)$ over $\mathcal{O}_v$. Similarly for $v \in S \backslash \mathrm{Ram}(G)$, let us choose hyperspecial points x_v so that the corresponding group schemes $\widehat{\mathfrak{G}}(v)$ over $\mathcal{O}_v$ are reductive.

According to [12, Prop D.4,p. 147] the obvious functor from the category of affine $\mathcal{O}_{(v)}$-schemes to that of triples $(X, \widehat{\mathfrak{X}}(v), f)$ where X is an affine F-scheme, $\widehat{\mathfrak{X}}(v)$ is an affine $\mathcal{O}_v$-scheme and $f: X \times_F F_v \simeq \widehat{\mathfrak{X}}(v) \times_{\mathcal{O}_v} F_v$ is an equivalence. (The notion of morphisms is obvious in each category.) Thanks to its functorial nature, the same functor defines an equivalence when restricted to group objects in each category. For $v \in \mathrm{Ram}(G)$, apply this functor to the Bruhat-Tits group scheme $\widehat{\mathfrak{G}}(v)$ over $\mathcal{O}_v$ equipped with $G \times_F F_v \simeq \widehat{\mathfrak{G}}(v) \times_{\mathcal{O}_v} F_v$ to obtain a group scheme $\mathfrak{G}(v)$ over $\mathcal{O}_{(v)}$.

An argument analogous to that on page 14 of [12] shows that the obvious functor between the following categories is an equivalence: from the category of finite-type $\mathcal{O}_F$-schemes to that of triples $(X, \{\mathfrak{X}(v)\}_{v \in S}, \{f_v\}_{v \in S})$ where X is a finite-type $\mathcal{O}_F[1/S]$-scheme, $\mathfrak{X}(v)$ is a finite-type $\mathcal{O}_{(v)}$-scheme and $f_v: X \times_{\mathcal{O}_F[1/S]} F \simeq \mathfrak{X}(v) \times_{\mathcal{O}_{(v)}} F$ is an isomorphism. Again this induces an equivalence when restricted to group objects in each category. In particular, there exists a group scheme $\mathfrak{G}$ over $\mathcal{O}_F$ with isomorphisms $\mathfrak{G} \times_{\mathcal{O}_F} \mathcal{O}_F[1/S] \simeq \mathfrak{G}'$ and $\mathfrak{G} \times_{\mathcal{O}_F} \mathcal{O}_{(v)} \simeq \mathfrak{G}(v)$ for $v \in S$ which are compatible with the isomorphisms between $\mathfrak{G}'$ and $\mathfrak{G}(v)$ over F. By construction $\mathfrak{G}$ satisfies the first two properties of the proposition.

We will be done if $\Xi_F \colon G \hookrightarrow GL_m$ over F extends to an embedding of group schemes over $\mathcal{O}_F$. It is evident from the construction of $\mathfrak{G}'$ that Ξ_F extends to $\Xi' \colon G \hookrightarrow GL_m$ over $\mathcal{O}_F[1/S]$. For each $v \in S$, Ξ_F extends to $\Xi(v) \colon \mathfrak{G}(v) \hookrightarrow GL_m$ over $\mathcal{O}_v$ thanks to [17, Prop 1.7.6], which can be defined over $\mathcal{O}_{(v)}$ using the first of the above equivalences. Then the second equivalence allows us to glue Ξ' and $\{\Xi(v)\}_{v \in S}$ to produce an $\mathcal{O}_F$-embedding $\Xi \colon G \hookrightarrow GL_m$. $\qquad\square$

For each finite place $v \notin \mathrm{Ram}(G)$, $\mathfrak{G}$ defines a reductive group scheme over $\mathcal{O}_v$, so $K_v := \mathfrak{G}(\mathcal{O}_v)$ is a hyperspecial subgroup of $G(F_v)$. Fix a maximal F_v-split torus A_v of G which contains A_{M_0} such that the hyperspecial point for K_v belongs to the apartment of A_v. For each Levi subgroup M of G whose center is contained in A_v, define a hyperspecial subgroup $K_{M,v} := K_v \cap M(F_v)$ of $M(F_v)$. At such a $v \notin \mathrm{Ram}(G)$ define $\mathcal{H}^{\mathrm{ur}}(G(F_v))$ (resp. $\mathcal{H}^{\mathrm{ur}}(M(F_v))$). The constant term (Sect. 6.1) of a function in $C_c^\infty(G(F_v))$ (resp. $C_c^\infty(M(F_v))$) will be taken relative to K_v (resp. $K_{M,v}$). When $P = MN$ is a Levi decomposition, we have Haar measures on K_v, $M(F_v)$ and $N(F_v)$ such that the product measure equals μ_v^{can} on $G(F_v)$ (cf. Sect. 6.1) and the Haar measure on $M(F_v)$ is the canonical measure of Sect. 6.6. In particular when G is unramified at v,

$$\mathrm{vol}(K_v \cap N(F_v)) = 1 \tag{8.1}$$

with respect to the measure on $N(F_v)$.

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ and v a finite place of F. Let $v(\mathfrak{n}) \in \mathbb{Z}_{\geqslant 0}$ be the integer determined by $\mathfrak{n}\mathcal{O}_v = \varpi_v^{v(\mathfrak{n})}\mathcal{O}_v$. Define $K_v(\varpi_v^s)$ to be the Moy-Prasad subgroup $G(F_v)_{x_v,s}$ of $G(F_v)$ by using Yu's minimal congruent filtration as in [108] (which is slightly different from the original definition of Moy and Prasad). Yu has shown that $G(F_v)_{x_v,s} = \ker(\mathfrak{G}(\mathcal{O}_v) \to \mathfrak{G}(\mathcal{O}_v/\varpi_v^s))$ in [108, Cor 8.8]. Set

$$K^{S,\infty}(\mathfrak{n}) := \prod_{v \notin S \cup S_\infty} \ker(\mathfrak{G}(\mathcal{O}_v) \to \mathfrak{G}(\mathcal{O}_v/\mathfrak{n})) = \prod_{v \notin S \cup S_\infty} K_v\left(\varpi_v^{v(\mathfrak{n})}\right),$$

to be considered the level $\mathfrak{n}$-subgroup of $G(\mathbb{A}^{S,\infty})$.

Fix a maximal torus T_0 of G over $\overline{F}$ and an $\mathbb{R}$-basis $\mathcal{B}_0$ of $X_*(T_0)_\mathbb{R}$, which induces a function $\|\cdot\|_{\mathcal{B}_0,G} \colon X_*(T_0)_\mathbb{R} \to \mathbb{R}_{\geqslant 0}$ as in Sect. 2.5. For any other maximal torus T, there is an inner automorphism of G inducing $T_0 \simeq T$, so $X_*(T)_\mathbb{R}$ has an $\mathbb{R}$-basis $\mathcal{B}$ induced from $\mathcal{B}_0$, well defined up to the action by $\Omega = \Omega(G, T)$. Therefore $\|\cdot\|_{\mathcal{B},G} \colon X_*(T)_\mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is defined without ambiguity. As it depends only on the initial choice of $\mathcal{B}_0$ (and T_0), let us write $\|\cdot\|$ for $\|\cdot\|_{\mathcal{B},G}$ when there is no danger of confusion.

Let v be a finite place of G, and T_v a maximal torus of $G \times_F \overline{F}_v$ (which may or may not be defined over F_v). Then $\|\cdot\| : X_*(T_v)_{\mathbb{R}} \to \mathbb{R}_{\geqslant 0}$ is defined without ambiguity via $T_v \simeq T_0 \times_{\overline{F}} \overline{F}_v$ by a similar consideration as above. Now assume that G is unramified at v. For any maximal F_v-split torus $A \subset G$ and a maximal torus T containing A over F_v, the function $\|\cdot\|_{\mathcal{B}_0}$ is well defined on $X_*(T)_{\mathbb{R}}$ (resp. $X_*(A)_{\mathbb{R}}$) and invariant under the Weyl group Ω (resp. Ω_F). Hence for every v where G is unramified, the Satake isomorphism allows us to define $\mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$ as well as $\mathcal{H}^{\mathrm{ur}}(M(F_v))^{\leqslant \kappa}$ for every Levi subgroup M of G over F_v. When G is unramified at S, we put $\mathcal{H}^{\mathrm{ur}}(G(F_S))^{\leqslant \kappa} := \otimes_{v \in S} \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$ and define $\mathcal{H}^{\mathrm{ur}}(M(F_S))^{\leqslant \kappa}$ similarly.

For the group GL_m with any $m \geqslant 1$, we use the diagonal torus and the standard basis to define $\|\cdot\|_{\mathrm{GL}_m}$ on the cocharacter groups of maximal tori of GL_m (cf. Sect. 2.4). For $\Xi : G \hookrightarrow \mathrm{GL}_m$ introduced above, define

$$B_\Xi := \max_{e \in \mathcal{B}_0} \|\Xi(e)\|_{\mathrm{GL}_m}. \tag{8.2}$$

8.2 z-Extensions

A surjective morphism $\alpha : H \to G$ of connected reductive groups over F is said to be a z-extension if the following three conditions are satisfied: H^{der} is simply connected, $\ker \alpha \subset Z(H)$, and $\ker \alpha$ is isomorphic to a finite product $\prod \mathrm{Res}_{F_i/F} \mathrm{GL}_1$ for finite extensions F_i of F. Writing $Z := \ker \alpha$, we often represent such an extension by an exact sequence of F-groups $1 \to Z \to H \to G \to 1$. By the third condition and Hilbert 90, $\alpha : H(F) \to G(F)$ is surjective.

Lemma 8.3 *For any G, a z-extension $\alpha : H \to G$ exists. Moreover, if G is unramified outside a finite set S, where $S_\infty \subset S \subset \mathcal{V}_F$, then H can be chosen to be unramified outside S.*

Proof It is shown in [76, Prop 3.1] that a z-extension exists and that if G splits over a finite Galois extension E of F then H can be chosen to split over E. By the assumption on G, it is possible to find such an E which is unramified outside S. Since the preimage of a Borel subgroup of G in H is a Borel subgroup of H, we see that H is quasi-split outside S. □

8.3 Rational conjugacy classes intersecting a small open compact subgroup

Throughout this subsection $S = S_0 \coprod S_1$ is a finite subset of $\mathcal{V}_F^\infty$ and it is assumed that $S_0 \supset \mathrm{Ram}(G)$. Fix compact subgroups U_{S_0} and U_∞ of $G(F_{S_0})$ and $G(F \otimes_{\mathbb{Q}} \mathbb{R})$, respectively. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ as before, now assumed to be coprime to S, with absolute norm $\mathbb{N}(\mathfrak{n}) \in \mathbb{Z}_{\geqslant 1}$.

Lemma 8.4 *Let* $U_{S_1} := \operatorname{supp} \mathcal{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leqslant \kappa}$. *There exists* $c_\Xi > 0$ *independent of* S, κ *and* $\mathfrak{n}$ *(but depending on* G, Ξ, U_{S_0} *and* U_∞*) such that for all* $\mathfrak{n}$ *satisfying*

$$\mathbb{N}(\mathfrak{n}) \geqslant c_\Xi q_{S_1}^{B_\Xi m \kappa},$$

the following holds: if $\gamma \in G(F)$ *and* $x^{-1} \gamma x \in K^{S,\infty}(\mathfrak{n}) U_{S_0} U_{S_1} U_\infty$ *for some* $x \in G(\mathbb{A}_F)$ *then* γ *is unipotent.*

Proof Let $\gamma' = x^{-1} \gamma x$. We keep using the embedding $\Xi \colon \mathfrak{G} \hookrightarrow \mathrm{GL}_m$ over $\mathcal{O}_F$ of Proposition 8.1. (For the lemma, an embedding away from the primes in S_0 or dividing $\mathfrak{n}$ is enough.) At each finite place $v \notin S_0$ and $v \nmid \mathfrak{n}$, Lemma 2.17 allows us to find $\Xi'_v \colon \mathfrak{G} \hookrightarrow \mathrm{GL}_m$ over $\mathcal{O}_v$ which is $\mathrm{GL}_m(\mathcal{O}_v)$-conjugate to $\Xi \times_{\mathcal{O}_F} F_v$ such that Ξ'_v sends A_v into the diagonal torus of GL_m.

Write $\det(\Xi(\gamma) - (1 - X)) = X^m + a_{m-1}(\gamma) X^{m-1} + \cdots + a_0(\gamma)$, where $a_i(\gamma) \in F$ for $0 \leqslant i \leqslant m - 1$. Our goal is to show that $a_i(\gamma) = 0$ for all i. To this end, assuming $a_i(\gamma) \neq 0$ for some fixed i, we will estimate $|a_i(\gamma)|_v$ at each place v and draw a contradiction.

First consider $v \in S_1$. We claim that

$$v(a_i(\gamma)) \geqslant -B_\Xi m \kappa$$

for every γ that is conjugate to an element of $\operatorname{supp} \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$. To prove the claim we examine the eigenvalues of $\Xi'_v(\gamma')$, which is conjugate to γ. We know γ' belongs to $\operatorname{supp} \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$, so $\Xi'_v(\gamma') \in \mathrm{GL}_m(\mathcal{O}_v) \Xi'_v(\mu(\varpi_v)) \mathrm{GL}_m(\mathcal{O}_v)$ for some $\mu \in X_*(A_v)$ with $\|\mu\| \leqslant \kappa$. Then $\|\Xi'_v(\mu)\|_{\mathrm{GL}_m} \leqslant B_\Xi \kappa$. [A priori this is true for $B_{\Xi'_v}$ defined as in (8.2), but $B_{\Xi'_v} = B_\Xi$ as Ξ'_v and Ξ are conjugate.] Let $k_1, k_2 \in \mathrm{GL}_m(\mathcal{O}_v)$ be such that $\Xi'_v(\gamma') = k_1 \Xi'_v(\mu(\varpi_v)) k_2$. Lemma 2.15 shows that every eigenvalue λ of $\Xi'_v(\mu(\varpi_v)) k_2 k_1$ [equivalently of $\Xi'_v(\gamma')$] satisfies $v(\lambda) \geqslant -B_\Xi \kappa$. If $\lambda \neq 1$, we must have $v(1 - \lambda) \geqslant -B_\Xi \kappa$. This shows that $v(a_i(\gamma)) \geqslant -B_\Xi i \kappa$ for any i such that $a_i(\gamma) \neq 0$. Hence the claim is true.

At infinity, by the compactness of U_∞, there exists $c_\Xi > 0$ such that

$$|a_i(\gamma)|_\infty < c_\Xi$$

whenever a conjugate of $\gamma \in G_\infty$ belongs to U_∞.

Now suppose that v is a finite place such that $v \notin S_1$ and $v \nmid \mathfrak{n}$. (This includes $v \in S_0$.) Then a conjugate of $\Xi(\gamma)$ lies in an open compact subgroup of $\mathrm{GL}_m(F_v)$. Therefore the eigenvalues of $\Xi(\gamma)$ are in $\mathcal{O}_v$ and

$$|a_i(\gamma)|_v \leqslant 1.$$

Finally at $v|\mathfrak{n}$, we have $\Xi(x^{-1}\gamma x)-1 \in \ker(\mathrm{GL}_m(\mathcal{O}_v) \to \mathrm{GL}_m(\mathcal{O}_v/\varpi_v^{v(\mathfrak{n})}))$. Therefore

$$|a_i(\gamma)|_v = |a_i(x^{-1}\gamma x)|_v \leqslant (|\mathfrak{n}|_v)^{m-i}.$$

Now assume that $\mathbb{N}(\mathfrak{n}) \geqslant c_\Xi q_{S_1}^{-B_\Xi m\kappa}$. We assert that $a_i(\gamma) = 0$ for all i. Indeed, if $a_i(\gamma) \neq 0$ for some i then the above inequalities imply that

$$1 = \prod_v |a_i(\gamma)|_v < \left(\prod_{v \in S_1} q_v^{-B_\Xi m\kappa}\right) c_\Xi \prod_{v|\mathfrak{n}} |\mathfrak{n}|_v^{m-i} \leqslant q_{S_1}^{-B_\Xi m\kappa} c_\Xi \mathbb{N}(\mathfrak{n})^{-1} \leqslant 1$$

which is clearly a contradiction. The proof of lemma is finished. $\qquad\square$

8.4 Bounding the number of rational conjugacy classes

We begin with a basic lemma, which is a quantitative version of the fact that F^r is discrete in $\mathbb{A}_F^r$.

Lemma 8.5 *Suppose that $\{\delta_v \in \mathbb{R}_{>0}\}_{v \in V_F}$ satisfies the following: $\delta_v = 1$ for all but finitely many v and $\prod_v \delta_v < 2^{-|S_\infty|}$. Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}_F^r$. Consider the following compact neighborhood of α*

$$\mathcal{B}(\alpha, \delta) := \{(x_1, \ldots, x_r) \in \mathbb{A}_F^r : |x_{i,v} - \alpha_{i,v}|_v \leqslant \delta_v, \quad \forall v, \forall 1 \leqslant i \leqslant r\}.$$

Then $\mathcal{B}(\alpha, \delta) \cap F^r$ has at most one element.

Proof Suppose $\beta = (\beta_i)_{i=1}^r$, $\gamma = (\gamma_i)_{i=1}^r \in \mathcal{B}(\alpha, \delta) \cap F^r$. By triangular inequalities,

$$|\beta_{i,v} - \gamma_{i,v}|_v \leqslant \begin{cases} \delta_v, & v \nmid \infty, \\ 2\delta_v, & v|\infty \end{cases}$$

for each i. Hence $\prod_v |\beta_{i,v} - \gamma_{i,v}|_v < 1$. Since $\beta_i, \gamma_i \in F$, the product formula forces $\beta_i = \gamma_i$. Therefore $\beta = \gamma$. $\qquad\square$

The next lemma measures the difference between $G(F)$-conjugacy and $G(\mathbb{A}_F)$-conjugacy.

Lemma 8.6 *Let X_G (resp. $\mathscr{X}_G$) be the set of semisimple $G(F)$-(resp. $G(\mathbb{A}_F)$-)conjugacy classes in $G(F)$. For any $[\gamma] \in \mathscr{X}_G$, there exist at most $(w_G s_G)^{r_G+1}$ elements in X_G mapping to $[\gamma]$ under the natural surjection $X_G \to \mathscr{X}_G$.*

Proof Let $[\gamma] \in \mathscr{X}_G$ be an element defined by a semisimple $\gamma \in G(F)$. Denote by X_γ the preimage of $[\gamma]$ in X_G. There is a natural bijection

$$X_\gamma \leftrightarrow \ker(\ker^1(F, I_\gamma) \to \ker^1(F, G)).$$

Since $|\ker^1(F, I_\gamma)| = |\ker^1(F, Z(\widehat{I_\gamma}))|$ by [62, §4.2], we have $|X_\gamma| \leqslant |\ker^1(F, Z(\widehat{I_\gamma}))|$.

Let T be a maximal torus in I_γ defined over F. Lemma 6.5 tells us that T becomes split over a finite extension E/F such that $[E:F] \leqslant w_G s_G$. Then $\mathrm{Gal}(\overline{F}/E)$ acts trivially on $\widehat{T}$ and $Z(\widehat{I_\gamma})$. The group $\ker^1(E, Z(\widehat{I_\gamma}))$ consists of continuous homomorphisms $\mathrm{Gal}(\overline{F}/E) \to Z(\widehat{I_\gamma})$ which are trivial on all local Galois groups. Hence $\ker^1(E, Z(\widehat{I_\gamma}))$ is trivial. This and the inflation–restriction sequence show that $\ker^1(F, Z(\widehat{I_\gamma}))$ is the subset of locally trivial elements in $H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma}))$, where we have written $\Gamma_{E/F}$ for $\mathrm{Gal}(E/F)$. In particular,

$$|X_\gamma| \leqslant |H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma}))|.$$

Let $d := |\mathrm{Gal}(E/F)|$ and denote by $[d]$ the d-torsion subgroup. The long exact sequence arising from $0 \to Z(\widehat{I_\gamma})[d] \to Z(\widehat{I_\gamma}) \xrightarrow{d} d(Z(\widehat{I_\gamma})) \to 0$ gives rise to an exact sequence

$$H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma})[d]) \to H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma})) = H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma}))[d] \to 0.$$

Let $\boldsymbol{\mu}_d$ denote the order d cyclic subgroup of $\mathbb{C}^\times$. Then $Z(\widehat{I_\gamma})[d] \hookrightarrow \widehat{T}[d] \simeq (\boldsymbol{\mu}_d)^{\dim T}$. Hence

$$|X_\gamma| \leqslant |H^1(\Gamma_{E/F}, Z(\widehat{I_\gamma})[d])| \leqslant |\Gamma_{E/F}| \cdot |Z(\widehat{I_\gamma})[d]|$$
$$\leqslant d \cdot (d)^{\dim T} \leqslant (w_G s_G)^{\dim T + 1}.$$

$\square$

For the proposition below, we fix a finite subset $S_0 \subset \mathcal{V}_F^\infty$ containing $\mathrm{Ram}(G)$. Also fix compact subsets $U_{S_0} \subset G(F_{S_0})$ and $U_\infty \subset G(F_\infty)$. As usual we will write S for $S_0 \coprod S_1$.

Proposition 8.7 *Let* $\kappa \in \mathbb{Z}_{\geqslant 0}$. *Let* $S_1 \subset \mathcal{V}_F^\infty \backslash S_0$ *be a finite subset such that* G *is unramified at all* $v \in S_1$. *Set* $U_{S_1} := \mathrm{supp}\, \mathcal{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leqslant \kappa}$, $U^{S,\infty} := \prod_{v \notin S \cup S_\infty} K_v$ *and* $U := U_{S_0} U_{S_1} U^{S,\infty} U_\infty$. *Define* $\mathscr{Y}_G$ *to be the set of semisimple* $G(\mathbb{A}_F)$-*conjugacy classes of* $\gamma \in G(F)$ *which meet* U. *Then there exist constants* $A_3, B_3 > 0$ *such that for all* S_1 *and* κ *as above,*

$$|\mathscr{Y}_G| = O\left(q_{S_1}^{A_3 + B_3 \kappa}\right)$$

[*In other words, the implicit constant for $O(\cdot)$ is independent of S_1 and κ.*]

Remark 8.8 By combining the proposition with Lemma 8.4 we can deduce the following. Under the same assumption but with $U := K^{S,\infty}(\mathfrak{n})U_{S_0}U_{S_1}U_\infty$ we have

$$|\mathscr{Y}_G| = 1 + O\left(q_{S_1}^{A+B\kappa}\mathbb{N}(\mathfrak{n})^{-C}\right).$$

for some constants $A, B, C > 0$.

Proof Our argument will be a quantitative refinement of the proof of [63, Prop 8.2].

Step I. When G^{der} is simply connected.

Choose a smooth reductive integral model $\mathfrak{G}$ over $\mathcal{O}_F[\frac{1}{S_0}]$ for G and an embedding of algebraic groups $\Xi : \mathfrak{G} \to \mathrm{GL}_m$ defined over $\mathcal{O}_F[\frac{1}{S_0}]$ as in Proposition 8.1. Consider

$$G(\mathbb{A}_F) \xrightarrow{\Xi} \mathrm{GL}_m(\mathbb{A}_F) \to \mathbb{A}_F^m \tag{8.3}$$

where the latter map assigns the coefficients of the characteristic polynomial, and call the composite map Ξ'. Set $U' := \Xi'(U)$. Then $|U' \cap F^m| < \infty$ since it is discrete and compact. We would like to estimate the cardinality.

Fix $\{\delta_v\}$ such that $\delta_v = 1$ for all finite places v and $\prod_v \delta_v < 2^{-|S_\infty|}$ so that the assumption of Lemma 8.5 is satisfied. We will write $\mathcal{B}_v(x, r)$ for the ball with center x and radius r in F_v. Since Ξ is defined over $\mathcal{O}_F[\frac{1}{S_0}]$, clearly $\Xi(U^{S,\infty}) \subset \mathrm{GL}_m(\widehat{\mathcal{O}}_F^{S,\infty})$. Thus

$$\Xi'(U^{S,\infty}) \subset \left(\widehat{\mathcal{O}}_F^{S,\infty}\right)^m = \prod_{v \notin S \cup S_\infty} \mathcal{B}_v(0, 1).$$

Set $J^{S,\infty} := \{0\} \subset (\mathbb{A}_F^{S,\infty})^m$. Similarly as above, $\Xi'(U_{S_1}) \subset (\mathcal{O}_{F,S_1})^m$. By the compactness of U_{S_0} and U_∞, there exist finite subsets $J_{S_0} \subset F_{S_0}$ and $J_\infty \subset F_\infty$ such that

$$\Xi'(U_{S_0}) \subset \bigcup_{\beta_{S_0} \in J_{S_0}} \left(\prod_{v \in S_0} \mathcal{B}_v(\beta_v, 1)\right), \quad \Xi'(U_\infty) \subset \bigcup_{\beta_\infty \in J_\infty} \left(\prod_{v \in S_\infty} \mathcal{B}_v(\beta_v, \delta_v)\right).$$

Now we treat the places contained in S_1. Let T be a maximal torus of G over $\overline{F}$. Since the image of the composite map $T_{\overline{F}} \hookrightarrow G_{\overline{F}} \xrightarrow{\Xi} (\mathrm{GL}_m)_{\overline{F}}$ is contained in a maximal torus of GL_m, we can choose $g = (g_{ij})_{i,j=1}^m \in \mathrm{GL}_m(\overline{F})$ such that

$g \Xi(T_{\overline{F}})g^{-1}$ sits in the diagonal maximal torus $\mathbb{T}$ of GL_d. Fix the choice of T and g once and for all (independently of S_1 and κ) until the end of Step I. Set $v_{\min}(g) := \min_{i,j} v(g_{ij})$ and $v_{\max}(g) := \max_{i,j} v(g_{ij})$. There exists $B_6 > 0$ such that for any $\mu \in X_*(T)$ with $\|\mu\| \leqslant \kappa$, the element $g\Xi(\mu)g^{-1} \in X_*(\mathbb{T})$ satisfies $\|g\Xi(\mu)g^{-1}\| \leqslant B_6\kappa$. Let $\gamma_{S_1} = (\gamma_v)_{v \in S_1} \in U_{S_1}$. Each γ_v has the form $\gamma_v = k_1\mu(\varpi_v)k_2$ for some $\|\mu\| \leqslant \kappa$ and $k_1, k_2 \in G(\mathcal{O}_v)$. Since $\Xi(G(\mathcal{O}_v)) \subset \mathrm{GL}_m(\mathcal{O}_v)$, we see that $\Xi(\gamma_v)$ is conjugate to $\Xi(\mu(\varpi_v))k'$ in $\mathrm{GL}_m(F_v)$ for some $k' \in \mathrm{GL}_m(\mathcal{O}_v)$. Applying Lemma 2.15 to $(g\Xi(\mu(\varpi_v))g^{-1})(gk'g^{-1})$ with $u = gk'g^{-1}$ and noting that $v_{\min}(u) \geqslant v_{\min}(g) + v_{\min}(g^{-1})$, we conclude that each eigenvalue λ of $\Xi(\gamma_v)$ satisfies

$$v(\lambda) \geqslant -B_6\kappa + v_{\min}(g) + v_{\min}(g^{-1}).$$

Therefore the coefficients of its characteristic polynomial lie in $\varpi_v^{-m(B_6\kappa + A_4)}$ $\mathcal{O}_v$, where we have set $A_4 := -(v_{\min}(g) + v_{\min}(g^{-1})) \geqslant 0$. To put things together, we see that

$$\Xi'(U_{S_1}) \subset \prod_{v \in S_1} \left(\varpi_v^{-m(B_6\kappa + A_4)}\mathcal{O}_v\right)^m.$$

[A fortiori $\Xi'(U_{S_1}) \subset \prod_{v \in S_1} \prod_{i=1}^m \varpi_v^{-i(B_6\kappa + A_4)}\mathcal{O}_v$ holds as well.] The right hand side is equal to the union of $\prod_{v \in S_1} \mathcal{B}_v(\beta_v, 1)$, as $\{\beta_v\}_{v \in S_1}$ runs over $J_{S_1} = \prod_{v \in S_1} J_v$, where J_v is a set of representatives for $(\varpi_v^{-m(B_6\kappa + A_4)}\mathcal{O}_v/\mathcal{O}_v)^m$. Notice that $|J_{S_1}| = q_{S_1}^{m^2(B_6\kappa + A_4)}$. Finally, we see that

$$U' = \Xi'(U) \subset \bigcup_{\beta \in J} \mathcal{B}(\beta, \delta)$$

where $J = J_{S_0} \times J_{S_1} \times J^{S,\infty} \times J_\infty$. Lemma 8.5 implies that

$$|U' \cap F^m| \leqslant |J| = |J_{S_0}| \cdot |J_{S_1}| \cdot |J_\infty| = O\left(q_{S_1}^{m^2(B_6\kappa + A_4)}\right),$$

since $|J_{S_0}| \cdot |J_\infty|$ is a constant independent of κ and S_1.

For each $\beta \in U' \cap F^m$, we claim that there are at most $m!$ semisimple $G(\overline{F})$-conjugacy classes in $G(\overline{F})$ which map to β via $G(\overline{F}) \to \mathrm{GL}_m(\overline{F}) \to \overline{F}^m$, the map analogous to (8.3). Let us verify the claim. Let T' and $\mathbb{T}'$ be maximal tori in G and GL_m over $\overline{F}$, respectively, such that $\Xi(T') \subset \mathbb{T}'$. Then the set of semisimple conjugacy classes in $G(\overline{F})$ [resp. $\mathrm{GL}_m(\overline{F})$] is in a natural bijection with $T'(\overline{F})/\Omega$ [resp. $\mathbb{T}'(\overline{F})/\Omega_{\mathrm{GL}_m}$]. The map $\Xi|_{T'} : T' \to \mathbb{T}'$ induces a map $T'(\overline{F})/\Omega \to \mathbb{T}'(\overline{F})/\Omega_{\mathrm{GL}_m}$. Each fiber of the latter map has cardinality at most $m!$, hence the claim follows.

Fix $\beta \in U' \cap F^m$. We also fix $\gamma \in G(F)$ such that $\Xi'(\gamma) = \beta$. We assume the existence of such a γ; otherwise our final bound will only improve. We would like to bound the number of $G(\mathbb{A}_F)$-conjugacy classes in $G(F)$ which meet U and $G(\overline{F})$-conjugate to γ. Let Φ_γ denote the set of roots over $\overline{F}$ for any choice of maximal torus T_γ in G. Define $V'(\gamma)$ to be the set of places v of F such that $v \notin S \cup S_\infty$ and $\alpha(\gamma) \neq 1$ and $|1 - \alpha(\gamma)|_v < 1$ for at least one $\alpha \in \Phi_\gamma$. Since T_γ splits over an extension of F_v of degree at most $w_G s_G$ (Lemma 6.5), $1 - \alpha(\gamma)$ belongs to such an extension. Hence the inequality $|1 - \alpha(\gamma)|_v < 1$ implies that

$$|1 - \alpha(\gamma)|_v \leqslant q_v^{-\frac{1}{w_G s_G}} \leqslant 2^{-\frac{1}{w_G s_G}}.$$

Put $V(\gamma) := V'(\gamma) \cup S \cup S_\infty$. Clearly $|V(\gamma)| < \infty$. Moreover we claim that $|V(\gamma)| = O(1)$ (bounded independently of γ). Set

$$C_{S_0} := \sup_{\gamma \in U_{S_0} U_\infty} \left(\prod_{\alpha \in \Phi_\gamma} |1 - \alpha(\gamma)|_{S_0} |1 - \alpha(\gamma)|_{S_\infty} \right),$$

which is finite since $U_{S_0} U_\infty$ is compact. Then

$$1 = \prod_v \prod_{\alpha \in \Phi_\gamma} |1 - \alpha(\gamma)|_v = \left(\prod_{v \in V(\gamma)} \prod_{\alpha \in \Phi_\gamma} |1 - \alpha(\gamma)|_v \right)$$
$$\leqslant C_{S_0} \prod_{v \in V'(\gamma)} 2^{-\frac{1}{w_G s_G}} \leqslant C_{S_0} 2^{-\frac{|V'(\gamma)|}{w_G s_G}}.$$

Thus $|V'(\gamma)| = O(1)$ and also $|V(\gamma)| = O(1)$.

We are ready to bound the number of $G(\mathbb{A}_F)$-conjugacy classes in $G(F)$ which meet U and are $G(\overline{F})$-conjugate to γ. For any such conjugacy class of $\gamma' \in G(F)$, [63, Prop 7.1] shows that γ' is $G(\mathcal{O}_v)$-conjugate to γ whenever $v \notin V(\gamma)$. Hence the number of $G(\mathbb{A}_F)$-conjugacy classes of such γ' is at most $u_G^{|V(\gamma)|}$, where u_G is the constant of Lemma 8.11 below.

Putting all this together, we conclude that $|\mathscr{Y}_G| = O(q_{S_1}^{m^2(B_7 \kappa + A_5)})$ as S_1 and κ vary. The lemma is proved in this case.
Step II: general case.

Now we drop the assumption that G^{der} is simply connected. By Lemma 8.3, choose a z-extension

$$1 \to Z \to H \xrightarrow{\alpha} G \to 1.$$

Our plan is to argue as on page 391 of [63] with a specific choice of $\mathscr{C}_H$ and $\mathscr{C}_Z$ below (denoted C_H and C_Z by Kottwitz). In order to explain this choice, we need some preparation. If $v \notin S \cup S_\infty$, choose $K_{H,v}$ to be a hyperspecial subgroup of $H(F_v)$ such that $\alpha(K_{H,v}) = K_v$. (Such a $K_{H,v}$ exists by the argument of [63, p. 386].) We can find compact sets $U_{H,S_0} \subset H(F_{S_0})$ and $U_{H,\infty}$ of $H(F_\infty)$ such that $\alpha(U_{H,S_0}) = U_{S_0}$ and $\alpha(U_{H,\infty}) = U_\infty$. Moreover, in Lemma 8.9 below we prove the following: □

Claim *There exists a constant $\beta > 0$ independent of κ and S_1 with the following property: for any $\kappa \in \mathbb{Z}_{\geqslant 0}$, we can choose an open compact subset $U_{H,S_1} \subset \operatorname{supp} \mathcal{H}^{\mathrm{ur}}(H)^{\leqslant \beta \kappa}$ such that $\alpha(U_{H,S_1}) = U_{S_1}$.*

Now choose U_{Z,S_1} to be the kernel of $\alpha \colon U_{H,S_1} \to U_{S_1}$, which is compact and open in $Z(F_{S_1})$. Then choose a compact set $U_Z^{S_1}$ such that $U_{Z,S_1} U_Z^{S_1} Z(F) = Z(\mathbb{A})^1$. (This is possible since $Z(F)\backslash Z(\mathbb{A})^1$ is compact.[8]) Set

$$U_H := \left(\prod_{v \notin S \cup S_\infty} K_{H,v} \right) U_{H,S_0} U_{H,S_1} U_{H,\infty}, \quad U_Z := U_{Z,S_1} U_Z^{S_1}$$

and set $U_H^1 := U_H \cap H(\mathbb{A}_F)^1$, $U_Z^1 := U_Z \cap Z(\mathbb{A}_F)^1$. Let $\mathscr{Y}_H$ be defined as in the statement of Proposition 8.7 (with H and U_H^1 replacing G and U). Then page 391 of [63] shows that the natural map $\mathscr{Y}_H \to \mathscr{Y}$ is a surjection, in particular $|\mathscr{Y}| \leqslant |\mathscr{Y}_H|$. Since H^{der} is simply connected, the earlier proof implies that $|\mathscr{Y}_H| = O(q_{S_1}^{B_7 \beta \kappa + A_5})$ for some $B_7, A_5 > 0$. (To be precise, apply the earlier proof after enlarging U_{H,S_1} to $\operatorname{supp} \mathcal{H}^{\mathrm{ur}}(H)^{\leqslant \beta \kappa}$ in the definition of U_H. Such a replacement only increases $|\mathscr{Y}_H|$, so the bound on $|\mathscr{Y}_H|$ remains valid.) The proposition follows.

We have postponed the proof of a claim in the proof of Step II above, which we justify now. Simple as the lemma may seem, we apologize for not having found a simple proof.

Lemma 8.9 *Claim 8.4 above is true.*

Proof As the claim is concerned with places in S_1, which (may vary but) are contained in the set of places where G is unramified (thus quasi-split), we may assume that H and G are quasi-split over F by replacing H and G with their quasi-split inner forms.

---

[8] Choose $U_Z^{S_1}$ to be any open compact subgroup. Then $U_{Z,S_1} U_Z^{S_1} Z(F)$ has a finite index in $Z(\mathbb{A})^1$ by compactness. Then enlarge $U_Z^{S_1}$ without breaking compactness such that the equality holds.

Choose a Borel subgroup B_H of H, whose image $B = \alpha(B_H)$ is a Borel subgroup of G. The maximal torus $T_H \subset B_H$ maps to a maximal torus $T \subset B$ and there is a short exact sequence

$$1 \to Z \to T_H \xrightarrow{\alpha} T \to 1.$$

The action of $\mathrm{Gal}(\overline{F}/F)$ on $X_*(T_H)$ factors through a finite quotient. Let Σ be the quotient of $\mathrm{Gal}(\overline{F}/F)$ which acts faithfully on $X_*(T_H)$. If $v \notin S_0$ then G is unramified at v, so the geometric Frobenius at v defines a well-defined conjugacy class, say $\mathscr{C}_v$, in Σ. Let $A_{H,v}$ (resp. A_v) be the maximal split torus in T_H (resp. T) over F_v. Then $A_{H,v} \hookrightarrow T_H$ and $A_v \hookrightarrow T$ induce $X_*(A_{H,v}) \simeq X_*(T_H)^{\mathscr{C}_v}$ and $X_*(A_v) \simeq X_*(T)^{\mathscr{C}_v}$. We claim that $X_*(T_H) \to X_*(T)$ induces a surjective map $X_*(A_{H,v}) \to X_*(A_v)$.

$$
\begin{array}{ccccc}
X_*(T_H) & \longleftarrow & X_*(T_H)^{\mathscr{C}_v} & \overset{\sim}{\longleftarrow} & X_*(A_{H,v}) \simeq T_H(F_v)/T_H(\mathscr{O}_v) \\
\downarrow & & \downarrow & & \downarrow \\
X_*(T) & \longleftarrow & X_*(T)^{\mathscr{C}_v} & \overset{\sim}{\longleftarrow} & X_*(A_v) \simeq T(F_v)/T(\mathscr{O}_v)
\end{array}
$$

Indeed, we have an isomorphism $X_*(A_{H,v}) \simeq T_H(F_v)/T_H(\mathscr{O}_v)$ via $\mu \mapsto \mu(\varpi_v)$ and similarly $X_*(A_v) \simeq T(F_v)/T(\mathscr{O}_v)$. Further, $\alpha : T_H(F_v) \to T(F_v)$ is surjective since $H^1(\mathrm{Gal}(\overline{F}_v/F_v), Z(\overline{F}_v))$ is trivial (as Z is an induced torus).

Denote by $[\Sigma]$ the finite set of all conjugacy classes in Σ. For $\mathscr{C} \in [\Sigma]$, choose $\mathbb{Z}$-bases $\mathcal{B}_{H,\mathscr{C}}$ and $\mathcal{B}_{\mathscr{C}}$ for $X_*(T_H)^{\mathscr{C}}$ and $X_*(T)^{\mathscr{C}}$ respectively. [Note that the $\mathbb{Z}$-bases $\mathcal{B}_H$ for $X_*(T)$ and $\mathcal{B}$ for $X_*(T_H)$ are fixed once and for all.] An argument as in the proof of Lemma 2.3 shows that there exist constants $c(\mathcal{B}_{\mathscr{C}}), c(\mathcal{B}_{H,\mathscr{C}}) > 0$ such that for all $x \in X_*(T_H)_{\mathbb{R}}^{\mathscr{C}}$ and $y \in X_*(T)_{\mathbb{R}}^{\mathscr{C}}$,

$$|x|_{\mathcal{B}_{H,\mathscr{C}}} \geqslant c(\mathcal{B}_{H,\mathscr{C}}) \cdot \|x\|_{\mathcal{B}_H}, \quad |y|_{\mathcal{B}_{\mathscr{C}}} \leqslant c(\mathcal{B}_{\mathscr{C}}) \cdot \|y\|_{\mathcal{B}}. \tag{8.4}$$

Set $m_{\mathscr{C}} := \max_y(\min_x |x|_{\mathcal{B}_{H,\mathscr{C}}})$, where $y \in X_*(T)^{\mathscr{C}}$ varies subject to the condition $|y|_{\mathcal{B}_{\mathscr{C}}} \leqslant 1$ and $x \in X_*(T_H)^{\mathscr{C}}$ runs over the preimage of y. (It was shown above that the preimage is nonempty.) Then by construction, for every $y \in X_*(T)^{\mathscr{C}}$, there exists an x in the preimage of y such that $|x|_{\mathcal{B}_{H,\mathscr{C}}} \leqslant m_{\mathscr{C}} |y|_{\mathcal{B}_{\mathscr{C}}}$.

Recall that $U_{S_1} = \prod_{v \in S_1} U_v$ where $U_v = \cup_\mu K_v \mu(\varpi_v) K_v$, the union being taken over $\mu \in X_*(T)^{\mathscr{C}_v}$ such that $\|\mu\|_{\mathcal{B}} \leqslant \kappa$. We have seen that there exists $\mu_H \in X_*(T_H)^{\mathscr{C}_v}$ mapping to μ and $|\mu_H|_{\mathcal{B}_{H,\mathscr{C}_v}} \leqslant m_{\mathscr{C}_v} |\mu|_{\mathcal{B}_{\mathscr{C}_v}}$. By (8.4),

$$\|\mu_H\|_{\mathcal{B}_H} \leqslant m_{\mathscr{C}_v} c(\mathcal{B}_{H,\mathscr{C}_v})^{-1} c(\mathcal{B}_{\mathscr{C}_v}) \|\mu\|_{\mathcal{B}}.$$

Take $\beta := \max_{\mathscr{C} \in [\Sigma]} (m_{\mathscr{C}} c(\mathcal{B}_{H,\mathscr{C}})^{-1} c(\mathcal{B}_{\mathscr{C}}))$. Clearly β is independent of S_1 and κ. Notice that $\|\mu_H\|_{\mathcal{B}_H} \leqslant \beta \|\mu\|_{\mathcal{B}} \leqslant \beta \kappa$.

For each $\mu \in X_*(T)^{\mathscr{C}_v}$ such that $\|\mu\|_{\mathcal{B}} \leqslant \kappa$, we can choose a preimage μ_H of μ such that $\|\mu_H\|_{\mathcal{B}_H} \leqslant \beta \kappa$. Take $U_{H,v}$ to be the union of $K_{H,v} \mu_H(\varpi_v) K_{H,v}$ for those μ_H's. By construction $\alpha(U_{H,v}) = U_v$. Hence $U_{H,S_1} := \prod_{v \in S_1} U_{H,v}$ is the desired open compact subset in the claim of Lemma 8.9. □

Corollary 8.10 *In the setting of Proposition* 8.7, *let Y_G be the set of all semisimple $G(F)$-conjugacy (rather than $G(\mathbb{A}_F)$-conjugacy) classes whose $G(\mathbb{A}_F)$-conjugacy classes intersect U. Then there exist constants $A_6, B_8 > 0$ such that $|Y_G| = O(q_{S_1}^{B_8\kappa + A_6})$ as S_1 and κ vary.*

Proof Immediate from Lemma 8.6 and Proposition 8.7.

The following lemma was used in Step I of the proof of Proposition 8.7 and will be applied again to obtain Corollary 8.12 below.

Lemma 8.11 *Assume that G^{der} is simply connected. For each $v \in \mathcal{V}_F$ and each semisimple $\gamma \in G(F)$, let $n_{v,\gamma}$ be the number of $G(F_v)$-conjugacy classes in the stable conjugacy class of γ in $G(F_v)$. Then there exists a constant $u_G \geqslant 1$ (depending only on F and G) such that one has the uniform bound $n_{v,\gamma} \leqslant u_G$ for all v and γ.*

Proof Put $\Gamma(v) := \mathrm{Gal}(\overline{F}_v/F_v)$. It is a standard fact that $n_{v,\gamma}$ is the cardinality of $\ker(H^1(F_v, I_\gamma) \rightarrow H^1(F_v, G))$. By [63], $H^1(F_v, I_\gamma)$ is isomorphic to the dual of $\pi_0(Z(\widehat{I_\gamma})^{\Gamma(v)})$. Hence $n_{v,\gamma} \leqslant |\pi_0(Z(\widehat{I_\gamma})^{\Gamma(v)})|$. It suffices to show that a uniform bound for $|\pi_0(Z(\widehat{I_\gamma})^{\Gamma(v)})|$ exists.

By Lemma 6.5, there exists a finite Galois extension E/F with $[E : F] \leqslant w_G s_G$ such that I_γ splits over E. Then $\mathrm{Gal}(\overline{F}/F)$ acts on $Z(\widehat{I_\gamma})$ through $\mathrm{Gal}(E/F)$. In particular $\Gamma(v)$ acts on $Z(\widehat{I_\gamma})$ through a group of order $\leqslant w_G s_G$. Denote the latter group by $\Gamma(v)'$.

Note that there is a uniform bound on the number of connected components $[Z(\widehat{I_\gamma}) : Z(\widehat{I_\gamma})^0]$ as v and γ vary. Indeed it suffices to observe that there are only finitely many isomorphism classes of root data for I_γ over $\overline{F}$ (hence also for $\widehat{I_\gamma}$). This is easily seen from the fact that the roots of I_γ (for a maximal torus containing γ) are exactly the roots α of G such that $\alpha(\gamma) = 1$. Write $Z(\widehat{I_\gamma})^{0,\Gamma(v)}$ for the $\Gamma(v)$-invariants in $Z(\widehat{I_\gamma})^0$. Since

$$[\pi_0(Z(\widehat{I_\gamma})^{\Gamma(v)}) : \pi_0(Z(\widehat{I_\gamma})^{0,\Gamma(v)})] \leqslant [Z(\widehat{I_\gamma})^{\Gamma(v)} : Z(\widehat{I_\gamma})^{0,\Gamma(v)}]$$
$$\leqslant [Z(\widehat{I_\gamma}) : Z(\widehat{I_\gamma})^0],$$

it is enough to show that $|\pi_0(Z(\widehat{I_\gamma})^{0,\Gamma(v)})|$ is uniformly bounded.

Now consider the set of pairs

$$\mathscr{T} = \{(\Delta, \widehat{T}) : |\Delta| \leqslant w_G s_G, \ \dim \widehat{T} \leqslant r_G\}$$

consisting of a $\mathbb{C}$-torus $\widehat{T}$ with an action by a finite group Δ. Two pairs $(\Delta, \widehat{T})$ and $(\Delta', \widehat{T}')$ are equivalent if there are isomorphisms $\Delta \simeq \Delta'$ and $\widehat{T} \simeq \widehat{T}'$ such that the group actions are compatible. Note that

$$(\Gamma(v)', Z(\widehat{I_\gamma})^0) \in \mathcal{T}$$

and that $\mathcal{T}$ depends only on G and F. Clearly $|\pi_0(\widehat{T}^\Delta)|$ depends only on the equivalence class of $(\Delta, \widehat{T}) \in \mathcal{T}$. Hence the proof will be complete if $\mathcal{T}$ consists of finitely many equivalence classes.

Clearly there are finitely many isomorphism classes for Δ appearing in $\mathcal{T}$. So we may fix Δ and prove the finiteness of isomorphism classes of $\mathbb{C}$-tori with Δ-action. By dualizing, it is enough to show that there are finitely many isomorphism classes of $\mathbb{Z}[\Delta]$-modules whose underlying $\mathbb{Z}$-modules are free of rank at most r_G. This is a result of [36, §79]. $\qquad \square$

Corollary 8.12 *There exists a constant $c > 0$ (depending only on G) such that for every semisimple $\gamma \in G(F)$, $|\pi_0(Z(\widehat{I_\gamma})^\Gamma)| < c$. (We do not assume that G^{der} is simply connected.)*

Proof Suppose that G^{der} is simply connected. The proof of Lemma 8.11 shows that $(\mathrm{Gal}(E/F), Z(\widehat{I_\gamma})) \in \mathcal{T}$ in the notation there, thus there exists $c > 0$ such that $|\pi_0(Z(\widehat{I_\gamma})^\Gamma)| < c$ for all semisimple γ.

In general, let $1 \to Z \to H \to G \to 1$ be a z-extension over F so that Z is a product of induced tori and H^{der} is simply connected. Since $H(F) \twoheadrightarrow G(F)$, we may choose a semisimple γ_H mapping to γ. Let I_{γ_H} denote the centralizer of γ_H in H. (Since H^{der} is simply connected, I_{γ_H} is connected.) By the previous argument there exists $c_H > 0$ such that $|\pi_0(Z(\widehat{I_{\gamma_H}})^\Gamma)| < c_H$ for any semisimple γ_H. The obvious short exact sequence $1 \to Z \to I_{\gamma_H} \to I_\gamma \to 1$ over F gives rise (Sect. 2.1) to a Γ-equivariant short exact sequence

$$1 \to Z(\widehat{I_\gamma}) \to Z(\widehat{I_{\gamma_H}}) \to \widehat{Z} \to 1,$$

hence by [62, Cor 2.3],

$$0 \to \mathrm{coker}\,(X_*(Z(\widehat{I_{\gamma_H}}))^\Gamma \to X_*(\widehat{Z})^\Gamma) \to \pi_0(Z(\widehat{I_\gamma})^\Gamma) \to \pi_0(Z(\widehat{I_{\gamma_H}})^\Gamma)$$
$$\to \pi_0(\widehat{Z}^\Gamma) = 0. \tag{8.5}$$

On the other hand, the inclusions $Z \to I_{\gamma_H} \to H$ induce Γ-equivariant maps $Z(\widehat{H}) \to Z(\widehat{I_{\gamma_H}}) \to \widehat{Z}$. The map $Z(\widehat{H}) \to Z(\widehat{I_{\gamma_H}})$ is constructed by [63, 4.2], whereas $Z(\widehat{I_{\gamma_H}}) \to \widehat{Z}$ and $Z(\widehat{H}) \to \widehat{Z}$ are given by Sect. 2.1. (The distinction comes from the fact that typically $I_{\gamma_H} \to H$ is not normal.) The three maps are compatible in the obvious sense. By the functoriality of $X_*(\cdot)^\Gamma$, there is a natural surjection

$$\mathrm{coker}\,(X_*(Z(\widehat{H}))^\Gamma \to X_*(\widehat{Z})^\Gamma) \twoheadrightarrow \mathrm{coker}\,(X_*(Z(\widehat{I}_{\gamma_H}))^\Gamma \to X_*(\widehat{Z})^\Gamma).$$

The left hand side is finite because it embeds into the finite group $\pi_0(Z(\widehat{G})^\Gamma)$, again by [62, Cor 2.3]. Going back to (8.5), we deduce

$$\left|\pi_0(Z\left(\widehat{I}_\gamma\right)^\Gamma)\right| \leqslant \left|\pi_0\left(Z(\widehat{I}_{\gamma_H})^\Gamma\right)\right| \cdot \left|\mathrm{coker}\,(X_*(Z(\widehat{H}))^\Gamma \to X_*(\widehat{Z})^\Gamma)\right|$$
$$< c_H \cdot \left|\pi_0(Z(\widehat{G})^\Gamma)\right|.$$

The proof is complete as the far right hand side is independent of γ. $\qquad\square$

For a cuspidal group and conjugacy classes which are elliptic at infinity, a more precise bound can be obtained by a simpler argument, which would be worth recording here.

Lemma 8.13 *Let G be a cuspidal F-group. For any $\gamma \in G(F)$ such that $\gamma \in G(F_\infty)$ is elliptic,*

$$\left|\pi_0(Z(\widehat{I}_\gamma)^\Gamma)\right| \leqslant 2^{\mathrm{rk}(G/A_G)}.$$

Proof Via restriction of scalars, we may assume that $F = \mathbb{Q}$ without losing generality. Let us prove the lemma when A_G is trivial. By assumption there exists an $\mathbb{R}$-anisotropic torus T in $G(\mathbb{R})$ containing γ. Thus $T \simeq U(1)^{\mathrm{rk}(G)}$ and $T \hookrightarrow I_\gamma$ over $\mathbb{R}$. The former tells us that $\widehat{T}^{\Gamma(\infty)} \simeq \{\pm 1\}^{\mathrm{rk}(G)}$ and the latter gives rise to $Z(\widehat{I}_\gamma)^{\Gamma(\infty)} \hookrightarrow \widehat{T}^{\Gamma(\infty)}$ [63, §4]. Hence the assertion follows from

$$Z(\widehat{I}_\gamma)^\Gamma \hookrightarrow Z(\widehat{I}_\gamma)^{\Gamma(\infty)} \hookrightarrow \widehat{T}^{\Gamma(\infty)} \simeq \{\pm 1\}^{\mathrm{rk}(G)}.$$

In general when A_G is not trivial, consider the exact sequence of $\mathbb{Q}$-groups $1 \to A_G \to I_\gamma \to I_\gamma/A_G \to 1$, whose dual is the Γ-equivariant exact sequence of $\mathbb{C}$-groups

$$1 \to Z(\widehat{I_\gamma/A_G}) \to Z(\widehat{I}_\gamma) \to \widehat{A}_G \to 1.$$

Thanks to [62, Cor 2.3], we obtain the following exact sequence:

$$X^*(\widehat{A}_G)^\Gamma \to \pi_0\left(Z(\widehat{I_\gamma/A_G})^\Gamma\right) \to \pi_0\left(Z(\widehat{I}_\gamma)^\Gamma\right) \to \pi_0(\widehat{A}_G)^\Gamma = 1.$$

Hence $|\pi_0(Z(\widehat{I}_\gamma)^\Gamma)| \leqslant |\pi_0(Z(\widehat{I_\gamma/A_G})^\Gamma)|$, and the latter is at most $2^{\mathrm{rk}(G/A_G)}$ by the preceding argument. $\qquad\square$

9 Automorphic Plancherel density theorem with error bounds

The local components of automorphic representations at a fixed finite set of primes tend to be equidistributed according to the Plancherel measure on the unitary dual, namely the error tends to zero in a family of automorphic representations (cf. Corollary 9.22 below). The main result of this section (Theorems 9.16, 9.19) is a bound on this error in terms of the primes in the fixed set as well as the varying parameter (level or weight) in the family. A crucial assumption for us is that the group G is cuspidal (Definition 9.7), which allows the use of a simpler version of the trace formula. For the proof we interpret the problem as bounding certain expressions on the geometric side of the trace formula and apply various technical results from previous sections. One main application is a proof of the Sato–Tate conjecture for families formulated in Sect. 5.4 under suitable conditions on the parameters involved. In turn the result will be applied to the question on low-lying zeros in later sections.

9.1 Sauvageot's density theorem on unitary dual

We reproduce a summary of Sauvageot's result [91] from [99, §2.3] as it can be used to effectively prescribe local conditions in our problem. The reader may refer to either source for more detail.

Let G be a connected reductive group over a number field F. Use v to denote a finite place of F. When M is a Levi subgroup of G over F_v, write $\Psi_u(M(F_v))$ (resp. $\Psi(M(F_v))$) for the real (resp. complex) torus whose points parametrize unitary (complex-valued) characters of $M(F_v)$ trivial on any compact subgroup of $M(F_v)$. The normalized parabolic induction of an admissible representation σ of $M(F_v)$ is denoted n-ind$_M^G(\sigma)$.

Denote by $\mathscr{B}_c(G(F_v)^\wedge)$ the space of bounded $\widehat{\mu}_v^{\mathrm{pl}}$-measurable functions $\widehat{f_v}$ on $G(F_v)^\wedge$ whose support has compact image in the Bernstein center, which is the set of $\mathbb{C}$-points of an (infinite) product of varieties. A measure on $G(F_v)^\wedge$ will be thought of as a linear functional on the space $\mathscr{F}(G(F_v)^\wedge)$ consisting of $\widehat{f_v} \in \mathscr{B}_c(G(F_v)^\wedge)$ such that for every F_v-rational Levi subgroup M of G and every discrete series σ of $M(F_v)$,

$$\Psi_u(M(F_v)) \to \mathbb{C} \quad \text{given by } \chi \mapsto \widehat{f_v}(\text{n-ind}_M^G(\sigma \otimes \chi))$$

is a function whose points of discontinuity are contained in a measure zero set. (Here n-ind denotes the normalized parabolic induction.) Now for any finite set S of finite places of F, one can easily extend the above definition to $\mathscr{F}(G(F_S)^\wedge)$ so that $\widehat{f_S}(\pi_S) \in \mathbb{C}$ makes sense for $\widehat{f_S} \in \mathscr{F}(G(F_S)^\wedge)$ and $\pi_S \in G(F_S)^\wedge$. We have a map

$$C_c^\infty(G(F_S)) \to \mathscr{F}(G(F_S)^\wedge), \quad \phi_S \mapsto \widehat{\phi}_S \colon \pi_S \mapsto \operatorname{tr} \pi_S(\phi_S),$$

as follows from Proposition 9.6 below. Harish-Chandra's Plancherel theorem states that

$$\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S) = \phi_S(1).$$

Our notational convention is that $\widehat{\phi}_S$ often signifies an element in the image of the above map whereas $\widehat{f}_S$ stands for a general element of $\mathscr{F}(G(F_S)^\wedge)$. Sauvageot's theorem allows us to approximate any $\widehat{f}_S \in \mathscr{F}(G(F_S)^\wedge)$ with elements of $C_c^\infty(G(F_S))$.

Proposition 9.1 [91, Thm 7.3] *Let* $\widehat{f}_S \in \mathscr{F}(G(F_S)^\wedge)$. *For any* $\epsilon > 0$, *there exist* $\phi_S, \psi_S \in C_c^\infty(G(F_S))$ *such that*

$$\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\psi}_S) \leqslant \epsilon \quad \text{and} \quad \forall \pi_S \in G(F_S)^\wedge, \ |\widehat{f}_S(\pi_S) - \widehat{\phi}_S(\pi_S)| \leqslant \widehat{\psi}_S(\pi_S).$$

Conversely, any $\widehat{f}_S \in \mathscr{B}_c(\widehat{G(F_S)})$ *with the above property belongs to* $\mathscr{F}(G(F_S)^\wedge)$.

Remark 9.2 It is crucial that $\widehat{f}_S \in \mathscr{F}(G(F_S)^\wedge)$ has the set of discontinuity in a measure zero set. Otherwise we could take $\widehat{f}_S$ to be the characteristic function on the set of points of $G(F_S)^\wedge$ which arise as the S-components of some $\pi \in \mathcal{AR}_{\mathrm{disc},\chi}(G)$ with nonzero Lie algebra cohomology. Note that the latter function typically lies outside $\mathscr{F}(G(F_S)^\wedge)$. The conclusions of Theorems 9.26, 9.27 and Corollary 9.22 are false in general if such an $\widehat{f}_S$ is placed at S_0. Namely in that case $\widehat{\mu}_{\mathcal{F}_k,S_1}(\widehat{\phi}_{S_1})$ is often far from zero but $\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S)$ always vanishes.

From here until the end of this subsection let us suppose that G is unramified at S. It will be convenient to introduce $\mathcal{F}(G(F_S)^{\wedge,\mathrm{ur}})$ and its subspace $\mathcal{F}(G(F_S)^{\wedge,\mathrm{ur},\mathrm{temp}})$ in order to state the Sato–Tate theorem in Sect. 9.7. The former (resp. the latter) consists of $\widehat{f}_S \in \mathcal{F}(G(F_S)^\wedge)$ such that the support of $\widehat{f}_S$ is contained in $G(F_S)^{\wedge,\mathrm{ur}}$ [resp. $G(F_S)^{\wedge,\mathrm{ur},\mathrm{temp}}$]. Denote by $\mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$ the space of bounded $\widehat{\mu}_\theta^{\mathrm{ST}}$-measurable functions on $\widehat{T}_{c,\theta}/\Omega_{c,\theta}$ whose points of discontinuity are contained in a $\widehat{\mu}_\theta^{\mathrm{ST}}$-measure zero set. Define $\mathcal{F}(\prod_{v \in S} \widehat{T}_{c,\theta_v}/\Omega_{c,\theta_v})$ in the obvious analogous way. By using the topological Satake isomorphism for tempered spectrum [cf. (5.2)]

$$\prod_{v \in S} \widehat{T}_{c,\theta_v}/\Omega_{c,\theta_v} \simeq G(F_S)^{\wedge,\mathrm{ur},\mathrm{temp}}$$

and extending by zero outside the tempered spectrum, one obtains

$$\mathcal{F}\left(\prod_{v \in S} \widehat{T}_{c,\theta_v}/\Omega_{c,\theta_v}\right) \simeq \mathcal{F}(G(F_S)^{\wedge,\mathrm{ur},\mathrm{temp}}) \hookrightarrow \mathcal{F}(G(F_S)^{\wedge,\mathrm{ur}}). \quad (9.1)$$

Although the first two $\mathcal{F}(\cdot)$ above are defined with respect to different measures $\prod_{v \in S} \widehat{\mu}_{\theta_v}^{\mathrm{ST}}$ and $\widehat{\mu}_S^{\mathrm{pl}}$, the isomorphism is justified by the fact that the ratio of the two measures is uniformly bounded above and below by positive constants (depending on q_S) in view of Proposition 3.3 and Lemma 5.2. Note that the space of continuous functions on $\prod_{v \in S} \widehat{T}_{c,\theta_v}/\Omega_{c,\theta_v}$ [resp. on $G(F_S)^{\wedge,\mathrm{ur},\mathrm{temp}}$] is contained in the first (resp. second) term of (9.1), and the two subspaces correspond under the isomorphism.

Corollary 9.3 *Let* $\widehat{f_S} \in \mathcal{F}(G(F_S)^{\wedge,\mathrm{ur}})$. *For any* $\epsilon > 0$, *there exist* $\phi_S, \psi_S \in \mathcal{H}^{\mathrm{ur}}(G(F_S))$ *such that (i)* $\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\psi}_S) \leqslant \epsilon$ *and (ii)* $\forall \pi_S \in G(F_S)^{\wedge,\mathrm{ur}}$, $|\widehat{f_S}(\pi_S) - \widehat{\phi}_S(\pi_S)| \leqslant \widehat{\psi}_S(\pi_S)$.

Proof Let $\phi_S, \psi_S \in C_c^\infty(G(F_S))$ be the functions associated to $\widehat{f_S}$ as in Proposition 9.1. Then it is enough to replace ϕ_S and ψ_S with their convolution products with the characteristic function on $\prod_{v \in S} K_v$. □

The following proposition will be used later in Sect. 9.7. For each $v \in \mathcal{V}_F(\theta)$, the image of $\widehat{f}$ in $\mathcal{F}(G(F_v)^{\wedge,\mathrm{ur}})$ via (9.1) will be denoted $\widehat{f}_v$.

Proposition 9.4 *Let* $\widehat{f} \in \mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$ *and* $\epsilon > 0$. *There exists an integer* $\kappa \geqslant 1$ *and for all places* $v \in \mathcal{V}_F(\theta)$, *there are bounded functions* $\phi_v, \psi_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$ *such that* $\widehat{\mu}_v^{\mathrm{pl}}(\widehat{\psi}_v) \leqslant \epsilon$ *and* $|\widehat{f}_v(\pi) - \widehat{\phi}_v(\pi)| \leqslant \widehat{\psi}_v(\pi)$ *for all* $\pi \in G(F_v)^{\wedge,\mathrm{ur}}$.

Proof This is no more than Corollary 9.3 if we only required $\phi_v, \psi_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))$ without the superscript $\leqslant \kappa$. So we may disregard finitely many v by considering the subset $\mathcal{V}_F(\theta)^{\geqslant Q}$ of $\mathcal{V}_F(\theta)$ consisting of v such that $q_v \geqslant Q$ for some $Q > 0$. In view of Proposition 5.3, we may choose $Q \in \mathbb{Z}_{>0}$ that

$$\forall v \in \mathcal{V}_F(\theta)^{\geqslant Q}, \ \forall \widehat{f} \in \mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta}), \ \frac{1}{2}\widehat{\mu}_\theta^{\mathrm{ST}}(|\widehat{f}|)$$
$$\leqslant \widehat{\mu}_v^{\mathrm{pl},\mathrm{ur}}(|\widehat{f}_v|) \leqslant 2\widehat{\mu}_\theta^{\mathrm{ST}}(|\widehat{f}|). \quad (9.2)$$

Fix any $w \in \mathcal{V}_F(\theta)^{\geqslant Q}$. Corollary 9.3 allows us to find $\phi_w, \psi'_w \in \mathcal{H}^{\mathrm{ur}}(G(F_w))$ such that

$$\widehat{\mu}_w^{\mathrm{pl}}(\widehat{\psi}'_w) \leqslant \epsilon/8 \ \ \text{and} \ \ \forall \pi_w \in G(F_w)^{\wedge,\mathrm{ur}}, \ |\widehat{f}_w(\pi_w) - \widehat{\phi}_w(\pi_w)| \leqslant \widehat{\psi}'_w(\pi_w). \quad (9.3)$$

Let $\kappa_0 \in \mathbb{Z}_{\geqslant 0}$ be such that $\phi_w, \psi'_w \in \mathcal{H}^{\mathrm{ur}}(G(F_w))^{\leqslant \kappa_0}$. Now recall that for every $v \in \mathcal{V}_F(\theta)$ there is a canonical isomorphism [cf. (2.2), Lemma 3.2] between $\mathcal{H}^{\mathrm{ur}}(G(F_v))$ and the space of regular functions in the complex variety $\widehat{T}_\theta / \Omega_\theta$. Using the latter as a bridge, we may transport ϕ_w, ψ'_w to $\phi_v, \psi'_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))$ for every $v \in \mathcal{V}_F(\theta)$. Clearly $\phi_v, \psi'_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa_0}$ from the definition of Sect. 2.3. Moreover (9.2) and (9.3) imply that for all $v \in \mathcal{V}_F(\theta)^{\geqslant Q}$,

$$\widehat{\mu}_v^{\mathrm{pl}}(\widehat{\psi}'_v) \leqslant \epsilon/2 \quad \text{and} \quad \forall \pi_v \in G(F_v)^{\wedge, \mathrm{ur}, \mathrm{temp}}, \ |\widehat{f}_v(\pi_v) - \widehat{\phi}_v(\pi_v)| \leqslant \widehat{\psi}'_v(\pi_v).$$

[Observe that $\widehat{\mu}_v^{\mathrm{pl}}(\widehat{\psi}'_v) \leqslant 2\widehat{\mu}_\theta^{\mathrm{ST}}(\widehat{\psi}'_v) = 2\widehat{\mu}_\theta^{\mathrm{ST}}(\widehat{\psi}'_w) \leqslant 4\widehat{\mu}_w^{\mathrm{pl}}(\widehat{\psi}'_w) \leqslant \epsilon/2$ to justify the first inequality.]

To achieve the latter inequality for non-tempered $\pi_v \in G(F_v)^{\wedge, \mathrm{ur}}$, we would like to perturb ψ'_v in a way independent of v while not sacrificing the former inequality. Since $\widehat{f}_v(\pi_v) = 0$ for such π_v, what we need to establish is that $|\widehat{\phi}_v(\pi_v)| \leqslant \widehat{\psi}_v(\pi_v)$ for all non-tempered $\pi_v \in G(F_v)^{\wedge, \mathrm{ur}}$. To this end, we use the fact that there is a compact subset $\mathcal{K}$ of $\widehat{T}_\theta / \Omega_\theta$ such that $G(F_v)^{\wedge, \mathrm{ur}}$ is contained in $\mathcal{K}$ for every $v \in \mathcal{V}_F(\theta)$ (cf. [11, Thm XI.3.3]). By using the Weierstrass approximation theorem, we find $\psi''_w \in \mathcal{H}^{\mathrm{ur}}(G(F_w))$ such that

$$\widehat{\mu}_w^{\mathrm{pl}}(\widehat{\psi}''_w) \leqslant \epsilon/8,$$

$$\forall \pi_w \in \mathcal{K} \backslash G(F_w)^{\wedge, \mathrm{ur}, \mathrm{temp}}, \ |\widehat{\psi}'_w(\pi_w)| + |\widehat{\phi}_w(\pi_w)| \leqslant \widehat{\psi}''_w(\pi_w),$$
$$\forall \pi_w \in G(F_w)^{\wedge, \mathrm{ur}, \mathrm{temp}}, \ \widehat{\psi}''_w(\pi_w) \geqslant 0.$$

Choose $\kappa \geqslant \kappa_0$ such that $\psi''_w \in \mathcal{H}^{\mathrm{ur}}(G(F_w))^{\leqslant \kappa}$ and put $\psi_w := \psi'_w + \psi''_w$ so that $\widehat{\mu}_w^{\mathrm{pl}}(\widehat{\psi}_w) \leqslant \epsilon/4$ and $\psi_w \in \mathcal{H}^{\mathrm{ur}}(G(F_w))^{\leqslant \kappa}$. For each $v \in \mathcal{V}_F(\theta)^{\geqslant Q}$, let ψ_v denote the transport of ψ_w just as ψ'_v was the transport of ψ'_w in the preceding paragraph. Then $\widehat{\mu}_v^{\mathrm{pl}}(\widehat{\psi}_v) \leqslant \epsilon$ and $\psi_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \kappa}$ as before. Moreover

$$\forall \pi_v \in G(F_v)^{\wedge, \mathrm{ur}, \mathrm{temp}}, \ |\widehat{f}_v(\pi_v) - \widehat{\phi}_v(\pi_v)| \leqslant \widehat{\psi}'_v(\pi_v) \leqslant \widehat{\psi}_v(\pi_v)$$

and for $\pi_v \in G(F_v)^{\wedge, \mathrm{ur}} \backslash G(F_v)^{\wedge, \mathrm{ur}, \mathrm{temp}}$,

$$\left| \widehat{f}_v(\pi_v) - \widehat{\phi}_v(\pi_v) \right| = \left| \widehat{\phi}_v(\pi_v) \right| \leqslant \widehat{\psi}''_v(\pi_v) - |\widehat{\psi}'_v(\pi_v)| \leqslant \widehat{\psi}_v(\pi_v),$$

the last inequality following from $\widehat{\psi}_v = \widehat{\psi}'_v + \widehat{\psi}''_v$. □

Remark 9.5 A more direct approach to (9.3) that wouldn't involve Corollary 9.3 would be to use Weierstrass approximation to find polynomials ϕ and ψ on $\widehat{T}_{c,\theta} / \Omega_{c,\theta}$ of degree $\leqslant \kappa$ such that $|\widehat{f} - \widehat{\phi}| \leqslant \widehat{\psi}$ and then the isomorphism (9.1) to transport ϕ and ψ at the place v.

We note [91, Lemme 3.5] that for any $\phi_v \in C_c^\infty(G(F_v))$ there exists a $\phi_v' \in C_c^\infty(G(F_v))$ such that $|\widehat{\phi}_v(\pi_v)| \leqslant \widehat{\phi}'_v(\pi_v)$ for all $\pi_v \in G(F_v)^\wedge$. This statement is elementary, e.g. it follows from the Dixmier–Malliavin decomposition theorem. In fact we have the following stronger result due to Bernstein [8].

Proposition 9.6 (Uniform admissibility theorem) *For any $\phi_v \in C_c^\infty(G(F_v))$ there exists $C > 0$ such that $|\mathrm{tr}\,\pi(\phi_v)| \leqslant C$ for all $\pi \in G(F_v)^\wedge$.*

9.2 Automorphic representations and a counting measure

Now consider a string of complex numbers

$$\mathcal{F} = \{a_\mathcal{F}(\pi) \in \mathbb{C}\}_{\pi \in \mathcal{AR}_{\mathrm{disc},\chi}(G)}$$

such that $a_\mathcal{F}(\pi) = 0$ for all but finitely many π. We think of $\mathcal{F}$ as a multi-set by viewing $a_\mathcal{F}(\pi)$ as multiplicity, or more appropriately as a density function with finite support in $\mathcal{F}$ as $a_\mathcal{F}(\pi)$ is allowed to be in $\mathbb{C}$. There are obvious meanings when we write $\pi \in \mathcal{F}$ and $|\mathcal{F}|$ (we could have written $\pi \in \mathrm{supp}\,\mathcal{F}$ for the former):

$$\pi \in \mathcal{F} \overset{\text{def}}{\Leftrightarrow} a_\mathcal{F}(\pi) \neq 0, \quad |\mathcal{F}| := \sum_{\pi \in \mathcal{F}} a_\mathcal{F}(\pi).$$

In order to explain our working hypothesis, we recall a definition.

Definition 9.7 Let H be a connected reductive group over $\mathbb{Q}$. The maximal $\mathbb{Q}$-split torus in $Z(H)$ is denoted A_H. We say H is *cuspidal* if $(H/A_H) \times_\mathbb{Q} \mathbb{R}$ contains a maximal $\mathbb{R}$-anisotropic torus.

If H is cuspidal then $H(\mathbb{R})$ has discrete series representations. (We remind the reader that discrete series always mean "relative discrete series" for us, i.e. those whose matrix coefficients are square-integrable modulo center.) The converse is true when H is semisimple but not in general. Throughout this section the following will be in effect:

Hypothesis 9.8 $\mathrm{Res}_{F/\mathbb{Q}}G$ is a cuspidal group.

Let $S = S_0 \coprod S_1 \subset \mathcal{V}_F^\infty$ be a nonempty finite subset and $\widehat{f}_{S_0} \in \mathscr{F}(G(F_{S_0})^\wedge)$. (It is allowed that either S_0 or S_1 is empty.) Let

– (level) $U^{S,\infty}$ be an open compact subset of $G(\mathbb{A}^{S,\infty})$,
– (weight) $\xi = \otimes_{v|\infty}\xi_v$ be an irreducible algebraic representation of

$$G_\infty \times_\mathbb{R} \mathbb{C} = (\mathrm{Res}_{F/\mathbb{Q}}G) \times_\mathbb{Q} \mathbb{C} = \prod_{v|\infty} G \times_{F,v} \mathbb{C}.$$

Denote by $\chi \colon A_{G,\infty} \to \mathbb{C}^\times$ the restriction of the central character for $\xi^\vee$. Define

$$\mathcal{F} = \mathcal{F}(U^{S,\infty}, \widehat{f}_{S_0}, S_1, \xi) \quad \text{by}$$

$$a_{\mathcal{F}}(\pi) := (-1)^{q(G)} m_{\mathrm{disc},\chi}(\pi) \dim(\pi^{S,\infty})^{U^{S,\infty}} \widehat{f}_{S_0}(\pi_{S_0})$$
$$\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1}) \chi_{\mathrm{EP}}(\pi_\infty \otimes \xi) \in \mathbb{C}. \tag{9.4}$$

Note that $\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1})$ equals 1 if π_{S_1} is unramified and 0 otherwise, and that $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi) = 0$ unless π_∞ has the same infinitesimal character as $\xi^\vee$. The set of π such that $a_{\mathcal{F}}(\pi) \neq 0$ is finite by Harish-Chandra's finiteness theorem. Let us define measures $\widehat{\mu}_{\mathcal{F},S_1}$ and $\widehat{\mu}^\natural_{\mathcal{F},S_1}$ associated with $\mathcal{F}$ on the unramified unitary dual $G(F_{S_1})^{\wedge,\mathrm{ur}}$, motivated by the trace formula. Put $\tau'(G) := \overline{\mu}^{\mathrm{can},\mathrm{EP}}(G(F)A_{G,\infty} \backslash G(\mathbb{A}_F))$. For any function $\widehat{f}_{S_1}$ on $G(F_{S_1})^{\wedge,\mathrm{ur}}$ which is continuous outside a measure zero set, define

$$\widehat{\mu}_{\mathcal{F},S_1}(\widehat{f}_{S_1}) := \frac{\mu^{\mathrm{can}}(U^{S,\infty})}{\tau'(G) \dim \xi} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc},\chi}(G)} a_{\mathcal{F}}(\pi) \widehat{f}_{S_1}(\pi_{S_1}). \tag{9.5}$$

The sum is finite because $a_{\mathcal{F}}$ is supported on finitely many π. Now the key point is that the right hand side can be identified with the spectral side of Arthur's trace formula with the Euler–Poincaré function at infinity as in Sect. 6.5 when $\widehat{f}_{S_1} = \widehat{\phi}_{S_1}$ for some $\phi_{S_1} \in \mathcal{H}^{\mathrm{ur}}(G(F_{S_1}))$ ([3, pp. 267–268], cf. proof of [99, Prop 4.1]). So to speak, if we write $\phi^\infty = \phi_{S_0} \phi_{S_1} \phi^{S,\infty}$,

$$\widehat{\mu}_{\mathcal{F},S_1}(\widehat{\phi}_{S_1}) = (-1)^{q(G)} \frac{I_{\mathrm{spec}}(\phi^\infty \phi_\xi, \mu^{\mathrm{can},\mathrm{EP}})}{\tau'(G) \dim \xi}$$
$$= (-1)^{q(G)} \frac{I_{\mathrm{geom}}(\phi^\infty \phi_\xi, \mu^{\mathrm{can},\mathrm{EP}})}{\tau'(G) \dim \xi} \tag{9.6}$$

where I_{spec} (resp. I_{geom}) denotes the spectral (resp. geometric) side Arthur's the invariant trace formula with respect to the measure $\mu^{\mathrm{can},\mathrm{EP}}$. Finally if $\widehat{f}_{S_0}$ has the property that $\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{f}_{S_0}) \neq 0$ then put

$$\widehat{\mu}^\natural_{\mathcal{F},S_1} := \widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{f}_{S_0})^{-1} \widehat{\mu}_{\mathcal{F},S_1}.$$

Remark 9.9 The measure $\widehat{\mu}^\natural_{\mathcal{F},S_1}$ is asymptotically the same as the counting measure

$$\widehat{\mu}^{\mathrm{count}}_{\mathcal{F},S_1}(\widehat{f}_{S_1}) = \frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc},\chi}(G)} a_{\mathcal{F}}(\pi) \widehat{f}_{S_1}(\pi_{S_1}).$$

associated with the S_1-components of $\mathcal{F}$ (assuming $|\mathcal{F}| \neq 0$). More precisely if $\{\mathcal{F}_k\}_{\geqslant 1}$ is a family of Sect. 9.3 below, then $\widehat{\mu}^{\mathrm{count}}_{\mathcal{F}_k, S_1} / \widehat{\mu}^{\natural}_{\mathcal{F}_k, S_1}$ is a constant tending to 1 as $k \to \infty$ by Corollary 9.25.

Example 9.10 Let $\pi \in \mathcal{AR}_{\mathrm{disc}, \chi}(G)$. Suppose that the highest weight of ξ is *regular* and that $S_0 = \emptyset$. Then π belongs to $\mathcal{F}$ if and only if the following three conditions hold: $(\pi^{S,\infty})^{U^{S,\infty}} \neq 0$, π is unramified at S, and $\pi_\infty \in \Pi_{\mathrm{disc}}(\xi^\vee)$. When $\pi_\infty \in \Pi_{\mathrm{disc}}(\xi^\vee)$, (9.4) simplifies as

$$a_{\mathcal{F}}(\pi) = m_{\mathrm{disc}, \chi}(\pi) \dim(\pi^{S,\infty})^{U^{S,\infty}}.$$

Example 9.11 Let $\widehat{f}_{S_0}$ be a characteristic function on some relatively compact $\widehat{\mu}^{\mathrm{pl}}_S$-measurable subset $\widehat{U}_{S_0} \subset G(F_{S_0})^\wedge$. Assume that S_0 is large enough such that G and all members of $\mathcal{F}$ are unramified outside S_0. Take $U^{S_0,\infty}$ to be the product of K_v over all finite places $v \notin S_0$. Then for each $\pi \in \mathcal{AR}_{\mathrm{disc}, \chi}(G)$,

$$a_{\mathcal{F}}(\pi) = (-1)^{q(G)} \chi_{\mathrm{EP}}(\pi_\infty \otimes \xi) m_{\mathrm{disc}, \chi}(\pi) \tag{9.7}$$

if $\pi^{S_0,\infty}$ is unramified, $\pi_{S_0} \in \widehat{U}_{S_0}$ (in which case $a_{\mathcal{F}}(\pi) \neq 0$ if moreover $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi) \neq 0$; otherwise $a_{\mathcal{F}}(\pi) = 0$). If the highest weight of ξ is regular, $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi) \neq 0$ exactly when $\pi_\infty \in \Pi_{\mathrm{disc}}(\xi^\vee)$, in which case (9.7) simplifies as

$$a_{\mathcal{F}}(\pi) = m_{\mathrm{disc}, \chi}(\pi).$$

Compare this with Example 9.10. (The analogy in the case of modular forms is that π as newforms are counted in the current example whereas old-forms are also counted in Example 9.10.) Finally we observe that since the highest weight of ξ is regular and $\pi_\infty \in \Pi_{\mathrm{disc}}(\xi^\vee)$, the discrete automorphic representation π is automatically cuspidal [107, Thm. 4.3]. In the present example the discrete multiplicity coincides with the cuspidal multiplicity.

Remark 9.12 As the last example shows, the main reason to include S_0 is to prescribe local conditions at finitely many places (namely at S_0) on automorphic families. For instance one can take $\widehat{f}_{S_0} = \widehat{\phi}_{S_0}$ where ϕ_{S_0} is a pseudo-coefficient of a supercuspidal representation (or a truncation thereof if the center of G is not anisotropic over F_{S_0}). Then it allows us to consider a family of π whose S_0-components are a particular supercuspidal representation (or an unramified character twist thereof). By using various $\widehat{f}_{S_0}$ [which are in general not equal to $\widehat{\phi}_{S_0}$ for any $\phi_{S_0} \in C_c^\infty(G(F_{S_0}))$] one obtains great flexibility in prescribing a local condition as well as imposing weighting factors for a family.

9.3 Families of automorphic representations

Continuing from the previous subsection (in particular keeping Hypothesis 9.8) let us introduce two kinds of families $\{\mathcal{F}_k\}_{k\geqslant 1}$ which will be studied later on. We will measure the size of ξ in the following way. Let T_∞ be a maximal torus of G_∞ over $\mathbb{R}$. For a B-dominant $\lambda \in X^*(T_\infty)$, set $m(\lambda) := \min_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle$. For ξ with B-dominant highest weight λ_ξ, define $m(\xi) := m(\lambda_\xi)$.

Let $\phi_{S_0} \in C_c^\infty(G(F_{S_0}))$. [More generally we will sometimes prescribe a local condition at S_0 by $\widehat{f}_{S_0} \in \mathscr{F}(G(F_{S_0})^\wedge)$ rather than ϕ_{S_0}.] In the remainder of Sect. 9 we mostly focus on families in the level or weight aspect, respectively described as the following:

*Example 9.13 (*Level aspect: *varying level, fixed weight)* Let $\mathfrak{n}_k \subset \mathcal{O}_F$ be a nonzero ideal prime to S for each $k \geqslant 1$ such that $\mathbb{N}(\mathfrak{n}_k) = [\mathcal{O}_F : \mathfrak{n}_k]$ tends to ∞ as $k \to \infty$. Take

$$\mathcal{F}_k := \mathcal{F}\left(K^{S,\infty}(\mathfrak{n}_k), \widehat{\phi}_{S_0}, S_1, \xi\right).$$

Then $|\mathcal{F}_k| \to \infty$ as $k \to \infty$.

*Example 9.14 (*Weight aspect: *fixed level, varying weight)* For our study of weight aspect it is always supposed that $Z(G) = 1$ so that $A_{G,\infty} = 1$ and $\chi = 1$ in order to eliminate the technical problem with central character when weight varies.[9] Let $\{\xi_k\}_{k\geqslant 1}$ be a sequence of irreducible algebraic representations of $G_\infty \times_\mathbb{R} \mathbb{C}$ such that $m(\xi_k) \to \infty$ as $k \to \infty$. Take

$$\mathcal{F}_k := \mathcal{F}\left(U^{S,\infty}, \widehat{\phi}_{S_0}, S_1, \xi_k\right).$$

Then $|\mathcal{F}_k| \to \infty$ as $k \to \infty$.

Remark 9.15 Sarnak proposed a definition of families of automorphic representations (or automorphic L-functions) in [87]. The above two examples fit in his definition.

9.4 Level aspect

We are in the setting of Example 9.13. Recall that $\mathrm{Res}_{F/\mathbb{Q}}G$ is assumed to be cuspidal. Fix $\Xi \colon G \hookrightarrow \mathrm{GL}_m$ as in Proposition 8.1 and let B_Ξ and c_Ξ be as in (8.2) and Lemma 8.4. Write $\mathscr{L}_c(M_0)$ for the set of F-rational cuspidal Levi subgroups of G containing the minimal Levi M_0.

---

[9] Without the hypothesis that the center is trivial, one should work with fixed central character and apply the trace formula in such a setting. Then our results and arguments in the weight aspect should remain valid without change.

Theorem 9.16 *Fix $\phi_{S_0} \in C_c^\infty(G(F_{S_0}))$ and ξ. Let $S_1 \subset \mathcal{V}_F^\infty$ be a subset where G is unramified. Let $\phi_{S_1} \in \mathcal{H}^{ur}(G(F_{S_1}))^{\leqslant \kappa}$ be such that $|\phi_{S_1}| \leqslant 1$ on $G(F_{S_1})$. If $\mathscr{L}_c(M_0) = \{G\}$ (in particular if G is abelian) then $\widehat{\mu}_{\mathcal{F}_k, S_1}(\widehat{\phi}_{S_1}) = \widehat{\mu}_S^{pl}(\widehat{\phi}_S)$. Otherwise there exist constants $A_{lv}, B_{lv} > 0$ and $C_{lv} \geqslant 1$ such that*

$$\widehat{\mu}_{\mathcal{F}_k, S_1}(\widehat{\phi}_{S_1}) - \widehat{\mu}_S^{pl}(\widehat{\phi}_S) = O\left(q_{S_1}^{A_{lv} + B_{lv}\kappa} \mathbb{N}(\mathfrak{n})^{-C_{lv}}\right) \tag{9.8}$$

as $\mathfrak{n}, \kappa \in \mathbb{Z}_{\geqslant 1}$, S_1 and ϕ_{S_1} vary subject to the following conditions:

(i) $\mathbb{N}(\mathfrak{n}) \geqslant c_\Xi q_{S_1}^{B_\Xi m \kappa}$,
(ii) *no prime divisors of $\mathfrak{n}$ are contained in S_1.*

[*The implicit constant in $O(\cdot)$ is independent of $\mathfrak{n}$, κ, S_1 and ϕ_{S_1}.*]

Remark 9.17 When $\widehat{\mu}_{S_0}^{pl}(\widehat{\phi}_{S_0}) \neq 0$ (9.8) is equivalent to

$$\widehat{\mu}_{\mathcal{F}, S_1}^{\natural}(\widehat{\phi}_{S_1}) - \widehat{\mu}_{S_1}^{pl}(\widehat{\phi}_{S_1}) = O\left(q_{S_1}^{A_{lv} + B_{lv}\kappa} \mathbb{N}(\mathfrak{n})^{-C_{lv}}\right)$$

.
Remark 9.18 One can choose A_{lv}, B_{lv}, C_{lv} to be explicit integers. See the proof below. For instance $C_{lv} \geqslant n_G$ for n_G defined in Sect. 1.8.

Proof Put $\phi^{S,\infty} := \mathbf{1}_{K^{S,\infty}(\mathfrak{n})}$. The right hand side of (9.6) is expanded as in [3, Thm 6.1] as shown by Arthur. Arguing as at the start of the proof of [99, Thm 4.4], we obtain from Lemma 8.4 in view of the imposed lower bound on $N(\mathfrak{n})$ that

$$\widehat{\mu}_{\mathcal{F}, S_1}(\widehat{\phi}_{S_1}) - \widehat{\mu}_S^{pl}(\widehat{\phi}_S) = \sum_{M \in \mathscr{L}_c(M_0) \backslash \{G\}} a_M \cdot \phi_{S_0, M}(1)\phi_{S_1, M}(1)\phi_M^{S,\infty}(1)$$
$$\times \frac{\Phi_M^G(1, \xi)}{\dim \xi}, \tag{9.9}$$

where the sum runs over proper cuspidal Levi subgroups of G containing a fixed minimal F-rational Levi subgroup (see [45, p. 539] for the reason why only cuspidal Levi subgroups contribute) and $a_M \in \mathbb{C}$ are explicit constants depending only on M and G. A further explanation of (9.9) needs to be given. Since only semisimple conjugacy classes contribute to Arthur's trace formula for each M, Lemma 8.4 tells us that any contribution from non-identity elements vanishes. Note that $\widehat{\mu}_S^{pl}(\widehat{\phi}_S)$ comes from the $M = G$ term on the right hand side.

The first assertion of the theorem follows immediately from (9.9). Henceforth we may assume that $\mathscr{L}_c(M_0) \backslash \{G\} \neq \emptyset$.

Clearly $\phi_{S_0,M}(1)$ and $\Phi_M^G(1,\xi)/\dim\xi$ are constants. It was shown in Lemma 2.14 that $|\phi_{S_1,M}(1)| = O(q_{S_1}^{d_G+r_G+b_G\kappa})$ for $b_G > 0$ in that lemma. We take

$$A_{\mathrm{lv}} := d_G + r_G \quad \text{and} \quad B_{\mathrm{lv}} := b_G.$$

We will be done if it is checked that $|\phi_M^{S,\infty}(1)| = O(\mathbb{N}(\mathfrak{n})^{-C_{\mathrm{lv}}})$ for some $C_{\mathrm{lv}} \geqslant 1$. Let $P = MN$ be a parabolic subgroup with Levi decomposition where M is as above. Then

$$0 \leqslant \phi_M^{S,\infty}(1) = \int_{N(\mathbb{A}_F^{S,\infty})} \phi^{S,\infty}(n)dn$$

$$= \prod_{\substack{v\notin S \\ v|\mathfrak{n} \text{ or } v\in\mathrm{Ram}(G)}} \mathrm{vol}(K_v(\varpi_v^{v(\mathfrak{n})}) \cap N(F_v))$$

$$= \prod_{\substack{v\notin S \\ v|\mathfrak{n} \text{ or } v\in\mathrm{Ram}(G)}} \mathrm{vol}(N(F_v)_{x,v(\mathfrak{n})})$$

$$= \left(\prod_{\substack{v|\mathfrak{n} \\ v\notin S}} q_v^{-v(\mathfrak{n})\dim N}\right) \prod_{\substack{v\in\mathrm{Ram}(G) \\ v\notin S}} \mathrm{vol}(K_v \cap N(F_v)).$$

The last equality uses the standard fact about the filtration that $\mathrm{vol}(N(F_v)_{x,v(\mathfrak{n})}) = |\varpi_v|^{v(\mathfrak{n})\dim N}\mathrm{vol}(N(F_v)_{x,0})$ and the fact (8.1) that $\mathrm{vol}(N(F_v)_{x,0}) = \mathrm{vol}(N(F_v) \cap K_v) = 1$ when G is unramified at v. Take

$$C_{\mathrm{lv}} := \min_{\substack{M\in\mathscr{L}_c(M_0)\setminus\{G\} \\ P=MN}} (\dim N)$$

to be the minimum dimension of the unipotent radical of a proper parabolic subgroup of G with cuspidal Levi part. Then $|\phi_M^{S,\infty}(1)| \leqslant \mathbb{N}(\mathfrak{n})^{-C_{\mathrm{lv}}}\prod_{v\in\mathrm{Ram}(G)} \mathrm{vol}(K_v \cap N(F_v))$ for every M in (9.9). $\qquad\square$

9.5 Weight aspect

We put ourselves in the setting of Example 9.14 and exclude the uninteresting case of $G = \{1\}$. By the assumption $Z(G) = \{1\}$, for every $\gamma \neq 1 \in G(F)$ the connected centralizer I_γ has a strictly smaller set of roots so that $|\Phi_{I_\gamma}| < |\Phi|$. Our next task is to prove a similar error bound as in the last subsection.

Theorem 9.19 *Fix $\phi_{S_0} \in C_c^\infty(G(F_{S_0}))$ and $U^{S,\infty} \subset G(\mathbb{A}^{S,\infty})$. There exist constants $A_{\mathrm{wt}}, B_{\mathrm{wt}} > 0$ and $C_{\mathrm{wt}} \geqslant 1$ satisfying the following: for*

– *any $\kappa \in \mathbb{Z}_{>0}$,*
– *any finite subset $S_1 \subset \mathcal{V}_F^\infty$ disjoint from S_0 and S_{bad} (Sect. 7.2) and*
– *any $\phi_{S_1} \in \mathcal{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leqslant \kappa}$ such that $|\phi_{S_1}| \leqslant 1$ on $G(F_{S_1})$,*

$$\widehat{\mu}_{\mathcal{F}, S_1}(\widehat{\phi}_{S_1}) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S) = O\left(q_{S_1}^{A_{\mathrm{wt}} + B_{\mathrm{wt}}\kappa} m(\xi)^{-C_{\mathrm{wt}}}\right)$$

where the implicit constant in $O(\cdot)$ is independent of κ, S_1 and ϕ_{S_1}. (Equivalently, $\widehat{\mu}_{\mathcal{F}, S_1}^{\natural}(\widehat{\phi}_{S_1}) - \widehat{\mu}_{S_1}^{\mathrm{pl}}(\widehat{\phi}_{S_1}) = O(q_{S_1}^{A_{\mathrm{wt}} + B_{\mathrm{wt}}\kappa} m(\xi)^{-C_{\mathrm{wt}}})$ if $\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\phi}_{S_0}) \neq 0$.)

Remark 9.20 We always assume that S_0 and S_1 are disjoint. So the condition on S_1 is really that it stays away from the finite set S_{bad}. This enters the proof where a uniform bound on orbital integrals from Sect. 7.2 is applied to the places in S_1.

Remark 9.21 Again $A_{\mathrm{wt}}, B_{\mathrm{wt}}, C_{\mathrm{wt}}$ can be chosen explicitly as can be seen from the proof below. For instance a choice can be made such that $C_{\mathrm{wt}} \geqslant n_G$ for n_G defined in Sect. 1.8.

Proof We can choose a sufficiently large finite set $S_0' \supset S_0 \cup \mathrm{Ram}(G)$ in the complement of $S_1 \cup S_\infty$ such that $U^{S,\infty}$ is a finite disjoint union of groups of the form $(\prod_{v \notin S_0' \cup S_1 \cup S_\infty} K_v) \times U_{S_0' \backslash S_0}$ for open compact subgroups $U_{S_0' \backslash S_0}$ of $G(\mathbb{A}_{F, S_0' \backslash S_0})$. By replacing S_0 with S_0' (and thus S with $S_0' \coprod S_1$), we reduce the proof to the case where $U^{S,\infty} = \prod_{v \notin S \cup S_\infty} K_v$.

For an F-rational Levi subgroup M of G, let Y_M be as in Proposition 8.7, where κ, S_0 and S_1 are as in the theorem. (So the set Y_M varies as κ and S_1 vary.) Take (9.6) as a starting point. Arthur's trace formula ([3, Thm 6.1]) and the argument in the proof of [99, Thm 4.11] show (note that our Y_M contains Y_M of [99] but could be strictly bigger):

$$\widehat{\mu}_{\mathcal{F}, S_1}(\widehat{\phi}_{S_1}) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S)$$

$$= \sum_{\gamma \in Y_G \backslash \{1\}} a_{G,\gamma} \cdot |\iota^G(\gamma)|^{-1} O_\gamma^{G(\mathbb{A}_F^\infty)}(\phi^\infty) \frac{\mathrm{tr}\, \xi_n(\gamma)}{\dim \xi_n}$$

$$+ \sum_{M \in \mathscr{L}_c \backslash \{G\}} \sum_{\gamma \in Y_M} a_{M,\gamma} \cdot |\iota^M(\gamma)|^{-1} O_\gamma^{M(\mathbb{A}_F^\infty)}(\phi_M^\infty) \frac{\Phi_M^G(\gamma, \xi_n)}{\dim \xi_n} \quad (9.10)$$

where $a_{M,\gamma}$ (including $M = G$) is given by

$$a_{M,\gamma} \;=\; \tau'(G)^{-1} \frac{\overline{\mu}^{\mathrm{can,EP}}\left(I_\gamma^M(F)\backslash I_\gamma^M(\mathbb{A}_F)/A_{I_\gamma^M,\infty}\right)}{\overline{\mu}^{\mathrm{EP}}\left(I_\gamma^M(F_\infty)/A_{I_\gamma^M,\infty}\right)}$$

$$\overset{\mathrm{Cor\ 6.13}}{=} \frac{\tau\left(I_\gamma^M\right)}{\tau'(G)} \frac{|\Omega_{I_\gamma^M}|}{|\Omega_{I_\gamma^M,c}|} \frac{L(\mathrm{Mot}_{I_\gamma^M})}{e\left(I_{\gamma,\infty}^M\right) 2^{[F:\mathbb{Q}]r_G}}.$$

Let us work with one cuspidal Levi subgroup M at a time. Observe that clearly $|\Omega_{I_\gamma^M}|/|\Omega_{I_\gamma^M,c}| \leqslant |\Omega|$ and that $\tau(I_\gamma^M)$ is bounded by a constant depending only on G in view of (6.3) and Corollary 8.12 or Lemma 8.13.

By Corollary 6.17, there exist constants c_2, $A_2 > 0$ such that

$$|a_{M,\gamma}| \leqslant c_2 \prod_{v \in \mathrm{Ram}(I_\gamma^M)} q_v^{A_2}$$

It is convenient to define the following finite subset of $\mathcal{V}_F^\infty$ for each $\gamma \in Y_M$. We fix a maximal torus T_γ^M in M over $\overline{F}$ containing γ and write $\Phi_{M,\gamma}$ for the set of roots of T_γ^M in M. (A different choice of T_γ^M does not affect the argument.)

$$S_{M,\gamma} := \left\{ v \in \mathcal{V}_F^\infty \backslash S : \exists \alpha \in \Phi_{M,\gamma},\ \alpha(\gamma) \neq 1 \text{ and } |1 - \alpha(\gamma)|_v \neq 1 \right\}.$$

(If γ is in the center of $M(F)$ then $S_{M,\gamma} = \emptyset$ and $q_{S_{M,\gamma}} = 1$.)

We know that $O_\gamma^{M(F_v)}(\mathbf{1}_{K_{M,v}}) = 1$ for $v \notin S \cup S_{M,\gamma} \cup S_\infty$ and that $S_{M,\gamma} \supset \mathrm{Ram}(I_\gamma^M)$ from [63, Cor 7.3]. According to Lemma 6.2 $\phi_v = \mathbf{1}_{K_v}$ implies $\phi_{v,M} = \mathbf{1}_{K_{M,v}}$. Hence

$$|a_{M,\gamma}| \leqslant c_2 \cdot (q_{S_{M,\gamma}})^{A_2} \tag{9.11}$$

$$O_\gamma^{M(\mathbb{A}_F^\infty)}(\phi_M^\infty) = O_\gamma^{M(F_S)}(\phi_{S,M}) \prod_{v \in S_{M,\gamma}} O_\gamma^{M(F_v)}(\mathbf{1}_{K_{M,v}}).$$

By Theorem 13.1, there exists a constant $c(\phi_{S_0,M}) > 0$ such that

$$O_\gamma^{M(F_{S_0})}(\phi_{S_0,M}) \leqslant c(\phi_{S_0,M}) \prod_{v \in S_0} D_v^M(\gamma)^{-1/2}, \quad \forall \gamma \in Y_M.$$

By Theorem 7.3, there exist $a, b, c, e_G \in \mathbb{R}_{\geqslant 0}$ (independent of γ, S_1, κ and k) such that

$$O_\gamma^{M(F_{S_1})}(\phi_{S_1,M}) \leqslant q_{S_1}^{a+b\kappa} \prod_{v \in S_1} D_v^M(\gamma)^{-e_G/2}, \tag{9.12}$$

$$O_\gamma^{M(F_v)}(\mathbf{1}_{K_{M,v}}) \leqslant q_v^c D_v^M(\gamma)^{-e_G/2}, \quad \forall v \in S_{M,\gamma}. \tag{9.13}$$

[To obtain (9.12) and (9.13), apply Theorem 7.3 to $v \in S_1$ and $v \in S_{M,\gamma}$.]
  Hence

$$O_\gamma^{M(\mathbb{A}_F^\infty)}(\phi_M^\infty) \leqslant c(\phi_{S_0,M}) q_{S_1}^{a+b\kappa} q_{S_{M,\gamma}}^c \left(\prod_{v \nmid \infty} D_v^M(\gamma)^{-1/2} \right)$$

$$\times \prod_{v \in S_1 \cup S_{M,\gamma}} D_v^M(\gamma)^{(1-e_G)/2}$$

$$= c(\phi_{S_0,M}) q_{S_1}^{a+b\kappa} q_{S_{M,\gamma}}^c \prod_{v | \infty} D_v^M(\gamma)^{1/2}$$

$$\times \prod_{v \in S_1 \cup S_{M,\gamma}} D_v^M(\gamma)^{(1-e_G)/2} \tag{9.14}$$

On the other hand there exist $\delta_{S_0}, \delta_\infty, \delta_{S_1} \geqslant 1$ such that for every $\gamma \in Y_M$ with $\alpha(\gamma) \neq 1$,

- $|1 - \alpha(\gamma)|_{S_0} \leqslant \delta_{S_0}$. (compactness of supp ϕ_{S_0})
- $|1 - \alpha(\gamma)|_\infty \leqslant \delta_\infty$. (compactness of U_∞)
- $|1 - \alpha(\gamma)|_{S_1} \leqslant \delta_{S_1} q_{S_1}^{B_5\kappa}$. (Lemma 2.18 and Remark 2.20 explains the independence of B_1 of S_1.)

[When $\alpha(\gamma) = 1$, our convention is that $|1 - \alpha(\gamma)|_v = 1$ for every v to be consistent with the first formula of Appendix A.] Hence, together with the product formula for $1 - \alpha(\gamma)$,

$$1 = \prod_v |1 - \alpha(\gamma)|_v \leqslant \delta_{S_0} \delta_\infty \delta_{S_1} q_{S_1}^{B_5\kappa} \prod_{v \in S_{M,\gamma}} |1 - \alpha(\gamma)|_v.$$

Set $\delta := \delta_{S_0} \delta_\infty \delta_{S_1}$. Note that $|1 - \alpha(\gamma)|_v \leqslant 1$ for all $\alpha \in \Phi_{M,\gamma}$ and all $v \in S_{M,\gamma}$. If $\gamma \in Z(M)(F)$ then $q_{S_{M,\gamma}} = 1$. Otherwise for each $v \in S_{M,\gamma}$, we may choose $\alpha \in \Phi_{M,\gamma}$ such that $|1 - \alpha(\gamma)|_v \neq 1$. Then $|1 - \alpha(\gamma)|_v \leqslant q_v^{-1/w_G s_G}$ (for the same reason as in the proof of Proposition 8.7, Step I) so

$$q_v \leqslant \left(\delta q_{S_1}^{B_5\kappa}\right)^{w_G s_G}, \quad v \in S_{M,\gamma}. \tag{9.15}$$

In particular the crude bound $\max_{v \in S_{M,\gamma}} q_v \geqslant 2|S_{M,\gamma}|$ holds, hence

$$|S_{M,\gamma}| \leqslant \frac{1}{2}\left(\left(\delta q_{S_1}^{B_5\kappa}\right)^{w_G s_G} + 1\right) =: \delta'. \tag{9.16}$$

Notice that the upper bound is independent of γ (and depends only on the fixed data). Keep assuming that γ is not central in M and that $\alpha(\gamma) \neq 1$. Again by the product formula $\prod_{v \in S_1 \cup S_{M,\gamma}} |1 - \alpha(\gamma)|_v = \prod_{v \in S_0 \cup S_\infty} |1 - \alpha(\gamma)|_v^{-1} \geqslant (\delta_{S_0} \delta_\infty)^{-1}$, thus

$$\prod_{v \in S_1 \cup S_{M,\gamma}} D_v^M(\gamma)^{-1} \leqslant \delta_{S_0} \delta_\infty. \tag{9.17}$$

The above holds also when γ is central in M, in which case the left hand side equals 1.

Now (9.14), (9.15), (9.16), and (9.17) imply

$$O_\gamma^{M(\mathbb{A}_F^\infty)}\left(\phi_M^\infty\right) \leqslant c(\phi_{S_0,M})\delta^{cw_G s_G \delta'}(\delta_{S_0}\delta_\infty)^{(e_G-1)/2}q_{S_1}^{a+b\kappa+cB_5 w_G s_G \delta'\kappa} \\ \times \prod_{v|\infty} D_v^M(\gamma)^{1/2}. \tag{9.18}$$

Lemma 6.11 gives a bound on the stable discrete series character:

$$\frac{\left|\Phi_M^G(\gamma,\xi)\right|}{\dim \xi} \leqslant c_0 \frac{\prod_{v|\infty} D_v^M(\gamma)^{-1/2}}{m(\xi)^{|\Phi^+|-|\Phi_{I_\gamma^M}^+|}}. \tag{9.19}$$

Multiplying (9.11), (9.18) and (9.19) altogether (and noting $|\iota^M(\gamma)|^{-1} \leqslant 1$), the absolute value of the summand for γ in (9.10) (including $M = G$) is

$$O\left(m(\xi)^{-(|\Phi^+|-|\Phi_{I_\gamma^M}^+|)}q_{S_1}^{a+b\kappa+cB_5 w_G s_G \delta'\kappa+A_2}\right).$$

All in all, $|\widehat{\mu}_{\mathcal{F},S}(\widehat{\phi}_S) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S)|$ is

$$\left(|Y_G| - 1 + \sum_{M \in \mathscr{L}_c \setminus \{G\}} |Y_M|\right) O\left(m(\xi)^{-(|\Phi^+|-|\Phi_{I_\gamma^M}^+|)}q_{S_1}^{a+b\kappa+cB_5 w_G s_G \delta'\kappa+A_2}\right).$$

Set (excluding $\gamma = 1$ in the second minimum when $M = G$)

$$C_{\text{wt}} := \min_{M \in \mathscr{L}_c(M_0)} \min_{\substack{\gamma \in M(F) \\ \text{ell.in} M(F_\infty)}} (|\Phi^+| - |\Phi^+_{I^M_\gamma}|)$$

Note that C_{wt} depends only on G. It is automatic that $|\Phi^+| - |\Phi^+_{I^M_\gamma}| \geqslant 1$ on $Y_G \backslash \{1\}$ and Y_M for $M \in \mathscr{L}_c(M_0) \backslash \{G\}$. The proof is concluded by invoking Corollary 8.10 (applied to Y_G and Y_M) with the choice

$$A_{\text{wt}} := a + A_2 + A_6, \quad B_{\text{wt}} := b + cB_5 w_G s_G \delta' + B_8.$$

$\square$

9.6 Automorphic Plancherel density theorem

In the situation of either Examples 9.13 or 9.14, let us write $\mathcal{F}_k(\phi_{S_0})$ for $\mathcal{F}_k$ in order to emphasize the dependence on ϕ_{S_0}. Take $S_1 = \emptyset$ so that $S = S_0$. Then $\widehat{\mu}_{\mathcal{F}_k(\phi_S),\emptyset}$ may be viewed as a complex number (as it is a measure on a point). In fact we can consider $\mathcal{F}_k(\widehat{f_S})$, a family whose local condition at S is prescribed by $\widehat{f_S} \in \mathscr{F}(G(F_S)^\wedge)$, even if $\widehat{f_S}$ does not arise from any ϕ_S in $C_c^\infty(G(F_S))$. Put $\widehat{\mu}_k(\widehat{f_S}) := \widehat{\mu}_{\mathcal{F}_k(\widehat{f_S}),\emptyset} \in \mathbb{C}$. We recover the automorphic Plancherel density theorem [99, Thms 4.3, 4.7].

Corollary 9.22 *Consider families $\mathcal{F}_k$ in level or weight aspect as above. In level aspect assume that the highest weight of ξ is regular. (No assumption is necessary in the weight aspect.) For any $\widehat{f_S} \in \mathscr{F}(G(F_S)^\wedge)$,*

$$\lim_{k \to \infty} \widehat{\mu}_k(\widehat{f_S}) = \widehat{\mu}^{\text{pl}}_S(\widehat{f_S}).$$

Proof Theorems 9.16 and 9.19 tell us that

$$\lim_{k \to \infty} \widehat{\mu}_k(\widehat{\phi_S}) = \widehat{\mu}^{\text{pl}}_S(\widehat{\phi_S}). \tag{9.20}$$

(Even though there was a condition on S_1, note that there was no condition on S_0 in either theorem.)

We would like to improve (9.20) to allow more general test functions. What needs to be shown [cf. (9.21) below] is that for every $\epsilon > 0$,

$$\limsup_{k \to \infty} \left| \widehat{\mu}_k(\widehat{f_S}) - \widehat{\mu}^{\text{pl}}_S(\widehat{f_S}) \right| \leqslant 4\epsilon.$$

Thanks to Proposition 9.1 there exist $\phi_S, \psi_S \in \mathcal{H}^{\text{ur}}(G(F_S))$ such that $|\widehat{f_S} - \widehat{\phi_S}| \leqslant \widehat{\psi_S}$ on $G(F_S)^\wedge$ and $\widehat{\mu}^{\text{pl}}_S(\widehat{\psi_S}) \leqslant \epsilon$. Then [cf. (9.22) below]

$$\left|\widehat{\mu}_k(\widehat{f}_S) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{f}_S)\right| \leqslant |\widehat{\mu}_k(\widehat{f}_S - \widehat{\phi}_S)|$$
$$+ \left|\widehat{\mu}_k(\widehat{\phi}_S) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S)\right| + \left|\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S - \widehat{f}_S)\right|.$$

Now $|\widehat{\mu}_S^{\mathrm{pl}}(\widehat{f}_S - \widehat{\phi}_S)| \leqslant |\widehat{\mu}_S^{\mathrm{pl}}(\widehat{\psi}_S)| \leqslant \epsilon$, and $|\widehat{\mu}_k(\widehat{\phi}_S) - \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\phi}_S)| \leqslant \epsilon$ for $k \gg 1$ by (9.20). Finally $\widehat{\mu}_k$ is a positive measure since the highest weight of ξ is regular (see Example 9.11), and we get

$$\left|\widehat{\mu}_k(\widehat{f}_S - \widehat{\phi}_S)\right| \leqslant \widehat{\mu}_k\left(|\widehat{f}_S - \widehat{\phi}_S|\right) \leqslant \widehat{\mu}_k(\widehat{\psi}_S).$$

[To see the positivity of $\widehat{\mu}_k$, notice that $\widehat{\mu}_k(\widehat{f}_S - \widehat{\phi}_S)$ is unraveled via (9.4) and (9.5) as a sum of $(\widehat{f}_S - \widehat{\phi}_S)(\pi)$ with coefficients having nonnegative signs. This is because $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi)$ is either 0 or $(-1)^{q(G)}$ when ξ has regular highest weight, cf. Sect. 6.5.] According to (9.20), $\lim_{k\to\infty} \widehat{\mu}_k(\widehat{\psi}_S) = \widehat{\mu}_S^{\mathrm{pl}}(\widehat{\psi}_S) \leqslant \epsilon$. In particular $|\widehat{\mu}_k(\widehat{f}_S - \widehat{\phi}_S)| \leqslant 2\epsilon$ for $k \gg 1$. The proof is complete. □

Remark 9.23 If G is anisotropic modulo center over F so that the trace formula for compact quotients is available, or if a further local assumption at finite places is imposed so as to avail the simple trace formula, the regularity condition on ξ can be removed by an argument of De George and Wallach [38] and Clozel [22]. The main point is to show that the contribution of (ξ-cohomological) non-tempered representations at ∞ to the trace formula is negligible compared to the contribution of discrete series. Their argument requires some freedom of choice of test functions at ∞, so it breaks down in the general case since one has to deal with new terms in the trace formula which disappear when Euler–Poincaré functions are used at ∞. In other words, it seems necessary to prove analytic estimates on more terms (if not all terms) in the trace formula than we did in order to get rid of the assumption on ξ. (This remark also applies to the same condition on ξ in Sects. 9.7 and 9.8 for level aspect families.) We may return to this issue in future work.

Remark 9.24 In the case of level aspect families [99, Thm 4.3] assumes that the level subgroups form a chain of decreasing groups whose intersection is the trivial group. The above corollary deals with some new cases as it assumes only that $\mathbb{N}(\mathfrak{n}_k) \to \infty$.

Corollary 9.25 *Keep assuming that $S_1 = \emptyset$. Let $(U_k^{S,\infty}, \xi_k) = (K^{S,\infty}(\mathfrak{n}_k), \xi)$ or $(U^{S,\infty}, \xi_k)$ in Examples 9.13 or 9.14, respectively, but prescribe local conditions at S by $\widehat{f}_S$ rather than ϕ_S. Then*

$$\lim_{k\to\infty} \frac{\mu^{\mathrm{can}}\left(U_k^{S,\infty}\right)}{\tau'(G)\dim\xi_k} |\mathcal{F}_k| = \widehat{\mu}_S^{\mathrm{pl}}(\widehat{f}_S).$$

Proof The corollary results from Corollary 9.22 since

$$\frac{\mu^{\mathrm{can}}\left(U_k^{S,\infty}\right)}{\tau'(G)\dim\xi_k}|\mathcal{F}_k| = \frac{\mu^{\mathrm{can}}\left(U_k^{S,\infty}\right)}{\tau'(G)\dim\xi_k}\sum_{\pi\in\mathcal{AR}_{\mathrm{disc},\chi_k}(G)}a_{\mathcal{F}_k}(\pi) = \widehat{\mu}_{\mathcal{F}_k,\emptyset}(\widehat{f}_S).$$

$\square$

9.7 Application to the Sato–Tate conjecture for families

As an application of Theorems 9.16 and 9.19, we are about to fulfill the promise of Sect. 5.4 by showing that the Satake parameters in the automorphic families $\{\mathcal{F}_k\}$ are equidistributed according to the Sato–Tate measure in a suitable sense (cf. Conjecture 5.9).

The notation and convention of Sect. 5 are retained here. Let $\theta\in\mathscr{C}(\Gamma_1)$ and $\widehat{f}\in\mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$. For each $v\in\mathcal{V}_F(\theta)$, the image of $\widehat{f}$ in $\mathcal{F}(G(F_v)^{\wedge,\mathrm{ur}})$ via (9.1) will be denoted $\widehat{f}_v$.

Theorem 9.26 (Level aspect) *Pick any $\theta\in\mathscr{C}(\Gamma_1)$ and let $\{v_j\}_{j\geqslant 1}$ be a sequence in $\mathcal{V}_F(\theta)$ such that $q_{v_j}\to\infty$ as $j\to\infty$. Suppose that*

- $\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\phi}_{S_0})\neq 0$ *and*
- ξ *has regular highest weight.*

Then for every $\widehat{f}\in\mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$,

$$\lim_{(j,k)\to\infty}\widehat{\mu}_{\mathcal{F}_k,v_j}^{\natural}\left(\widehat{f}_{v_j}\right) = \widehat{\mu}_\theta^{\mathrm{ST}}(\widehat{f})$$

where the limit is taken over (j,k) subject to the following conditions:

- $\mathrm{N}(\mathfrak{n}_k)q_{v_j}^{-B_\Xi m\kappa}\geqslant c_\Xi^{-1}$,
- $v_j\nmid\mathfrak{n}_k$,
- $q_{v_j}^N\mathrm{N}(\mathfrak{n}_k)^{-1}\to 0$ *for all $N>0$.*

Proof Fix $\widehat{f}$. We are done if $\limsup_{(j,k)\to\infty}|\widehat{\mu}_{\mathcal{F}_k,v_j}^{\natural}(\widehat{f}_{v_j}) - \widehat{\mu}_\theta^{\mathrm{ST}}(\widehat{f})|\leqslant 4\epsilon$ for every $\epsilon>0$. By Proposition 5.3, $|\widehat{\mu}_{v_j}^{\mathrm{pl}}(\widehat{f}_{v_j}) - \widehat{\mu}_\theta^{\mathrm{ST}}(\widehat{f})|\leqslant\epsilon$ for sufficiently large j. So it is enough to show that

$$\limsup_{(j,k)\to\infty}\left|\widehat{\mu}_{\mathcal{F}_k,v_j}^{\natural}(\widehat{f}_{v_j}) - \widehat{\mu}_{v_j}^{\mathrm{pl}}(\widehat{f}_{v_j})\right|\leqslant 3\epsilon. \tag{9.21}$$

For every $j\geqslant 1$, Proposition 9.4 allows us to find $\phi_{v_j},\psi_{v_j}\in\mathcal{H}^{\mathrm{ur}}(G(F_{v_j}))^{\leqslant\kappa}$ such that $|\widehat{f}_{v_j}-\widehat{\phi}_{v_j}|\leqslant\widehat{\psi}_{v_j}$ on $G(F_{v_j})^{\wedge}$ and $\widehat{\mu}_{v_j}^{\mathrm{pl}}(\widehat{\psi}_{v_j})\leqslant\epsilon$.

For each $j \geqslant 1$,

$$
\begin{aligned}
\left|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j}) - \widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{f}_{v_j})\right| \leqslant{} & \left|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j} - \widehat{\phi}_{v_j})\right| \\
& + \left|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}(\widehat{\phi}_{v_j}) - \widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{\phi}_{v_j})\right| + \left|\widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{\phi}_{v_j} - \widehat{f}_{v_j})\right|.
\end{aligned}
\tag{9.22}
$$

Since $\widehat{\mu}^{\mathrm{pl}}_{v_j}$ is a positive measure,

$$
\left|\widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{\phi}_{v_j} - \widehat{f}_{v_j})\right| \leqslant \widehat{\mu}^{\mathrm{pl}}_{v_j}\left(|\widehat{\phi}_{v_j} - \widehat{f}_{v_j}|\right) \leqslant \widehat{\mu}^{\mathrm{pl}}_{v_j}\left(\widehat{\psi}_{v_j}\right) \leqslant \epsilon.
$$

Theorem 9.16 and the assumptions of the theorem imply that for sufficiently large (j, k), $|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}(\widehat{\phi}_{v_j}) - \widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{\phi}_{v_j})| \leqslant \epsilon$. So we will be done if for sufficiently large (j, k),

$$
\left|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}\left(\widehat{f}_{v_j} - \widehat{\phi}_{v_j}\right)\right| \leqslant \epsilon.
\tag{9.23}
$$

Arguing as in the proof of Corollary 9.22 we deduce the following: when $\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j} - \widehat{\phi}_{v_j})$ is unraveled as a sum over π [cf. (9.4) and (9.5)], each summand is $\widehat{\phi}_{S_0}(\pi_{S_0})(\widehat{f}_{v_j} - \widehat{\phi}_{v_j})(\pi_{v_j})$ times a nonnegative real number. (This uses the regularity assumption on ξ. Certainly the absolute value of the sum does not get smaller when every summand is replaced with (something greater than or equal to) its absolute value, i.e.

$$
\left|\widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}\left(\widehat{f}_{v_j} - \widehat{\phi}_{v_j}\right)\right| \leqslant \widehat{\mu}^{\natural}_{\mathcal{F}_k(|\widehat{\phi}_{S_0}|), v_j}\left(|\widehat{f}_{v_j} - \widehat{\phi}_{v_j}|\right) \leqslant \widehat{\mu}^{\natural}_{\mathcal{F}_k(|\widehat{\phi}_{S_0}|), v_j}(\widehat{\psi}_{v_j}).
$$

Now choose $\phi'_{S_0} \in C^\infty_c(G(F_{S_0}))$ according to Lemma 9.6 so that $|\phi_{S_0}(\pi_{S_0})| \leqslant \phi'_{S_0}(\pi_{S_0})$ for every $\pi_{S_0} \in G(F_{S_0})^\wedge$. Then

$$
\widehat{\mu}^{\natural}_{\mathcal{F}_k(|\widehat{\phi}_{S_0}|), v_j}(\widehat{\psi}_{v_j}) \leqslant \widehat{\mu}^{\natural}_{\mathcal{F}_k(\phi'_{S_0}), v_j}(\widehat{\psi}_{v_j}).
$$

Theorem 9.16 applied to $\widehat{\psi}_{v_j}$ and the inequality $\widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{\psi}_{v_j}) \leqslant \epsilon$ imply that

$$
\limsup_{(j,k) \to \infty} \widehat{\mu}^{\natural}_{\mathcal{F}_k(\phi'_{S_0}), v_j}\left(\widehat{\psi}_{v_j}\right) \leqslant \epsilon.
$$

This concludes the proof of (9.23), thus also (9.21). $\qquad\square$

Theorem 9.27 (Weight aspect) *Let $\theta \in \mathscr{C}(\Gamma_1)$ and $\widehat{\phi}_{S_0} \in C^\infty_c(G(F_{S_0}))$. Suppose that $\{v_j\}_{j \geqslant 1}$ is a sequence in $\mathcal{V}_F(\theta)$ such that $q_{v_j} \to \infty$ as $j \to \infty$ and that $\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{\phi}_{S_0}) \neq 0$. Then for every $\widehat{f} \in \mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$,*

$$\lim_{(j,k)\to\infty} \widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}\left(\widehat{f}_{v_j}\right) = \widehat{\mu}^{ST}_{\theta}(\widehat{f})$$

if $q_{v_j}^N m(\xi_k)^{-1} \to 0$ as $k \to \infty$ for any integer $N \geqslant 1$.

Proof Same as above, except that Theorem 9.19 is used instead of Theorem 9.16. □

Remark 9.28 As we have mentioned in Sect. 5.4, Theorems 9.26 and 9.27 indicate that $\{\mathcal{F}_k\}_{\geqslant 1}$ are "general" families of automorphic representations in the sense of Conjecture 5.9.

Corollary 9.29 *In the setting of Theorems 9.26 or 9.27, suppose in addition that $|\mathcal{F}_k| \neq 0$ for all $k \geqslant 1$. Then*

$$\lim_{(j,k)\to\infty} \widehat{\mu}^{count}_{\mathcal{F}_k, v_j}\left(\widehat{f}_{v_j}\right) = \widehat{\mu}^{ST}_{\theta}(\widehat{f}).$$

Proof Follows from Corollary 9.25 and the two preceding theorems (cf. Remark 9.9). □

Remark 9.30 The assumption that $|\mathcal{F}_k| \neq 0$ is almost automatically satisfied. Corollary 9.25 and the assumption that $\widehat{\mu}^{pl}_{S_0}(\widehat{\phi}_{S_0}) \neq 0$ imply that $|\mathcal{F}_k| \neq 0$ for any sufficiently large k.

9.8 More general test functions at S_0

So far we worked primarily with families of Examples 9.13 and 9.14. We wish to extend Theorems 9.26 and 9.27 when the local condition at S_0 is given by $\widehat{f}_{S_0}$, which may not be of the form $\widehat{\phi}_{S_0}$ for any $\phi_{S_0} \in C_c^{\infty}(G(F_{S_0}))$ (cf. Example 9.11 and Remark 9.12).

Corollary 9.31 *Let $\theta \in \mathscr{C}(\Gamma_1)$ and let $\{v_j\}_{j\geqslant 1}$ be a sequence of places in $\mathcal{V}_F(\theta)$ such that $q_{v_j} \to \infty$ as $j \to \infty$. Consider $\widehat{\mu}_{\mathcal{F}_k, v_j}$ where*

$$\mathcal{F}_k = \begin{cases} \mathcal{F}\left(K^{S,\infty}(\mathfrak{n}_k), \widehat{f}_{S_0}, v_j, \xi\right) & \text{level aspect, or} \\ \mathcal{F}\left(U^{S,\infty}, \widehat{f}_{S_0}, v_j, \xi_k\right) & \text{weight aspect} \end{cases}$$

satisfying the conditions of Theorems 9.26 or 9.27, respectively. Then

$$\lim_{(j,k)\to\infty} \widehat{\mu}^{\natural}_{\mathcal{F}_k, v_j}\left(\widehat{f}_{v_j}\right) = \widehat{\mu}^{ST}_{\theta}(\widehat{f})$$

where the limit is taken as in Theorem 9.26 (resp. Theorem 9.27).

Proof The basic strategy is to reduce to the case of $\widehat{\phi}$ and $\widehat{\phi}_{v_j}$ in place of $\widehat{f}$ and $\widehat{f}_{v_j}$ via Sauvageot's density theorem, as in the proof of Theorem 9.26. We can decompose $\widehat{f} = \widehat{f}^+ + \widehat{f}^-$ with $\widehat{f}^+, \widehat{f}^- \in \mathcal{F}(\widehat{T}_{c,\theta}/\Omega_{c,\theta})$ such that $\widehat{f}^+$ and $\widehat{f}^-$ are nonnegative everywhere. The corollary for $\widehat{f}$ is proved as soon as it is proved for $\widehat{f}^+$ and $\widehat{f}^-$. Thus we may assume that $\widehat{f} \geqslant 0$ from now on.

Fix any choice of $\epsilon > 0$. Proposition 9.1 ensures the existence of $\phi_{S_0}, \psi_{S_0} \in C_c^\infty(G(F_{S_0}))$ such that $\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\psi}_{S_0}) \leqslant \epsilon$ and $|\widehat{f}_{S_0}(\pi_{S_0}) - \widehat{\phi}_{S_0}(\pi_{S_0})| \leqslant \widehat{\psi}_{S_0}(\pi_{S_0})$ for all $\pi_{S_0} \in G(F_{S_0})^\wedge$. Of course we can guarantee in addition that $\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\phi}_{S_0}) \neq 0$. Put

$$\mathcal{F}_k(\widehat{\phi}_{S_0}) := \mathcal{F}\left(K^{S,\infty}(\mathfrak{n}_k), \widehat{\phi}_{S_0}, v_j, \xi\right)$$

$$\left(\text{resp. } \mathcal{F}_k(\widehat{\phi}_{S_0}) = \mathcal{F}(U^{S,\infty}, \widehat{\phi}_{S_0}, v_j, \xi_k)\right).$$

Likewise we define $\mathcal{F}_k(\widehat{\psi}_{S_0})$ and so on. Then (cf. a similar step in the proof of Theorem 9.26)

$$\left|\widehat{\mu}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j}) - \widehat{\mu}_{S_0 \cup \{v_j\}}^{\mathrm{pl}}\left(\widehat{f}_{S_0}\widehat{f}_{v_j}\right)\right| \leqslant \left|\widehat{\mu}_{\mathcal{F}_k(\widehat{\phi}_{S_0}), v_j}(\widehat{f}_{v_j}) - \widehat{\mu}_{S_0 \cup \{v_j\}}^{\mathrm{pl}}(\widehat{\phi}_{S_0}\widehat{f}_{v_j})\right|$$

$$+ \left|\widehat{\mu}_{\mathcal{F}_k(|\widehat{f}_{S_0} - \widehat{\phi}_{S_0}|)}(\widehat{f}_{v_j})\right| + \widehat{\mu}_{S_0 \cup \{v_j\}}^{\mathrm{pl}}\left(\left|\widehat{f}_{S_0} - \widehat{\phi}_{S_0}\right|\widehat{f}_{v_j}\right)$$

The first term on the right side tends to 0 as $(j, k) \to \infty$ by Theorems 9.26 and 9.27. The last term is bounded by $\widehat{\mu}_{S_0 \cup \{v_j\}}^{\mathrm{pl}}(\widehat{\psi}_{S_0}\widehat{f}_{v_j}) \leqslant \epsilon \widehat{\mu}_{v_j}^{\mathrm{pl}}(\widehat{f}_{v_j})$ using the fact that $\widehat{\mu}_{S_0}^{\mathrm{pl}}$ is a positive measure. In order to bound the second term, recall that we are either in the weight aspect, or in the level aspect with regular highest weight for ξ. Then $a_{\mathcal{F}_k(|\widehat{f}_{S_0} - \widehat{\phi}_{S_0}|)}(\pi)$ is a nonnegative multiple of $|\widehat{f}_{S_0}(\pi_{S_0}) - \widehat{\phi}_{S_0}(\pi_{S_0})|$ as in the proof of Theorem 9.26. Thus

$$\left|\widehat{\mu}_{\mathcal{F}_k(|\widehat{f}_{S_0} - \widehat{\phi}_{S_0}|)}(\widehat{f}_{v_j})\right| = \widehat{\mu}_{\mathcal{F}_k(|\widehat{f}_{S_0} - \widehat{\phi}_{S_0}|)}(\widehat{f}_{v_j}) \leqslant \widehat{\mu}_{\mathcal{F}_k(\widehat{\psi}_{S_0})}(\widehat{f}_{v_j}) \leqslant \epsilon \widehat{\mu}_{\mathcal{F}_k(\widehat{\psi}_{S_0})}^{\natural}(\widehat{f}_{v_j}),$$

the last inequality coming from the bound $\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\psi}_{S_0}) \leqslant \epsilon$.

Hence we have shown that

$$\limsup_{(j,k) \to \infty} \left|\widehat{\mu}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j}) - \widehat{\mu}_{S_0 \cup \{v_j\}}^{\mathrm{pl}}(\widehat{f}_{S_0}\widehat{f}_{v_j})\right|$$

$$\leqslant \epsilon \limsup_{(j,k) \to \infty} \times \left(\widehat{\mu}_{\mathcal{F}_k(\widehat{\psi}_{S_0})}^{\natural}(\widehat{f}_{v_j}) + \widehat{\mu}_{v_j}^{\mathrm{pl}}(\widehat{f}_{v_j})\right).$$

By Theorems 9.26 and 9.27 and the fact that $\lim_{j \to \infty} \widehat{\mu}^{\mathrm{pl}}_{v_j}(\widehat{f}_{v_j}) = \widehat{\mu}^{\mathrm{ST}}_\theta(\widehat{f})$, the right hand side is seen to be bounded by $2\epsilon\widehat{\mu}^{\mathrm{ST}}_\theta(\widehat{f})$. As we are free to choose $\epsilon > 0$, we deduce that

$$\lim_{(j,k) \to \infty} \widehat{\mu}_{\mathcal{F}_k, v_j}(\widehat{f}_{v_j}) = \widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{f}_{S_0})\widehat{\mu}^{\mathrm{ST}}_\theta(\widehat{f}).$$

□

Remark 9.32 It would be desirable to improve Theorems 9.16 and 9.19 similarly by prescribing conditions at S_0 in terms of $\widehat{f}_{S_0}$ rather than the less general $\widehat{\phi}_{S_0}$. Unfortunately the argument proving Corollary 9.31 does not carry over. For instance in the case of Theorem 9.16, one should know in addition that the multiplicative constant implicit in $O(q_{S_1}^{A_{\mathrm{lv}}+B_{\mathrm{lv}}\kappa}\mathbb{N}(\mathfrak{n}_k)^{-C_{\mathrm{lv}}})$ is bounded as a sequence of $\widehat{\phi}_{S_0}$ approaches $\widehat{f}_{S_0}$.

10 Langlands functoriality

Let $r \colon {}^L G \to \mathrm{GL}_d(\mathbb{C})$ be a representation of ${}^L G$. Let $\pi \in \mathcal{AR}_{\mathrm{disc}, \chi}(G)$ be such that with $\pi_v \in \Pi_{\mathrm{disc}}(\xi_v^\vee)$ for each $v|\infty$ (recall the notation from Sects. 6.4 and 9.2). The Langlands correspondence for $G(F_v)$ [71] associates an L-parameter $\varphi_{\xi_v^\vee} \colon W_{\mathbb{R}} \to {}^L G$ to the L-packet $\Pi_{\mathrm{disc}}(\xi_v^\vee)$, cf. Sect. 6.4.

The following asserts the existence of the functorial lift of π under r as predicted by the Langlands functoriality principle.

Hypothesis 10.1 There exists an automorphic representation Π of $\mathrm{GL}_d(\mathbb{A}_F)$ such that

(i) Π is isobaric,
(ii) $\Pi_v = r_*(\pi_v)$ [defined in (2.9)] when G, r and π are unramified at v,
(iii) Π_v corresponds to $r\varphi_{\xi_v^\vee}$ via the Langlands correspondence for $\mathrm{GL}_d(F_v)$ for all $v|\infty$.

If Π as above exists then it is uniquely determined by (i) and (ii) thanks to the strong multiplicity one theorem. Moreover

Lemma 10.2 *Hypothesis 10.1 (iii) implies that Π_v is tempered for all $v|\infty$.*

Proof Recall the following general fact from [71, §3, (vi)]: let φ be an L-parameter for a real reductive group and $\Pi(\varphi)$ its corresponding L-packet. Then φ has relatively compact image if and only if $\Pi(\varphi)$ contains a tempered representation if and only if $\Pi(\varphi)$ contains only tempered representations. In our case this implies that $\varphi_{\xi_v^\vee}$ has relatively compact image for every $v|\infty$, and the continuity of r shows that the image of $r\varphi_{\xi_v^\vee}$ is also relatively compact. The lemma follows. □

As before let $(\widehat{B}, \widehat{T}, \{X_\alpha\}_{\alpha \in \Delta^\vee})$ denote the $\mathrm{Gal}(\overline{F}/F)$-invariant splitting datum for $\widehat{G}$. Recall that $\lambda_{\xi_v^\vee} \in X^*(\widehat{T})^+$ designates the highest weight for $\xi_v^\vee$. Then $\varphi_{\xi_v^\vee}|_{W_\mathbb{C}}$ is described as

$$\varphi_{\xi_v^\vee}(z) = \left((z/\overline{z})^{\rho + \lambda_{\xi_v^\vee}}, z\right) \in \widehat{G} \times W_\mathbb{C}, \quad \forall z \in W_\mathbb{C} = \mathbb{C}^\times.$$

It is possible to extend $\varphi_{\xi_v^\vee}|_{W_\mathbb{C}}$ to the whole of $W_\mathbb{R}$ but this does not concern us. (The interested reader may consult pp. 183–184 of [65] for instance.) Let $\widehat{\mathbb{T}}$ be a maximal torus of $\mathrm{GL}_d(\mathbb{C})$ containing the image $r(\widehat{T})$, and $\widehat{\mathbb{B}}$ a Borel subgroup containing $\widehat{\mathbb{T}}$. Write $r|_{\widehat{G}} = \oplus_{i \in I} r_i$ as a sum of irreducible $\widehat{G}$-representations. For each $i \in I$, denote by $\lambda(r_i) \in X^*(\widehat{T})$ the $\widehat{B}$-positive highest weight for r_i. Write $\lambda(r_i) = \lambda_0(r_i) + \sum_{\alpha \in \Delta} a(r_i, \alpha) \cdot \alpha^\vee$ for $\lambda_0(r_i) \in X_*(Z(G))_\mathbb{Q}$ and $a(r_i, \alpha) \in \mathbb{Q}_{\geqslant 0}$. Put $|\lambda(r_i)| := \sum_{\alpha \in \Delta} a(r_i, \alpha)$ and

$$M(\xi_v) := \max_{\alpha \in \Delta} \langle \alpha, \lambda_{\xi_v^\vee} \rangle, \quad M(r) := \max_{i \in I} |\lambda(r_i)|.$$

Similarly define $m(\xi_v)$ and $m(r)$ by using minima in place of maxima. We are interested in the case where $\lambda_0(r_i)$ is trivial for every $i \in I$. This is automatically true if $Z(G)$ is finite. (Recall that we consistently assume $Z(G) = 1$ in the weight aspect.)

Lemma 10.3 *Suppose that $\lambda_0(r_i)$ is trivial for every $i \in I$. Hypothesis* 10.1 *(iii) implies that for each $v|\infty$,*

$$(2 + m(r)m(\xi_v))^{|I|} \leqslant C(\Pi_v) \leqslant (3 + 2M(r)M(\xi_v))^d.$$

In particular if $Z(G)$ is finite, then the following holds for any fixed L-morphism r.

$$1 + m(\xi_v) \ll_r C(\Pi_v) \ll_r M(\xi_v)^d$$

Proof First we recall a general fact about archimedean L-factors. Let $\varphi \colon W_\mathbb{R} \to \mathrm{GL}_N(\mathbb{C})$ be a tempered L-parameter and decompose $\varphi|_{W_\mathbb{C}}$ into GL_1-parameters as $\varphi|_{W_\mathbb{C}} = \oplus_{k=1}^N \chi_k$. The archimedean L-factor associated with φ may be written in the form [cf. (4.1)]

$$L(s, \varphi) = \prod_{k=1}^N \Gamma_\mathbb{R}(s - \mu_k(\varphi)). \tag{10.1}$$

For each k assume that $\chi_k(z) = (z/\overline{z})^{a_k}$ for some $a_k \in \frac{1}{2}\mathbb{Z}$. Then we have for every $1 \leqslant k \leqslant N$, $\mu_k(\varphi) \in \frac{1}{2}\mathbb{Z}_{\leqslant 0}$ and, after reordering $\mu_k(\varphi)$'s if necessary,

$$|a_k| \leqslant |\mu_k(\varphi)| \leqslant |a_k| + 1. \tag{10.2}$$

Indeed this comes from inspecting the definition of local L-factors as in of [102, 3.1, 3.3] for instance. (Use [102, 3.1] if $a_k = 0$ and [102, 3.3] otherwise.)

Returning to the setup of the lemma, we have by definition $L(s, \Pi_v) = L(s, r\varphi_{\xi_v})$. For each $i \in I$ we consider the composite complex L-parameter

$$W_{\mathbb{C}} \stackrel{\varphi_{\xi_v}|_{W_{\mathbb{C}}}}{\to} \widehat{G} \times W_{\mathbb{C}} \stackrel{(r_i, 1)}{\to} \mathrm{GL}_{\dim r_i}(\mathbb{C})$$

decompose it as $\oplus_{j=1}^{\dim r_i} \chi_{i,j}$. We can find $a_{i,j} \in \frac{1}{2}\mathbb{Z}$ such that $\chi_{i,j}(z) = (z/\overline{z})^{a_{i,j}}$. For each i, the highest weight theory tells us that $a_{i,j} = \langle \rho + \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle \geqslant 0$ for one j and $|a_{i,j'}| \leqslant a_{i,j}$ for the other $j' \neq j$. By (10.1) and (10.2), the analytic conductor for Π_v (introduced in Sect. 4.2) satisfies

$$C(\Pi_v) = \prod_{k=1}^{d}(2 + |\mu_k(\Pi_v)|) \leqslant \prod_{i \in I} \prod_{j=1}^{\dim r_i}(3 + |a_{i,j}|)$$
$$\leqslant \prod_{i \in I}(3 + \langle \rho + \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle)^{\dim r_i}.$$

Further $\langle \rho + \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle = \langle \rho, \lambda(r_i) \rangle + \langle \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle \leqslant |\lambda(r_i)| + |\lambda(r_i)| M(\xi_v) \leqslant M(r)(1 + M(\xi_v))$. Hence

$$C(\Pi_v) \leqslant \prod_{i \in I}((3 + M(r)(1 + M(\xi_v)))^{\dim r_i} = ((3 + M(r)(1 + M(\xi_v)))^d.$$

Now we establish a lower bound for $C(\Pi_v)$. For each i, we apply (10.2) to the unique $j = j(i)$ such that $a_{i,j} = \langle \rho + \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle$. Then

$$C(\Pi_v) \geqslant \prod_{i \in I}(2 + |a_{i,j(i)}|) = \prod_{i \in I}(2 + \langle \rho + \lambda_{\xi_v^\vee}, \lambda(r_i) \rangle)$$
$$\geqslant (2 + m(r)(1 + m(\xi_v)))^{|I|}.$$

$\square$

11 Statistics of low-lying zeros

As explained in the introduction an application of the quantitative Plancherel Theorems 9.16 and 9.19 is to the study the distribution of the low-lying zeros of families of L-functions $\Lambda(s, \Pi)$. The purpose of this section is to state the main results and make our working hypothesis precise.

11.1 The random matrix models

For the sake of completness we recall briefly the limiting 1-level density of normalized eigenvalues. We consider the three symmetry types $\mathcal{G}(N) = \mathrm{SO}(2N), \mathrm{U}(N), \mathrm{USp}(2N)$. For each integer $N \geqslant 1$ these groups are endowed with their Haar probability measure. For all matrices $A \in \mathcal{G}(N)$ we have a sequence $\vartheta_j = \vartheta_j(A)$ of normalized angles [59]

$$0 \leqslant \vartheta_1 \leqslant \vartheta_2 \leqslant \cdots \leqslant \vartheta_N \leqslant N. \tag{11.1}$$

Namely the eigenvalues of $A \in \mathrm{U}(N)$ are given by $e(\frac{\vartheta_j}{N}) = e^{2i\pi\vartheta_j/N}$. The eigenvalues of $A \in USp(2N)$ or $A \in \mathrm{SO}(2N)$ occur in conjugate pairs and are given by $e(\pm\frac{\vartheta_j}{2N})$.

The mean spacing of the sequence (11.1) is one. The 1-level density is defined by

$$W_{\mathcal{G}(N)}(\Phi) := \int_{\mathcal{G}(N)} \sum_{1 \leqslant j \leqslant N} \Phi(\vartheta_j(A)) dA.$$

The limiting density as $N \to \infty$ is given by the following [59, Theorem AD.2.2].

Proposition 11.1 *Let $\mathcal{G} = \mathrm{U}, \mathrm{SO}(even)$ or USp. For all Schwartz functions Φ on $\mathbb{R}_+$,*

$$\lim_{N\to\infty} W_{\mathcal{G}(N)}(\Phi) = \int_{\mathbb{R}_+} \Phi(x) W(\mathcal{G})(x) dx,$$

where the density functions $W(\mathcal{G})$ are given by (1.5).

The density functions $W(\mathcal{G})$ are defined a priori on $\mathbb{R}_+$. They are extended to $\mathbb{R}_-$ by symmetry, namely $W(\mathcal{G})(x) = W(\mathcal{G})(-x)$ for all $x \in \mathbb{R}$. For a Paley–Wiener function Φ whose Fourier transform $\widehat{\Phi}$ has support inside $(-1, 1)$, we have the identities

$$\int_{-\infty}^{\infty} \Phi(x) W(\mathcal{G})(x) dx = \begin{cases} \widehat{\Phi}(0) & \text{if } \mathcal{G} = \mathrm{U}, \\ \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) & \text{if } \mathcal{G} = \mathrm{USp}, \\ \widehat{\Phi}(0) - \frac{1}{2}\Phi(0) & \text{if } \mathcal{G} = \mathrm{SO}(even). \end{cases} \tag{11.2}$$

11.2 The 1-level density of low-lying zeros

Consider a family $\mathfrak{F} = (\mathfrak{F}_k)_{k \geqslant 1}$ of automorphic representations of $\mathrm{GL}(d, \mathbb{A}_F)$. The 1-level density of the low-lying zeros is defined by

$$D(\mathfrak{F}_k; \Phi) := \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \sum_j \Phi\left(\frac{\gamma_j(\Pi)}{2\pi} \log C(\mathfrak{F}_k)\right) \tag{11.3}$$

Here Φ is a Paley–Wiener function; we don't necessarily assume Φ to be even because the automorphic representations $\Pi \in \mathfrak{F}_k$ might not be self-dual. See also the discussion at the end of Sect. 4.4. The properties of the analytic conductor $C(\mathfrak{F}_k) \geqslant 2$ will be described in Sect. 11.5.

Since Φ decays rapidly at infinity, the zeros $\gamma_j(\Pi)$ of $\Lambda(s, \Pi)$ that contribute to the sum are within $O(1/\log C(\mathcal{F}_k))$ distance of the central point. Therefore the sum over j only captures a few zeros for each Π. The average over the family $\Pi \in \mathfrak{F}_k$ is essential to have a meaningful statistical quantity.

11.3 Properties of families of L-functions

Recall that in Sect. 9.3 we have defined two kinds of families $\mathcal{F} = (\mathcal{F}_k)_{k \geqslant 1}$ of automorphic representations on $G(\mathbb{A}_F)$. The families from Example 9.13 are varying in the level aspect: $\mathbb{N}(\mathfrak{n}_k) \to \infty$ while the families from Example 9.14 are varying in the weight aspect: $m(\xi_k) \to \infty$. In both cases we assume that $\phi_{S_0} \in C_c^\infty(G(F_{S_0}))$ is normalized such that

$$\widehat{\mu}_{S_0}^{\mathrm{pl}}(\widehat{\phi}_{S_0}) = \phi_{S_0} = 1. \tag{11.4}$$

For families in the weight aspect we assume from now the weights are bounded away from the walls. Namely we assume that we are given a fixed $\eta > 0$ and that

$$(\dim \xi_k)^\eta \leqslant m(\xi_k), \quad \forall k. \tag{11.5}$$

Given the continuous L-morphism $r: {}^L G \to \mathrm{GL}(d, \mathbb{C})$ we can construct a family $\mathfrak{F} = r_* \mathcal{F}$ of automorphic L-functions. Assuming the Langlands functoriality in the form of Hypothesis 10.1, for each $\pi \in \mathcal{F}_k$ there is a unique isobaric automorphic representation $\Pi = r_* \pi$ of $\mathrm{GL}(d, \mathbb{A}_F)$. We denote by $\mathfrak{F}_k = r_* \mathcal{F}_k$ the corresponding family of all such Π. Recall from Sect. 9.2 that $\mathcal{F}_k$ is a weighted set and that the weight of each representation π is denoted $a_{\mathcal{F}_k}(\pi)$. The same holds for $\mathfrak{F}_k$ and in particular we have

$$|\mathfrak{F}_k| = |\mathcal{F}_k| = \sum_{\pi \in \mathcal{F}_k} a_{\mathcal{F}_k}(\pi).$$

We have seen in Corollary 9.25 that $|\mathfrak{F}_k| \to \infty$ as $k \to \infty$.

By definition [see (9.4)], if $\pi \in \mathcal{F}_k$ then π_∞ has the same infinitesimal character as $\xi_k^\vee$, i.e. $\pi \in \Pi_{disc}(\xi_k)$. If $\Pi \in \mathfrak{F}_k$ then Π_∞ corresponds to the composition $r \circ \phi_{\xi_k}$ via the Langlands correspondence for $\mathrm{GL}_d(F_\infty)$ [This is Hypothesis 10.1 (iii)]. In particular Π_∞ is uniquely determined by ξ_k and r. It is identical for all $\Pi \in \mathfrak{F}_k$.

It is shown in Lemma 10.2 that Π_∞ is tempered. Therefore Proposition 4.1 applies and the bounds towards Ramanujan (4.6) are satisfied for all $\Pi \in \mathfrak{F}_k$.

To simplify notation throughout this and the next section, we use the convention of omitting the weight when writing a sum over $\mathfrak{F}_k$. If $l(\Pi)$ is a quantity that depends on $\Pi \in \mathfrak{F}_k$, we set

$$\sum_{\Pi \in \mathfrak{F}_k} l(\Pi) := \sum_{\pi \in \mathcal{F}_k} a_{\mathcal{F}_k}(\pi) l(r_* \pi).$$

This convention applied in particular to (11.3) above.

11.4 Occurrence of poles

We make the following hypothesis concerning poles of L-functions in our families.

Hypothesis 11.2 There is $C_{pole} > 0$ such that the following holds as $k \to \infty$:

$$\# \{\Pi \in \mathfrak{F}_k, \ \Lambda(s, \Pi) \text{ has a pole}\} \ll |\mathfrak{F}_k|^{1 - C_{pole}}.$$

The hypothesis is natural because it is related to the functoriality Hypothesis 10.1 in many ways. Of course it would be difficult to define the event that "$L(s, \Pi)$ has a pole" without assuming Hypothesis 10.1. Also when Functoriality is known unconditionally it is usually possible to establish the Hypothesis 11.2 unconditionally as well. We shall return to this question in a subsequent article.

11.5 Analytic conductors

As in [53] we define an analytic conductor $C(\mathfrak{F}_k)$ associated to the family. The significance of $C(\mathfrak{F}_k)$ is that each $\Pi \in \mathfrak{F}_k$ have an analytic conductor $C(\Pi)$ comparable to $C(\mathfrak{F}_k)$. The hypothesis in this subsection will ensure that $\log |\mathfrak{F}_k| \asymp \log C(\mathfrak{F}_k)$. We distinguish between families in the weight and level aspect.

11.5.1 Weight aspect

For families in the weight aspect we set $C(\mathfrak{F}_k)$ to be the analytic conductor $C(\Pi_\infty)$ of the archimedean factor Π_∞ (recall that Π_∞ is the same for all $\Pi \in \mathfrak{F}_k$). Then $C(\Pi) \asymp C(\mathfrak{F}_k)$ for all $\Pi \in \mathfrak{F}_k$.

From Corollary 9.25 we have that $|\mathfrak{F}_k| \asymp \dim \xi_k$ as $k \to \infty$. It remains to relate the quantities $C(\mathfrak{F}_k)$, $\dim \xi_k$ and $m(\xi_k)$, which is achieved in (11.6) and (11.7) below.

Lemma 11.3 *Let $v|\infty$. Let ξ_v be an irreducible finite dimensional algebraic representation of $G(F_v)$. Then $m(\xi_v)^{|\Phi^+|} \ll \dim \xi_v \ll M(\xi_v)^{|\Phi^+|}$. Also $M(\xi_v) \ll \dim \xi_v$.*

Proof This follows from Lemma 6.10. Recall the definition of $m(\xi_v)$ in Sect. 6.4 and $M(\xi_v)$ in Sect. 10. □

Because of (11.5) and the previous lemma we have that

$$m(\xi_k)^{|\Phi^+|} \ll \dim \xi_k \ll m(\xi_k)^{1/\eta}. \tag{11.6}$$

From Lemma 10.3 we deduce that there are positive constants C_1, C_2 such that

$$m(\xi_k)^{C_1} \ll C(\mathfrak{F}_k) \ll m(\xi_k)^{C_2}. \tag{11.7}$$

11.5.2 Level aspect

For families in the level aspect the situation is more complicated mainly because of the lack of knowledge of the local Langlands correspondence on general groups and the depth preservation under functoriality. We define $C(\mathfrak{F}_k)$ by the following

$$\log C(\mathfrak{F}_k) := \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \log C(\Pi),$$

and we introduce the following hypothesis.

Hypothesis 11.4 There are constants $C_3, C_4 > 0$ such that

$$\mathbb{N}(\mathfrak{n}_k)^{C_3} \ll C(\mathfrak{F}_k) \ll \mathbb{N}(\mathfrak{n}_k)^{C_4}.$$

11.6 Main result

We may now state our main results on low-lying zeros of the family $\mathfrak{F} = r_*\mathcal{F}$. The following is a precise version of Theorem 1.5 from the introduction [compare with (11.2)].

Theorem 11.5 *Assume Hypothesis 10.1 for individual representations as well as 11.2 and 11.4. There is $0 < \delta < 1$ such that for all Paley–Wiener functions Φ whose Fourier transform $\widehat{\Phi}$ has support in $(-\delta, \delta)$ the following holds:*

$$\lim_{k \to \infty} D(\mathfrak{F}_k, \Phi) = \widehat{\Phi}(0) - \frac{s(r)}{2}\Phi(0),$$

where $s(r) \in \{-1, 0, 1\}$ is the Frobenius–Schur indicator of $r: {}^L G \to \mathrm{GL}_d(\mathbb{C})$.

12 Proof of Theorem 11.5

The method of proof of the asymptotic distribution of the 1-level density of low-lying zeros of families of L-functions has appeared at many places in the literature and is by now relatively standard. However we must justify the details carefully as families of L-functions haven't been studied in such a general setting before. The advantage of working in that degree of generality is that we can isolate the essential mechanisms and arithmetic ingredients involved.

In order to keep the analysis concise we have introduced some technical improvements which can be helpful in other contexts: we use non-trivial bounds towards Ramanujan in a systematic way to handle ramified places; we clarify that it is not necessary to assume that the representation be self-dual or any other symmetry property to carry out the analysis; most importantly we exploit the properties of the Plancherel measure when estimating Satake parameters. Previous articles on the subject rely in a way or another on explicit Hecke relations which made the proof indirect and lengthy, although manageable for groups of low rank.

12.1 Notation

To formulate the main statements we introduce the following notation

$$\widehat{\mathcal{L}}_{k,v}(y) := \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \int_{-\infty}^{\infty} \frac{L'}{L}\left(\frac{1}{2} + ix, \Pi_v\right) e^{2\pi i y x} dx, \quad v \in \mathcal{V}_F, \ y \in \mathbb{R}.$$

$$(12.1)$$

We view $\widehat{\mathcal{L}}_{k,v}$ as a tempered distribution on $\mathbb{R}$. Note that when v is non-archimedean $\widehat{\mathcal{L}}_{k,v}$ is a signed measure supported on a discrete set inside $\mathbb{R}_{>0}$.

The proof of the main theorems will follow by a fine estimation of $\widehat{\mathcal{L}}_{k,v}(y)$ as $k \to \infty$. The uniformity in both the places $v \in \mathcal{V}_F$ and the parameter $y \in \mathbb{R}$ will play an important role. Typically q_v will be as large as $C(\mathfrak{F}_k)^{O(\delta)}$ and y will be of size proportional to $\log C(\mathfrak{F}_k)$.

The first step of the proof consists in applying the explicit formula (Proposition 4.4). There are terms coming from the poles of $L(s, \Pi)$ which we handle in Sect. 12.4. The second term in the right hand-side in Proposition 4.4 is expressed in terms of the arithmetic conductor $q(\Pi)$ and will yield a positive contribution in the limit for families in the level aspect. When evaluating the 1-level density $D(\mathfrak{F}_k, \Phi)$ it remains to consider the following sum over all places

$$\frac{1}{\log C(\mathfrak{F}_k)} \sum_{v \in \mathcal{V}_F} \left\langle \widehat{\mathcal{L}}_{k,v}(y), \widehat{\Phi}\left(\frac{2\pi y}{\log C(\mathfrak{F}_k)}\right) \right\rangle, \tag{12.2}$$

plus a conjugate expression, see Sect. 12.3.

Our convention on Fourier transforms is standard. Let Φ be a Schwartz function on $\mathbb{R}$. The Fourier transform is as in (4.7) and the inverse Fourier transform reads

$$\Phi(x) = \int_{-\infty}^{+\infty} \widehat{\Phi}(y) e^{2\pi i x y} dy.$$

Given two Schwartz functions Φ and Ψ we let

$$\langle \Phi, \Psi \rangle := \int_{-\infty}^{\infty} \Phi(x) \Psi(x) dx.$$

Sometimes we use the notation $\langle \Phi(x), \Psi(x) \rangle$ to put emphasize on the variable of integration. The Plancherel formula reads

$$\langle \Phi(x), \Psi(x) \rangle = \langle \widehat{\Phi}(y), \widehat{\Psi}(-y) \rangle. \tag{12.3}$$

We use the same conventions for tempered distributions. The Fourier transform of the pure phase function $x \mapsto e^{2i\pi a x}$ is the Dirac distribution $\delta(a)$ centered at the point a.

To condense notation we write

$$\Psi(y) := \widehat{\Phi}\left(\frac{2\pi y}{\log C(\mathfrak{F}_k)}\right)$$

and shall express our remainder terms with the quantities $\|\Psi\|_\infty \leqslant \|\widehat{\Phi}\|_\infty$ and $\|\widehat{\Psi}\|_1 \leqslant \|\Phi\|_1$. Since Φ is fixed these are uniformly bounded, independent of $k \to \infty$.

There are different kinds of estimates depending on the nature of the place $v \in \mathcal{V}_F$. We shall distinguish the following set of places:

(i) the archimedean places S_∞, the contribution of which is evaluated in Sect. 12.5;

(ii) a fixed set S_0 of non-archimedean places. These may be thought of as the "ramified places". Their contribution is negligible as shown in Sect. 12.7;

(iii) the set $\{v \mid \mathfrak{n}_k\}$ of places that divide the level. These play a role only when the level varies and we show in Sect. 12.10 that their contribution is negligible. We use the convention that for families in the weight aspect this set of places is empty;

(iv) the generic places S_{gen} which is the complement in $\mathcal{V}_F$ of the above three sets of places. This set will actually be decomposed in two parts:

$$S_{\text{gen}} = S_{\text{main}} \sqcup S_{\text{cut}},$$

(v) where the set S_{cut} is infinite and consists of those non-archimedean places $v \in S_{\text{gen}}$ such that $\frac{\log q_v}{2\pi}$ is large enough to be outside of the support of Ψ [see (12.18) below for the exact definition of S_{cut}]. Then the pairing in (12.2) vanishes;

(vi) the remaining set S_{main} is finite (but growing as $k \to \infty$). It will produce the main contribution of (12.2). For all places $v \in S_{\text{main}}$, each of G, r and π is unramified over F_v. Using the notation of Sect. 5 we split S_{main} further as the disjoint union of

$$S_{\text{main}} \cap \mathcal{V}_F(\theta), \quad \theta \in \mathscr{C}(\Gamma_1).$$

12.2 Outline

For non-archimedean places $v \in S_{\text{main}}$ we study in Sect. 12.6 various moments of Satake parameters. The quantity $\widehat{\mathcal{L}}_{\text{pl},v}$ in (12.11) below will be the analogue of (12.1) where the average over automorphic representations $\Pi \in \mathfrak{F}_k$ gets replaced by an average of Π_v against the unramified Plancherel measure. Our Plancherel equidistribution theorems for families (Theorems 9.16 and 9.19) imply that $\widehat{\mathcal{L}}_{k,v}$ is asymptotic to $\widehat{\mathcal{L}}_{\text{pl},v}$ as $k \to \infty$.

It is essential that our equidistribution theorems are quantitative in a strong polynomial sense. Details on handling the remainder terms are given in Sects. 12.8–12.10.

For the main term we then need need to show the existence of the limit of

$$\frac{1}{\log C(\mathfrak{F}_k)} \sum_{v \in S_{\text{main}}} \langle \widehat{\mathcal{L}}_{\text{pl},v}, \Psi \rangle \tag{12.4}$$

as $k \to \infty$. The evaluation of $\widehat{\mathcal{L}}_{\text{pl},v}$ is a nice argument in representation theory, see Sect. 12.12 where we shall see clearly the role of the two assumptions on r (that r is irreducible and does not factor through W_F). The evaluation of $\widehat{\mathcal{L}}_{\text{pl},v}$ can actually be quite complicated since it depends on the restriction of r to subgroups $\widehat{G} \rtimes W_{F_v}$ for varying $v \in S_{\text{main}}$ and on the Plancherel measure on $G(F_v)^{\wedge, \text{ur}}$. Fortunately the expression will simplify when summing over all places $v \in S_{\text{main}}$ and applying the Cebotarev density theorem (see Sect. 12.11).

The overall conclusion of the below analysis is that the limit of (12.4) as $k \to \infty$ is equal[10] to $-\frac{s(r)}{2}\Phi(0)$, where $s(r)$ is the Frobenius–Schur indicator

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[10] A quick explanation for the minus sign is as follows. A local L-factor is of the form $(1 - \alpha q^{-s})^{-1}$ with three minus signs thus its logarithmic derivative is $-\log q \sum_{v \geq 1} \alpha^v q^{-vs}$ with one minus sign.

of r. In the derivation of the one-level density there is an additional term $\widehat{\Phi}(0)$ which easily comes from the explicit formula and the contribution of the archimedean terms. Thereby we finish the proof of Theorem 11.5.

12.3 Explicit formula

We apply the explicit formula (Proposition 4.4) for each $\Pi \in \mathfrak{F}_k$ to obtain

$$D(\mathfrak{F}_k, \Phi) = D_{\mathrm{pol}}(\mathfrak{F}_k, \Phi) + \frac{\widehat{\Phi}(0)}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \frac{\log q(\Pi)}{\log C(\mathfrak{F}_k)} + \sum_{v \in \mathcal{V}_F} D_v(\mathfrak{F}_k, \Phi) + \overline{D_v}(\mathfrak{F}_k, \Phi).$$
(12.5)

Here $D_{\mathrm{pol}}(\mathfrak{F}_k, \Phi)$ denotes the contribution of the eventual poles. Also we have set

$$D_v(\mathfrak{F}_k, \Phi) := \frac{1}{2\pi \, |\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \int_{-\infty}^{\infty} \frac{L'}{L}\left(\frac{1}{2} + ix, \Pi_v\right) \Phi\left(\frac{x}{2\pi} \log C(\mathfrak{F}_k)\right) dx.$$

See also the remark in (4.9) explaining how to shift contours. The scaling factor $\frac{\log C(\mathfrak{F}_k)}{2\pi}$ comes from (11.3).

Applying Fourier duality (12.3) and the definition (12.1) implies the equality[11]

$$D_v(\mathfrak{F}_k, \Phi) = \frac{1}{\log C(\mathfrak{F}_k)} \left\langle \widehat{\mathcal{L}}_{k,v}(y), \widehat{\Phi}\left(\frac{2\pi y}{\log C(\mathfrak{F}_k)}\right) \right\rangle.$$
(12.6)

We have made a change of variable so as to make explicit the multiplicative factor $1/\log C(\mathfrak{F}_k)$ in front of the overall sum. Similarly we have

$$\overline{D_v}(\mathfrak{F}_k, \Phi) := \frac{1}{2\pi \, |\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \int_{-\infty}^{\infty} \overline{\frac{L'}{L}\left(\frac{1}{2} + ix, \Pi_v\right)} \Phi\left(\frac{x}{2\pi} \log C(\mathfrak{F}_k)\right) dx$$

$$= \frac{1}{\log C(\mathfrak{F}_k)} \left\langle \overline{\widehat{\mathcal{L}}_{k,v}}, \widehat{\Phi}\left(\frac{-2\pi y}{\log C(\mathfrak{F}_k)}\right) \right\rangle.$$

12.4 Contribution of the poles

The contribution of the poles in the explicit formula above is given by

$$D_{\mathrm{pol}}(\mathfrak{F}_k, \Phi) := \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \sum_j \Phi\left(\frac{r_j(\Pi)}{2\pi} \log C(\mathfrak{F}_k)\right).$$

---

[11] Note that the exponential in (12.1) is $e^{2i\pi xy}$ with a plus sign.

We bound the sum trivially and obtain

$$D_{\text{pol}}(\mathfrak{F}_k, \Phi) \ll \frac{\#\{\Pi \in \mathfrak{F}_k, \ L(s, \Pi) \text{ has a pole}\}}{|\mathfrak{F}_k|} C(\mathfrak{F}_k)^{O(\delta)},$$

where the last term comes from the exponential order of growth of Φ along the real axis because the Fourier transform $\widehat{\Phi}$ is supported in $(-\delta, \delta)$.

12.5 Archimedean places

In this subsection we handle the archimedean places $v \in S_\infty$. Recall from Lemma 10.2 that Π_∞ is tempered. In fact we shall only need here a bound towards Ramanujan $0 < \theta < \frac{1}{2}$ as in Sect. 4.2.

Lemma 12.1 *For all $\mu \in \mathbb{C}$ with $\Re e \, \mu \leqslant \theta$, and all Schwartz function Ψ, the following holds uniformly*

$$\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \mu + ix\right) \Psi(x)dx = \widehat{\Psi}(0) \log\left(\frac{1}{2} - \mu\right) + O(\|\Psi\|_1 + \|x\Psi(x)\|_1)$$

Proof We have the following Stirling approximation for the Digamma function [traditionally denoted $\psi(z)$]:

$$\frac{\Gamma'}{\Gamma}(z) = \log z + O(1) \tag{12.7}$$

uniformly in the angular region $|\arg z| \leqslant \pi - \epsilon$, see e.g. [51, Appendix B]. Since $\theta < 1/2$ all points $\frac{1}{2} - \mu + ix$ lie in the interior of the angular region and we can apply (12.7). We note also that uniformly

$$\log\left(\frac{1}{2} - \mu + ix\right) = \log\left(\frac{1}{2} - \mu\right) + O(\log(2 + |x|)),$$

and this conclude the proof of the proposition. $\qquad\square$

Remark 12.2 Note that the complete asymptotic expansion actually involves the Bernoulli numbers and is of the form

$$\frac{\Gamma'}{\Gamma}(z) = \log z + \frac{1}{2z} - \sum_{n=1}^{N} \frac{B_{2n}}{2nz^{2n}} + O\left(\frac{1}{z^{2N+2}}\right). \tag{12.8}$$

From (12.8) we have that

$$\frac{\Gamma'}{\Gamma}(\sigma + it) + \frac{\Gamma'}{\Gamma}(\sigma - it) = 2\frac{\Gamma'}{\Gamma}(\sigma) + O((t/\sigma)^2)$$

holds uniformly for σ and t real with $\sigma > 0$. As in [53, §4] this may be used when the test function Ψ is even (e.g. which is the typical case when all representations $\Pi \in \mathfrak{F}$ are self-dual). We don't make this assumption and therefore use (12.7) instead.

Corollary 12.3 *Uniformly for all archimedean places $v \in S_\infty$ and all Schwartz function Ψ, the following holds*

$$\left\langle \widehat{\mathcal{L}}_{k,v}, \Psi \right\rangle = \frac{\Psi(0)}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \sum_{i=1}^{d} \log_v \left(\frac{1}{2} - \mu_i(\Pi_v) \right) + O(\|\widehat{\Psi}\|_1).$$

Here we have set $\log_v z := \frac{1}{2}\log z$ when v is real and $\log_v z := \log z$ when v is complex.

Proof Recall the convention (4.1) on local L-factors at archimedean places $v \in S_\infty$. From Fourier duality (12.3) and the definition (12.1) we have

$$\left\langle \widehat{\mathcal{L}}_{k,v}, \Psi \right\rangle = \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \int_{-\infty}^{\infty} \frac{L'}{L} \left(\frac{1}{2} + ix, \Pi_v \right) \widehat{\Psi}(x)dx.$$

Note that

$$\frac{\Gamma_v'}{\Gamma_v}(s) = \begin{cases} -\frac{1}{2}\log \pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}), & \text{when } v \text{ is real,} \\ -\log(2\pi) + \frac{\Gamma'}{\Gamma}(s), & \text{when } v \text{ is complex.} \end{cases}$$

Applying Lemma 12.1, the estimate in the corollary follows. Recall from Proposition 4.1 that the bounds towards Ramanujan in Sect. 4.2 apply to all $\Pi \in \mathfrak{F}_k$. □

We may continue the analysis of the contribution of the archimedean places to the one-level density. For $v \in S_\infty$, the local L-function $L(s, \Pi_v)$ are the same for all $\Pi \in \mathfrak{F}_k$. We therefore conclude that

$$\sum_{v \in S_\infty} D_v(\mathfrak{F}_k, \Phi) + \overline{D_v(\mathfrak{F}_k, \Phi)} = \frac{\widehat{\Phi}(0)}{\log C(\mathfrak{F}_k)}$$

$$\times \left(\sum_{v \in S_\infty} \sum_{i=1}^{d} 2\log_v \left| \frac{1}{2} - \mu_i(\Pi_v) \right| + O(1) \right)$$

$$= \frac{\widehat{\Phi}(0)}{\log C(\mathfrak{F}_k)} \left(\sum_{v \in S_\infty} \log C(\Pi_v) + O(1) \right). \tag{12.9}$$

In the last line we used the definition of the analytic conductor at archimedean places from Sect. 4.2.

12.6 Moments of Satake parameters

Now let $v \in \mathcal{V}_F^\infty$ be a non-archimedean place. A straightforward computation shows that

$$\frac{L'}{L}(s, \Pi_v) = -\log q_v \sum_{\nu \geqslant 1} \beta^{(\nu)}(\Pi_v) q_v^{-\nu s}$$

where $\beta^{(\nu)}(\Pi_v) := \alpha_1(\Pi_v)^\nu + \cdots + \alpha_d(\Pi_v)^\nu$. Averaging over the family $\mathfrak{F}$ we let

$$\beta_v^{(\nu)}(\mathfrak{F}_k) := \frac{1}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \beta^{(\nu)}(\Pi_v), \quad v \in \mathcal{V}_F^\infty, \ \nu \geqslant 1.$$

The formula (12.1) becomes

$$\widehat{\mathcal{L}}_{k,v} = -\log q_v \sum_{\nu \geqslant 1} \beta_v^{(\nu)}(\mathfrak{F}_k) q_v^{-\nu/2} \delta\left(\frac{\nu}{2\pi} \log q_v\right), \tag{12.10}$$

where δ is Dirac distribution (see Sect. 12.1).

Similarly for all $v \in S_{\text{gen}}$ we let

$$\widehat{\mathcal{L}}_{\text{pl},v} := -\log q_v \sum_{\nu \geqslant 1} \beta_{\text{pl},v}^{(\nu)} q_v^{-\nu/2} \delta\left(\frac{\nu}{2\pi} \log q_v\right), \tag{12.11}$$

where the coefficients $\beta_{\text{pl},v}^{(\nu)}$ are defined locally as follows. Since $v \in S_{\text{gen}}$, the group G is unramified over F_v and that the restriction $r|_{\widehat{G} \rtimes W_{F_v}}$ is an unramified L-morphism, i.e. it factors through $\widehat{G} \rtimes W_{F_v}^{\text{ur}}$. Recall from Sect. 3 that $\widehat{\mu}_v^{\text{pl,ur}}$ is the restriction of the Plancherel measure $\widehat{\mu}_v^{\text{pl}}$ to $G(F_v)^{\wedge,\text{ur}}$. Then

$$\beta_{\text{pl},v}^{(\nu)} := \widehat{\mu}_v^{\text{pl,ur}}\left(r^*(Y_1^\nu + \cdots + Y_d^\nu)\right), \tag{12.12}$$

where we are using the convention in Sect. 2.3 for the L-morphism of unramified Hecke algebras $r^* : \mathcal{H}^{\text{ur}}(\text{GL}_d(F_v)) \to \mathcal{H}^{\text{ur}}(G(F_v))$ and the Satake isomorphism with the polynomial algebra in $Y_1, \ldots, Y_d$ (Sect. 2.4).

The supports of both measures $\widehat{\mathcal{L}}_{k,v}$ and $\widehat{\mathcal{L}}_{\text{pl},v}$ are contained in the discrete set $\frac{\log q_v}{2\pi} \mathbb{N}_{\geqslant 1}$. If q_v is large enough this is disjoint from the support of Ψ and thus all sums over places $v \in \mathcal{V}_F$ considered below shall be finitely supported.

12.7 General upper-bounds

Recall from Proposition 4.1 that the bounds towards Ramanujan apply to every $\Pi \in \mathfrak{F}_k$. Thus for every non-archimedean $v \in \mathcal{V}_F^\infty$, we have the upper bound $|\alpha_i(\Pi_v)| \leqslant q_v^\theta$ from which it follows that for every $\nu \geqslant 1$,

$$\left| \beta_v^{(\nu)}(\mathfrak{F}_k) \right| \leqslant d q_v^{\nu\theta}.$$

Proposition 12.4 *(i) For all $v \in \mathcal{V}_F^\infty$ and all continuous function Ψ,*

$$\left\langle \widehat{\mathcal{L}}_{k,v}, \Psi \right\rangle \ll q_v^{\theta - \frac{1}{2}} \log q_v \, \|\Psi\|_\infty .$$

(ii) For all $v \in S_{gen}$ and all continuous function Ψ,

$$\left\langle \widehat{\mathcal{L}}_{pl,v}, \Psi \right\rangle \ll q_v^{-\frac{1}{2}} \log q_v \, \|\Psi\|_\infty .$$

Proof (i) Inserting the above upper bound into (12.10) we have

$$\left\langle \widehat{\mathcal{L}}_{k,v}, \Psi \right\rangle \ll \log q_v \sum_{\nu \geqslant 1} q_v^{\nu(\theta - 1/2)} \left| \Psi \left(\frac{\nu}{2\pi} \log q_v \right) \right|.$$

Because $0 < \theta < \frac{1}{2}$, the conclusion easily follows.

(ii) The Plancherel measure $\widehat{\mu}^{\mathrm{pl,ur}}$ has total mass one and is supported on the tempered spectrum $\widehat{G}(F_v)^{\wedge,\mathrm{ur,temp}}$ (see Sect. 3.2). We deduce similarly that for every $\nu \geqslant 1$,

$$\left| \beta_{\mathrm{pl},v}^{(\nu)} \right| \leqslant d \tag{12.13}$$

Indeed the image of any unramified L-parameter $r \circ \varphi : W_{F_v}^{\mathrm{ur}} \to \mathrm{GL}_d(\mathbb{C})$ is bounded and all Frobenius eigenvalues have therefore absolute value one. □

12.8 Plancherel equidistribution

We are in position to apply the Plancherel equidistribution theorem for families established in Sect. 9. We shall derive uniform asymptotics as $k \to \infty$ for $\beta_v^{(\nu)}(\mathfrak{F}_k)$.

Proposition 12.5 *There exist constants $C_5 > 0$ and A_7, $B_9 < \infty$ such that the following holds uniformly on $\nu \geqslant 1$ and $v \in S_{gen}$*

$$\beta_v^{(\nu)}(\mathfrak{F}_k) = (1 + o(1))\beta_{\mathrm{pl},v}^{(\nu)} + O\left(q_v^{A_7 + B_9\nu} C(\mathfrak{F}_k)^{-C_5} \right). \tag{12.14}$$

Proof Let S_0 be a sufficiently large set of non-archimedean places which contains all places $v \in \mathcal{V}_F^\infty$ where G is ramified and where r is ramified. Let $S_1 := \{v\}$. We set

$$\widehat{\Phi}_v := r^*(Y_1^v + \cdots + Y_d^v) \in \mathcal{H}^{\mathrm{ur}}(G(F_v)).$$

The notation for the Satake isomorphism is as in Sects. 2.2 and 2.5. By definition we have that $\beta_{\mathrm{pl},v}^{(v)} = \widehat{\mu}_v^{\mathrm{pl}}(\widehat{\Phi}_v)$. Thanks to Lemma 2.6 we have that $\phi_v \in \mathcal{H}^{\mathrm{ur}}(G(F_v))^{\leqslant \beta v}$ and $|\phi_v| \ll 1$. Thus we are in position to apply the respective Theorems 9.16 (in the level aspect) and 9.19 (in the weight aspect).

Using the notation of Sect. 9.2, we have by construction

$$\beta_v^{(v)}(\mathfrak{F}_k) = \frac{1}{|\mathcal{F}_k|} \sum_{\pi \in \mathcal{F}_k} a_{\mathcal{F}_k}(\pi) \widehat{\Phi}_v(\pi_v)$$

$$= \widehat{\mu}_{\mathcal{F}_k,v}^{\mathrm{count}}(\widehat{\Phi}_v) = \frac{\tau'(G) \dim \xi_k}{\mu^{\mathrm{can}}(U_k^{S,\infty}) |\mathcal{F}_k|} \widehat{\mu}_{\mathcal{F}_k,v}(\widehat{\Phi}_v).$$

The Corollary 9.25 shows that $\dfrac{\tau'(G) \dim \xi_k}{\mu^{\mathrm{can}}(U_k^{S,\infty}) |\mathcal{F}_k|} = 1 + o(1)$ as $k \to \infty$. We shall now distinguish between the two types of families.

For families in the level aspect, the assumption (ii) in Theorem 9.16 is satisfied because $v \nmid \mathfrak{n}_k$ for all k and all $v \in S_{\mathrm{gen}}$. If the assumption (i) in Theorem 9.16 is not satisfied, then

$$C(\mathfrak{F}_k)^{\frac{1}{C_4}} \ll N(\mathfrak{n}_k) < c_\Xi q_v^{B_\xi m\kappa}$$

where the first inequality comes from Hypothesis 11.4. Thus the error term in (12.14) dominates if A_7 is chosen large enough. If the assumption (i) in Theorem 9.16 is satisfied, then from (9.8) we obtain the main term in (12.14) and the error term $O(q_v^{A_{\mathrm{lv}} + B_{\mathrm{lv}}\kappa} \mathbb{N}(\mathfrak{n}_k)^{-C_{\mathrm{lv}}})$. By Hypothesis 11.4 we may then choose then $C_5 := C_{\mathrm{lv}}/C_4$ to conclude the proof of (12.14).

For families in the weight aspect the assumptions in Theorem 9.19 are always satisfied. This yields the main term in (12.14) with the error term $O(q_{S_1}^{A_{\mathrm{wt}} + B_{\mathrm{wt}}\kappa} m(\xi_k)^{-C_{\mathrm{wt}}})$. By the estimate (11.7) we may choose $C_5 := C_{\mathrm{wt}}/C_2$ to conclude the proof of (12.14). $\qquad\square$

12.9 Main term

We deduce from Proposition 12.5 the following estimate for $\widehat{\mathcal{L}}_{k,v}$.

Proposition 12.6 *For all $A > 0$ there is $A_8 > 0$ such that the following holds uniformly for all $v \in S_{gen}$ and all continuous function Ψ:*

$$\langle \widehat{\mathcal{L}}_{k,v}, \Psi \rangle = \langle \widehat{\mathcal{L}}_{pl,v}, \Psi \rangle (1+o(1)) + O(q_v^{A_8} C(\mathfrak{F}_k)^{-C_5} \|\Psi\|_\infty) + O(q_v^{-A} \|\Psi\|_\infty),$$

Proof Let κ be a large enough integer. We apply the bounds towards Ramanujan in the form (12.7) to those term in (12.10) with $v > \kappa$. The contribution of those terms to $\langle \widehat{\mathcal{L}}_{k,v}, \Psi \rangle$ is uniformly bounded by

$$\ll q_v^{\kappa(\theta - \frac{1}{2})} \|\Psi\|_\infty .$$

We have that $A := \kappa(\frac{1}{2} - \theta)$ may be chosen as large as we want since $\theta < \frac{1}{2}$ is fixed and κ is arbitrary large.

For those terms in (12.10) with $v \leqslant \kappa$ we apply (12.14). Their contribution to $\langle \widehat{\mathcal{L}}_{k,v}, \Psi \rangle$ is equal to

$$- \log q_v \sum_{1 \leqslant v \leqslant \kappa} \beta_{pl,v}^{(v)} q_v^{-v/2} \Psi \left(\frac{v}{2\pi} \log q_v \right) + O \left(q_v^{A_7 + B_9 \kappa} \|\Psi\|_\infty C(\mathfrak{F}_k)^{-C_5} \right).$$

The next step is now to complete the v-sum. Applying (12.7) we see that the terms $v > \kappa$ yield another remainder term of the form $q_v^{-A} \|\Psi\|_\infty$ with A arbitrary large (again depending on κ). $\qquad \square$

12.10 Handling remainder terms

In this subsection we handle the various remainder terms and show that they don't contribute to $D(\mathfrak{F}_k, \Phi)$ in the limit when $k \to \infty$. We shall apply the above estimates to the function

$$\Psi(y) := \widehat{\Phi} \left(\frac{2\pi y}{\log C(\mathfrak{F}_k)} \right), \quad y \in \mathbb{R}. \tag{12.15}$$

Recall from (12.6) that $D_v(\mathfrak{F}_k, \Phi) = \langle \widehat{\mathcal{L}}_{k,v}, \Psi \rangle / \log C(\mathfrak{F}_k)$.

For archimedean places $v \in S_\infty$ we encountered in Sect. 12.5 the remainder term $O(\|\widehat{\Psi}\|_1)$. Because $\log C(\mathfrak{F}_k) \to \infty$, this remainder term is negligible for $D_v(\mathfrak{F}_k, \Phi)$ as $k \to \infty$.

For the non-archimedean places v such that $v \mid \mathfrak{n}_k$ or $v \in S_0$ we use the general bounds of Sect. 12.7 that imply

$$\sum_{v \in S_0, \text{ and } v | \mathfrak{n}_k} \left| \langle \widehat{\mathcal{L}}_{k,v}, \Psi \rangle \right| \ll \sum_{v \in S_0, \text{ and } v | \mathfrak{n}_k} q_v^{\theta - \frac{1}{2}} \log q_v \|\Psi\|_\infty \ll 1 + \# \{v \mid \mathfrak{n}_k\}. \tag{12.16}$$

In the last inequality we used the fact that S_0 is fixed and that $\theta < \frac{1}{2}$. Again the multiplication by $1/\log C(\mathfrak{F}_k)$ shows that these terms are negligible for $D(\mathfrak{F}_k, \Phi)$ as $k \to \infty$. Indeed it is easy to verify that

$$\# \{v \mid \mathfrak{n}_k\} = o(\log \mathbb{N}(\mathfrak{n}_k)), \quad \text{as } k \to \infty, \tag{12.17}$$

and we conclude using Hypothesis 11.4 that this is $o(\log C(\mathfrak{F}_k))$.

We partition the set of generic non-archimedean places S_{gen} into two disjoint sets S_{main} and S_{cut} where

$$S_{\text{cut}} := \left\{ v \in S_{\text{gen}} \colon q_v > C(\mathfrak{F}_k) \right\}. \tag{12.18}$$

Since the support of $\widehat{\Phi}$ is included in $(-\delta, \delta)$ we know that $\Psi(\nu \log q_v / 2\pi)$ vanishes for all $v \in S_{\text{cut}}$ and $\nu \geqslant 1$.

For the generic places $v \in S_{\text{main}}$ we use the estimate in Proposition 12.6. The second remainder term yields

$$\frac{1}{\log C(\mathfrak{F}_k)} \sum_{v \in S_{\text{main}}} q_v^{-A} \|\Psi\|_\infty = O\left(\frac{1}{\log C(\mathfrak{F}_k)}\right). \tag{12.19}$$

This is again negligible as $k \to \infty$. The first remainder term in Proposition 12.6 is negligible as well because

$$\sum_{v \in S_{\text{main}}} q_v^{A_8} \|\Psi\|_\infty C(\mathfrak{F}_k)^{-C_5} \ll C(\mathfrak{F}_k)^{\delta(A_8+1)-C_5} \tag{12.20}$$

and δ is chosen small small enough such that $\delta(A_8 + 1) < C_5$.

Finally we show that the contribution to $\langle \widehat{\mathcal{L}}_{\text{pl},v}, \Psi \rangle / \log C(\mathfrak{F}_k)$ of the higher moments $\nu \geqslant 3$ is negligible. Because of the definition (12.11) of $\widehat{\mathcal{L}}_{\text{pl},v}$ and the bound (12.13) for $\beta_{\text{pl},v}^{(\nu)}$, the contribution of the higher moments is uniformly bounded by

$$\sum_{v \in S_{\text{main}}} \log q_v \sum_{\nu \geqslant 3} q_v^{-\nu/2} \Psi\left(\frac{\nu \log q_v}{2\pi}\right) \ll \|\Psi\|_\infty \sum_{v \in \mathcal{V}_F^\infty} q_v^{-3/2} \log q_v \ll 1. \tag{12.21}$$

Therefore we can write the main contribution to $D(\mathfrak{F}_k, \Phi)$ as

$$\frac{1}{\log C(\mathfrak{F}_k)} \sum_{v \in S_{\text{main}}} \langle \widehat{\mathcal{L}}_{\text{pl},v}, \Psi \rangle = M^{(1)} + M^{(2)} + O\left(\frac{1}{\log C(\mathfrak{F}_k)}\right)$$

where for $\nu = 1, 2$ we define

$$M^{(\nu)} := - \sum_{v \in S_{\text{main}}} \frac{\log q_v}{\log C(\mathfrak{F}_k)} q_v^{-\nu/2} \beta_{\text{pl},v}^{(\nu)} \widehat{\Phi}\left(\frac{\nu \log q_v}{\log C(\mathfrak{F}_k)}\right). \qquad (12.22)$$

[recall the relation (12.15) between Φ and Ψ]

12.11 Sum over primes

It remains to estimate the above terms (12.22) which consist of sums over the places $v \in S_{\text{main}}$. We shall use the prime number theorem and the Cebotarev equidistribution theorem which we now proceed to recall, following e.g. [79, Chap. 7]. Let E/F be a finite Galois extension with Galois group $\Gamma = \text{Gal}(E/F)$. For all conjugacy class $\theta \in \mathscr{C}(\Gamma)$, recall that $\mathcal{V}_F(\theta)$ consists of those unramified places $v \in \mathcal{V}_F^\infty$ such that $\text{Fr}_v \in \theta$.

Proposition 12.7 (Prime number theorem) *Notation being as above,*

$$\#\left\{v \in \mathcal{V}_F^\infty, \; q_v \leqslant x\right\} \sim \frac{x}{\log x}, \quad \text{as } x \to \infty.$$

(*Cebotarev equidistribution theorem*) *For any $\theta \in \mathscr{C}(\Gamma)$,*

$$\#\{v \in \mathcal{V}_F(\theta), \; q_v \leqslant x\} \sim \frac{x}{\log x} \times \frac{|\theta|}{|\Gamma|}, \quad \text{as } x \to \infty.$$

As a corollary we deduce the following estimate for any $\theta \in \mathscr{C}(\Gamma)$

$$\sum_{v \in \mathcal{V}_F(\theta)} \frac{\log q_v}{\log C(\mathfrak{F}_k)} q_v^{-1} \widehat{\Phi}\left(\frac{\log q_v}{\log C(\mathfrak{F}_k)}\right) = \left(\frac{|\theta|}{|\Gamma|} + o(1)\right)$$

$$\times \int_0^\infty \widehat{\Phi}(y) dy, \quad \text{as } k \to \infty. \qquad (12.23)$$

This estimate will be used below to evaluate $M^{(2)}$.

Note that if we replace $\log q_v$ by $-\log q_v$, the same estimate holds with the integral on the right-hand side ranging from $-\infty$ to 0. We shall use this observation below when adding the contribution of $\overline{D_v}(\mathfrak{F}_k, \Phi)$ which will then produce produce the integral $\int_{-\infty}^\infty \widehat{\Phi}(y) dy = \Phi(0)$.

12.12 Computing the moments $M^{(1)}$ and $M^{(2)}$

Recall that by assumption

$$r : \widehat{G} \rtimes \mathrm{Gal}(\bar{F}/F) = {}^L G \to \mathrm{GL}_d(\mathbb{C})$$

is irreducible and does not factor through $\mathrm{Gal}(\bar{F}/F)$.

Lemma 12.8 *The restriction $r|_{\widehat{G}}$ does not contain the trivial representation.*

Proof If there were a non-zero vector in $\mathbb{C}^d$ invariant by $r(\widehat{G})$ then all its translates by $\mathrm{Gal}(\bar{F}/F)$ would still be invariant because $\widehat{G}$ is a normal subgroup of ${}^L G$. Because r is irreducible these translates generate $\mathbb{C}^d$ and thus the restriction $r|_{\widehat{G}}$ would be trivial [12] which yields a contradiction. For an extension of this argument see e.g. [93, Prop. 24, § I.8.1]. $\qquad\square$

Since $v \in S_{\mathrm{main}}$, the group G is unramified over F_v and the restriction $r|_{\widehat{G} \rtimes W_{F_v}}$ is an unramified L-morphism which factors through $\widehat{G} \rtimes W_{F_v}^{\mathrm{ur}}$. Note that this restriction might reducible in general.

Let A is a maximal F_v-split torus and Ω_{F_v} the F_v-rational Weyl group for $(G(F_v), A)$. Recall from Sect. 2 the Satake isomorphism

$$\mathcal{S} : \mathcal{H}^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathcal{H}^{\mathrm{ur}}(A)^{\Omega_{F_v}}.$$

For the group GL_d the right hand-side is identified with $\mathbb{C}[Y_1^{\pm}, \dots Y_d^{\pm}]^{\mathfrak{S}_d}$. We recall the morphism of unramified Hecke algebras $r^* : \mathcal{H}^{\mathrm{ur}}(\mathrm{GL}_d) \to \mathcal{H}^{\mathrm{ur}}(G(F_v))$ and the test functions:

$$\phi_v^{(\nu)} := r^*(Y_1^\nu + \dots + Y_d^\nu) \in \mathcal{H}^{\mathrm{ur}}(G(F_v)).$$

In view of (12.12) we have

$$\beta_{\mathrm{pl},v}^{(\nu)} = \widehat{\mu}_v^{\mathrm{pl}}\left(\widehat{\Phi}_v^{(\nu)}\right) = \phi_v^{(\nu)}(1).$$

Proposition 12.9 *The following estimate holds uniformly for all $v \in S_{main}$*

$$\beta_{\mathrm{pl},v}^{(1)} = O\left(q_v^{-1}\right).$$

Proof We decompose the restriction of r to $\widehat{G} \rtimes W_{F_v}^{\mathrm{ur}}$ into a direct sum of irreducible $\oplus_i r_i$. By Lemma 12.8 each $r_i|_{\widehat{G}}$ does not contain the trivial representation. In particular each r_i does not factor through $W_{F_v}^{\mathrm{ur}}$.

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[12] In the sense that $r|_{\widehat{G}}$ would be a direct sum of trivial representations. In the sequel we use this slight abuse of notation when saying that a representation is "trivial".

We can now apply Lemma 2.9 which shows that

$$\left| \phi_v^{(1)}(1) \right| \leqslant q_v^{-1} \left| \Omega_{F_v} \right| \max_{w \in \Omega_{F_v}} p(\lambda_i \star_w 0).$$

Here λ_i is as defined in Sect. 2.2. The two terms $\left| \Omega_{F_v} \right|$ and $p(\lambda_i \star_w 0)$ are easily seen to be bounded (uniformly with respect to $v \in S_{\mathrm{main}}$). $\qquad\square$

As a consequence of Proposition 12.9 we deduce that

$$M^{(1)} = O \left(\frac{1}{\log C(\mathfrak{F}_k)} \right) \tag{12.24}$$

because the summand over v in (12.22) is dominated by $q_v^{-3/2}$.

For the second moment $M^{(2)}$ we shall need a more refined estimate. Recall the finite extension F_1/F from Sect. 5. We also choose a finite extension F_2/F_1 such that r factors through $\widehat{G} \rtimes \mathrm{Gal}(F_2/F)$. Let $\Gamma_2 := \mathrm{Gal}(F_2/F)$ and denote by $\mathscr{C}(\Gamma_2)$ the set of conjugacy classes in Γ_2.

Proposition 12.10 (i) *For all $\theta \in \mathscr{C}(\Gamma_2)$ there is an algebraic integer $s(r, \theta)$ such that uniformly for all $v \in S_{\mathrm{main}}$,*

$$\beta_{\mathrm{pl},v}^{(2)} = s(r, [\mathrm{Fr}_v]) + O \left(q_v^{-1} \right). \tag{12.25}$$

Here $[\mathrm{Fr}_v] \in \mathscr{C}(\Gamma_2)$ is the conjugacy class of Fr_v in Γ_2.
(ii) *The following identity holds*

$$s(r) = \sum_{\theta \in \mathscr{C}(\Gamma_2)} \frac{|\theta|}{|\Gamma_2|} s(r, \theta)$$

where $s(r) \in \{-1, 0, 1\}$ is the Frobenius-Schur indicator of r.

Proof (i) We proceed in way similar to the proof of Proposition 12.9 above. We shall give an explicit formula (12.26) for $s(r, \theta)$.

We decompose $\mathrm{Sym}^2 r = \oplus \rho_i^+$ (resp. $\bigwedge^2 r = \oplus \rho_i^-$) into a direct sum of irreducible representation of $\widehat{G} \rtimes \mathrm{Gal}(E/F)$. Then we can decompose for each i the restriction $\rho_i^+|_{\widehat{G} \rtimes W_{F_v}^{\mathrm{ur}}} = \oplus_j \rho_{ij}^+$ as a direct sum of irreducible representations of $\widehat{G} \rtimes W_{F_v}^{\mathrm{ur}}$. Similarly we let $\rho_i^-|_{\widehat{G} \rtimes W_{F_v}^{\mathrm{ur}}} = \oplus_j \rho_{ij}^-$.

Let $\widehat{\Phi}_{ij}^+ := (\rho_{ij}^+)^*(Y_1^2 + \cdots + Y_{d_i}^2)$ and similarly for ϕ_{ij}^-. Then it is easily verified that

$$\widehat{\Phi}_v^{(2)} = \sum_{ij} \widehat{\Phi}_{ij}^+ - \sum_{ij} \widehat{\Phi}_{ij}^-.$$

We now distinguish two cases. In the first case, i is such that ρ_i^+ does not factor through $\mathrm{Gal}(E/F)$. Then by Lemma 12.8 the restriction $\rho_i^+|_{\widehat{G}}$ does not contain the trivial representation. Thus for all j, $\rho_{ij}^+|_{\widehat{G}}$ does not contain the trivial representation. In particular ρ_{ij}^+ does not factor through $W_{F_v}^{\mathrm{ur}}$. By Lemma 2.9 we deduce that $\phi_{ij}^+(1) = O(q_v^{-1})$. These representations ρ_i^+ only contribute to the error term in (12.25).

In the second case, i is such that ρ_i^+ does factor through $\mathrm{Gal}(E/F)$. Then for all j, ρ_{ij}^+ factors through W_F^{ur} (in particular it is 1-dimensional). We have that $\widehat{\Phi}_{ij}^+(1) = \rho_{ij}^+(\mathrm{Fr}_v)$. By linearity we deduce that $\sum_j \phi_{ij}^+(1) = \mathrm{tr}\,\rho_i^+(\mathrm{Fr}_v)$. This is an algebraic integer which depends only on the conjugacy class of Fr_v in Γ_2.

We proceed in the same way for ϕ_{ij}^-. We deduce that (12.25) holds with $\theta = [\mathrm{Fr}_v]$ and $s(r, \theta) = s^+(r, \theta) - s^-(r, \theta)$, where

$$s^+(r, \theta) := \sum_{\substack{\rho_i^+ \text{ factors} \\ \text{through } \mathrm{Gal}(E/F)}} \mathrm{tr}\,\rho_i^+(\theta) \tag{12.26}$$

and similarly for the definition of $s^-(r, \theta)$. This concludes the proof of assertion (i).

(ii) By orthogonality of characters we have for each i such that ρ_i^+ factors through $\mathrm{Gal}(E/F)$,

$$\sum_{\theta \in \mathscr{C}(\Gamma_2)} \frac{|\theta|}{|\Gamma_2|} \mathrm{tr}\,\rho_i^+(\theta) = \langle \mathbb{1}, \rho_i^+ \rangle = \begin{cases} 1, & \text{if } \rho_i^+ = \mathbb{1} \\ 0, & \text{otherwise.} \end{cases}$$

We deduce that

$$\sum_{\theta \in \mathscr{C}(\Gamma_2)} \frac{|\theta|}{|\Gamma_2|} s^+(r, \theta) = \langle \mathbb{1}, \mathrm{Sym}^2 r \rangle,$$

the multiplicity of the trivial representation $\mathbb{1}$ in $\mathrm{Sym}^2 r$ [as a representation of $\widehat{G} \rtimes \mathrm{Gal}(E/F)$]. The same identity holds for $s^-(r, \theta)$ and $\bigwedge^2 r$. From the definition of the Frobenius–Schur indicator $s(r)$ in Sect. 6.8 we conclude the proof of the proposition. $\qquad\square$

As a corollary we have the following estimate for the second moment:

$$M^{(2)} = - \sum_{\theta \in \mathscr{C}(\Gamma_2)} s(r, \theta) \sum_{v \in S_{\mathrm{main}} \cap \mathcal{V}_F(\theta)} \frac{\log q_v}{\log C(\mathfrak{F}_k)} q_v^{-1} \widehat{\Phi}$$

$$\times \left(\frac{v \log q_v}{\log C(\mathfrak{F}_k)} \right) + O\left(\frac{1}{\log C(\mathfrak{F}_k)} \right).$$

We can extend the sum to $v \in S_{\text{gen}} \cap \mathcal{V}_F(\theta)$ because

$$\sum_{v \in \mathcal{V}_F^\infty - S_{\text{main}}} \frac{\log q_v}{\log C(\mathfrak{F}_k)} q_v^{-1} \ll \frac{\log \log C(\mathfrak{F}_k)}{\log C(\mathfrak{F}_k)} = o(1)$$

uniformly as $k \to \infty$. Applying the Cebotarev equidistribution theorem we deduce that

$$M^{(2)} = - \sum_{\theta \in \mathscr{C}(\Gamma_2)} s(r, \theta) \left(\frac{|\theta|}{|\Gamma_2|} + o(1) \right) \frac{1}{2} \int_0^\infty \widehat{\Phi}(y) dy$$

$$= \left(-\frac{s(r)}{2} + o(1) \right) \int_0^\infty \widehat{\Phi}(y) dy. \tag{12.27}$$

The last line follows from Proposition 12.10 (ii) above.

12.13 Conclusion

We now gather all the estimates and conclude the proof of Theorem 11.5. The explicit formula (12.5) expresses $D(\mathfrak{F}_k, \Phi)$ as the sum of four terms. The term $D_{\text{pol}}(\mathfrak{F}_k, \Phi)$ goes to zero as $k \to \infty$ as consequence of Hypothesis 11.2, see Sect. 12.4.

The archimedean terms are evaluated in (12.9). In addition with the second term in (12.5) which involves $\log q(\Pi)$, these contribute

$$\frac{\widehat{\Phi}(0)}{|\mathfrak{F}_k|} \sum_{\Pi \in \mathfrak{F}_k} \frac{\log C(\Pi)}{\log C(\mathfrak{F}_k)} + o(1).$$

This is equal to $\widehat{\Phi}(0) + o(1)$ (using the Hypothesis 11.4 for families in the level aspect).

We now turn to the non-archimedean contribution. The places $v \in S_0$ and $v \mid \mathfrak{n}_k$ are negligible thanks to (12.16) and (12.17), respectively.

It remains the non-archimedean places $v \in S_{\text{gen}} = S_{\text{main}} \sqcup S_{\text{cut}}$. The contribution from $v \in S_{\text{cut}}$ is zero because the support of $\widehat{\Phi}$ is included in $(-\delta, \delta)$, see (12.18).

For each $v \in S_{\text{main}}$ we apply Proposition 12.6. The sum over $v \in S_{\text{main}}$ of the remainder terms is shown to be negligible in (12.19) and (12.20). For the main term the estimate (12.21) shows that the contribution of the higher moments is negligible. It remains the two terms $M^{(1)}$ and $M^{(2)}$ as defined in (12.22).

The asymptotic of $M^{(1)}$ and $M^{(2)}$ are given in (12.24) and (12.27) respectively. There is a similar contribution from the conjugate $\overline{D_v}(\mathfrak{F}_k, \Phi)$. Overall this yields

$$\sum_{v \in S_{\text{main}}} D_v(\mathfrak{F}_k, \Phi) + \overline{D_v}(\mathfrak{F}_k, \Phi) = -\frac{s(r)}{2} \int_0^\infty \widehat{\Phi}(y)dy$$

$$-\frac{s(r)}{2} \int_{-\infty}^0 \widehat{\Phi}(y)dy + o(1) = -\frac{s(r)}{2} \Phi(0) + o(1).$$

We can now conclude that

$$\lim_{k \to \infty} D(\mathfrak{F}_k, \Phi) = \widehat{\Phi}(0) - \frac{s(r)}{2} \Phi(0).$$

This is the statement of Theorem 11.5. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix A: By Robert Kottwitz

Let F be a finite extension of $\mathbb{Q}_p$, and G a connected reductive group over F. For each semisimple $\gamma \in G(F)$, define a positive real number

$$D^G(\gamma) := \left| \det(1 - Ad(\gamma)|_{\text{Lie } G / \text{Lie } G_\gamma^0}) \right|_v = \prod_{\substack{\alpha \in \Phi \\ \alpha(\gamma) \neq 1}} |1 - \alpha(\gamma)|_v. \quad (13.1)$$

[In particular if γ belongs to the center of $G(F)$ then $D^G(\gamma) = 1$.] We equip $G(F)$, as well as $I_\gamma(F)$ (the connected centralizer of γ) for each semisimple $\gamma \in G(F)$, with the Haar measures as in [47, §4]. The quotient measure on $I_\gamma(F)\backslash G(F)$ is used to define the orbital integral $O_\gamma(f)$.

Theorem 13.1 *For each $f \in C_c^\infty(G(F))$, there exists a constant $c(f) > 0$ such that for all semisimple $\gamma \in G(F)$,*

$$|O_\gamma(f)| \leqslant c(f) \cdot D^G(\gamma)^{-1/2}.$$

Proof There are only finitely many $G(F)$-conjugacy classes of maximal F-tori in G, so in proving the theorem we can fix a maximal F-torus T in G and restrict attention to elements γ lying in $T(F)$. Then we must show that the function $\gamma \mapsto D^G(\gamma)^{1/2} O_\gamma(f)$ is bounded on $T(F)$. Harish-Chandra proved that the restriction of this function to the set of regular elements in $T(F)$ is bounded, so we just need to check that his methods can be used to treat singular γ as well.

Since the function $\gamma \mapsto D^G(\gamma)^{1/2} O_\gamma(f)$ is compactly supported on $T(F)$, it is enough to show that it is also locally bounded. Harish-Chandra's method of semisimple descent reduces us to proving local boundedness in a neighborhood of $1 \in T(F)$, and then the exponential map reduces us to the analogous problem on the Lie algebra $\mathfrak{g}$ of G. The remainder of this appendix handles $\mathfrak{g}$, the main result being Theorem 13.11. □

A.1 Notation pertaining to the Lie algebra version of the problem

We write $\mathfrak{t}$ for the Lie algebra of T. We write R for the (absolute) root system of T in G. We often write G for the group of F-points of G, etc. We will follow closely the exposition of Harish-Chandra's work given in [66]. Most of the proofs are just the same as the ones there and will therefore be omitted. (Instead of a proof, the reader will find the words "same as usual.") However, a couple of additional ingredients will be needed; these are simple adaptations of ideas in Sparling's article [100].

A.2 Orbital integrals O_X for $X \in \mathfrak{t}$

Let $X \in \mathfrak{t}$. The centralizer of X in G is a connected reductive F-subgroup of G that we will denote by M_X. (The reason for using the letter M is that this subgroup is a twisted Levi subgroup of G, i.e. an F-subgroup that becomes a Levi subgroup after extending scalars to an algebraic closure of F; however this fact is not actually needed below.) The set $\mathcal{M}$ of subgroups obtained in this way (as X varies in $\mathfrak{t}$) is finite.

The following notation will be useful. Let $M \in \mathcal{M}$. We write R_M for the (absolute) root system of M (a subset of R). We write $\mathfrak{z}_M$ for the Lie algebra of the center of M; we then have

$$\mathfrak{z}_M = \{X \in \mathfrak{t} : \alpha(X) = 0 \quad \forall \alpha \in R_M\}$$

For $X \in \mathfrak{t}$ we have $M_X = M$ if and only if

$$\{\alpha \in R : \alpha(X) = 0\} = R_M$$

or, in other words, if and only if X lies in the open subset

$$\mathfrak{z}'_M := \{X \in \mathfrak{z}_M : \alpha(X) \neq 0 \quad \forall \alpha \in R \backslash R_M\}$$

of $\mathfrak{z}_M$. Obviously $\mathfrak{t}$ is the disjoint union of the locally closed subsets $\mathfrak{z}'_M$. For example we have $\mathfrak{z}'_G = \mathfrak{z}_G$, while $\mathfrak{z}'_T$ is the set of regular elements in $\mathfrak{t}$.

We fix a Haar measure dg on G. In addition, for each $M \in \mathcal{M}$ we fix a Haar measure dm on M. For instance one can use the canonical measures defined by Gross. In any case, for $X \in \mathfrak{z}'_M$ we define the orbital integral O_X by

$$O_X(f) := \int_{M \backslash G} f(g^{-1} X g) \, dg/dm. \tag{13.2}$$

Thus we now have a coherent definition of orbital integrals for all $X \in \mathfrak{t}$.

A.3 Preliminary definition of Shalika germs on $\mathfrak{g}$

There are finitely many nilpotent G-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ in $\mathfrak{g}$. We write $\mu_1, \ldots, \mu_r$ for the corresponding nilpotent orbital integrals. The distributions $\mu_1, \ldots, \mu_r$ are linearly independent.

Theorem 13.2 *There exist functions $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ on $\mathfrak{t}$ having the following property. For every $f \in C_c^\infty(\mathfrak{g})$ there exists an open neighborhood U_f of 0 in $\mathfrak{t}$ such that*

$$O_X(f) = \sum_{i=1}^{r} \mu_i(f) \cdot \Gamma_i(X) \tag{13.3}$$

for all $X \in U_f$. The germs about $0 \in \mathfrak{t}$ of the functions $\Gamma_1, \ldots, \Gamma_r$ are unique. We refer to Γ_i as the provisional Shalika germ for the nilpotent orbit $\mathcal{O}_i$.

Proof Same as usual. □

A Shalika germ is an equivalence class of functions on $\mathfrak{t}$. As we will see next, the homogeneity of Shalika germs makes it possible to single out one

particularly nice function Γ_i within its equivalence class. Once we have done this, Γ_i will from then on denote this function (whose germ about 0 is the old Γ_i).

A.4 Behavior under scaling

For $\beta \in F^{\times}$ and $f \in C_c^{\infty}(\mathfrak{g})$ we write f_{β} for the function on $\mathfrak{g}$ defined by

$$f_{\beta}(X) := f(\beta X). \tag{13.4}$$

Harish-Chandra proved that

$$\mu_{\mathcal{O}}(f_{\alpha^2}) = |\alpha|^{-\dim \mathcal{O}} \mu_{\mathcal{O}}(f) \tag{13.5}$$

for every nilpotent orbit $\mathcal{O}$ and $\alpha \in F^{\times}$. Moreover it is clear from (13.2) that

$$O_X(f_{\beta}) = O_{\beta X}(f) \tag{13.6}$$

for all $X \in \mathfrak{t}$ and all $\beta \in F^{\times}$.

A.5 Partial homogeneity of our provisional Shalika germs Γ_i

Let $\alpha \in F^{\times}$. Let $\mathcal{O}_i$ be one of our nilpotent orbits, let μ_i be the corresponding nilpotent orbital integral, and let Γ_i be the corresponding Shalika germ. Put $d_i := \dim \mathcal{O}_i$. We claim that

$$\Gamma_i(X) = |\alpha|^{d_i} \Gamma_i(\alpha^2 X), \tag{13.7}$$

where the equality means equality of germs about 0 of functions on $\mathfrak{t}$.

Indeed, as in the proof of the Shalika germ expansion on G, pick a function $f_i \in C_c^{\infty}(\mathfrak{g})$ such that

$$\mu_j(f_i) = \delta_{ij}. \tag{13.8}$$

Then $\Gamma_i(X)$ is the germ about 0 of the function

$$X \mapsto O_X(f_i) \tag{13.9}$$

on $\mathfrak{t}$. In fact during the remainder of our discussion of provisional germs, we will use always use (13.9) as our choice for a specific function Γ_i having the right germ.

In view of the homogeneity of nilpotent orbital integrals established above, $|\alpha|^{d_i} \cdot (f_i)_{\alpha^2}$ can also serve as f_i, so that $\Gamma_i(X)$ is also the germ about 0 of the function

$$X \mapsto O_X \left(|\alpha|^{d_i} \cdot (f_i)_{\alpha^2} \right) = |\alpha|^{d_i} \cdot O_{\alpha^2 X}(f_i) \qquad (13.10)$$

on $\mathfrak{t}$. Comparing (13.9), (13.10), we see that the germs of $\Gamma_i(X)$ and $|\alpha|^{d_i}\Gamma_i(\alpha^2 X)$ are equal, as desired.

A.6 Canonical Shalika germs

Let Γ_i be one of our germs. We are going to replace Γ_i by another function Γ_i^{new} on $\mathfrak{t}$ that has the same germ about 0 and is at the same time homogeneous.

Lemma 13.3 *There is a unique function Γ_i^{new} on $\mathfrak{t}$ which has the same germ about 0 as Γ_i and which satisfies (13.7) for all $\alpha \in F^\times$ and all $X \in \mathfrak{t}$. Moreover Γ_i^{new} is real-valued, translation invariant under the center of $\mathfrak{g}$, and invariant under conjugation by elements in the normalizer of T.*

Proof Same as usual. □

From now on we replace the germs Γ_i by the functions Γ_i^{new}, but we drop the superscript "new."

We also need a slight strengthening of the fact that Γ_i is translation invariant under the center $\mathfrak{z}$ of $\mathfrak{g}$. Let G' be the derived group of the algebraic group G. Then $G(\bar{F}) = G'(\bar{F})Z(\bar{F})$, but for F-points we have only that $G'Z$ is a normal subgroup of finite index in G. We denote by D the finite group $G/G'Z$. Each G-orbit $\mathcal{O}$ in $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ decomposes as a finite union of G'-orbits $\mathcal{O}'$, permuted transitively by D. We normalize the invariant measures on the orbits in such a way that

$$\int_{\mathcal{O}} = \sum_{x \in D} \int_{x^{-1}\mathcal{O}'x}. \qquad (13.11)$$

For a nilpotent G-orbit $\mathcal{O}$ (respectively, nilpotent G'-orbit $\mathcal{O}'$) we denote by $\Gamma_{\mathcal{O}}^G$ (respectively, $\Gamma_{\mathcal{O}'}^{G'}$) the corresponding Shalika germ on $\mathfrak{t}$ (respectively, $\mathfrak{g}' \cap \mathfrak{t}$).

Lemma 13.4 *Let $X \in \mathfrak{t}$ and decompose X as $X' + Z$ with $X' \in \mathfrak{g}' \cap \mathfrak{t}$ and $Z \in \mathfrak{z}$. Then*

$$\Gamma_{\mathcal{O}}^G(X) = \sum_{\mathcal{O}' \subset \mathcal{O}} \Gamma_{\mathcal{O}'}^{G'}(X'). \qquad (13.12)$$

Proof Same as usual, but note that there is a typo in the proof of the corresponding result in [66]: the functions f, f' occurring in formula (17.8.9) of that article should have a subscript $\mathcal{O}$.

A.7 Germ expansions about arbitrary central elements in $\mathfrak{g}$

We have been studying germ expansions about $0 \in \mathfrak{t}$. These involve orbital integrals for the nilpotent orbits $\mathcal{O}_i$. Now we consider germ expansions about an arbitrary element Z in the center of $\mathfrak{g}$. These will involve orbital integrals $\mu_{Z+\mathcal{O}_i}$ for the orbits $Z + \mathcal{O}_i$, but will involve exactly the same germs Γ_i as before.

Theorem 13.5 *Let Z be an element in the center of $\mathfrak{g}$. For every $f \in C_c^\infty(\mathfrak{g})$ there exists an open neighborhood U_f of Z in $\mathfrak{t}$ such that*

$$O_X(f) = \sum_{i=1}^{r} \mu_{Z+\mathcal{O}_i}(f) \cdot \Gamma_i(X) \tag{13.13}$$

for all $X \in U_f$.

Proof Same as usual. □

A.8 Germ expansions about arbitrary semisimple elements in $\mathfrak{g}$

We are going to use Harish-Chandra's theory of semisimple descent in order to obtain germ expansions about an arbitrary element $S \in \mathfrak{t}$. We fix such an element S and let $H := G_S$ denote the centralizer of S, a connected reductive subgroup of G.

Let $Y_1, \ldots, Y_s$ be a set of representatives for the nilpotent H-orbits in $\mathfrak{h}$. Let μ_{S+Y_i} denote the orbital integral on $\mathfrak{g}$ obtained by integration over the G-orbit of $S+Y_i$. Now T is also a maximal torus in H, so for each $1 \leqslant i \leqslant s$ we can consider the canonical Shalika germ Γ_i^H for H, $\mathfrak{t}$ and the nilpotent H-orbit of Y_i.

Theorem 13.6 *Let S, H be as above. For every $f \in C_c^\infty(\mathfrak{g})$ there exists an open neighborhood U_f of S in $\mathfrak{t}$ such that*

$$O_X(f) = \sum_{i=1}^{s} \mu_{S+Y_i}(f) \cdot \Gamma_i^H(X) \tag{13.14}$$

for all $X \in U_f$.

Proof Same as usual.

A.9 Normalized orbital integrals and Shalika germs

For $X \in \mathfrak{t}$ we put

$$D^G(X) = \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{m}_X)$$

($\mathfrak{m}_X$ being the Lie algebra of the centralizer M_X of X in G) and define the normalized orbital integral I_X by

$$I_X = |D^G(X)|^{1/2} O_X.$$

When we use I_X instead of O_X, we need to use the normalized Shalika germs $\bar{\Gamma}_i(X) := |D^G(X)|^{1/2} \Gamma_i(X)$ instead of the usual Shalika germs.

Clearly Theorem 13.2 remains valid when O_X, Γ_X are replaced by I_X, $\bar{\Gamma}_i$ respectively. Now consider the germ expansion about an arbitrary element $S \in \mathfrak{t}$. As usual put $H := G_S$. There exists a neighborhood of S in $\mathfrak{t}$ on which

$$|D^G(X)|^{1/2} = |D^H(X)|^{1/2} |\det(\mathrm{ad}(S); \mathfrak{g}/\mathfrak{h})|^{1/2}.$$

It then follows from Theorem 13.6 that

$$I_X(f) = |\det(\mathrm{ad}(S); \mathfrak{g}/\mathfrak{h})|^{1/2} \sum_{i=1}^{s} \mu_{S+Y_i}(f) \cdot \bar{\Gamma}_i^H(X) \qquad (13.15)$$

for all X in some sufficiently small neighborhood of S in $\mathfrak{t}$.

The homogeneity property (13.7) of the Shalika germs Γ_i implies the following homogeneity property for the normalized Shalika germs $\bar{\Gamma}_i$:

$$\bar{\Gamma}_i(\alpha^2 X) = |\alpha|^{\dim(G_{X_i}) - \dim(M_X)} \cdot \bar{\Gamma}_i(X) \qquad (13.16)$$

for all $\alpha \in F^\times$ and all $X \in \mathfrak{t}$. Here we have chosen $X_i \in \mathcal{O}_i$ and introduced its centralizer G_{X_i}.

The next proposition will be needed when we use (13.16) in the proof of boundedness of normalized Shalika germs. It is a simple adaptation of ideas from Sparling's article [100]. To formulate the proposition we need a definition. Consider the action morphism $G \times \mathfrak{g} \to \mathfrak{g}$ (given by $(g, X) \mapsto gXg^{-1}$); we are now thinking of G and $\mathfrak{g}$ as algebraic varieties over F. For $M \in \mathcal{M}$ we consider the image $V_M^0 \subset \mathfrak{g}$ of $G \times \mathfrak{z}_M'$ under this morphism. Obviously V_M^0 is an irreducible G-invariant subset of the variety $\mathfrak{g}$, so its Zariski closure V_M is a G-invariant irreducible subvariety of $\mathfrak{g}$. We say that a nilpotent orbit $\mathcal{O}$ is *relevant to M* if $\mathcal{O}$ is contained in V_M.

Proposition 13.7 *Let $M \in \mathcal{M}$ and let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Then the following two statements hold.*

(i) *If $\mathcal{O}$ is relevant to M, then for $Y \in \mathcal{O}$ we have $\dim G_Y \geqslant \dim M$, where G_Y denotes the centralizer of Y in G.*

(ii) *If $\mathcal{O}$ is not relevant to M, then the normalized Shalika germ $\overline{\Gamma}_\mathcal{O}$ vanishes identically on $\mathfrak{z}_M'$.*

Proof (1) Over V_M we have the group scheme whose fiber at $X \in V_M$ is the centralizer of X in G. At points in $\mathfrak{z}'_M$ this centralizer is M and at points of V^0_M it is some conjugate of M. Since V^0_M is dense in V_M, we conclude from SGA 3, Tome I, Exp. VI$_B$, Prop. 4.1 that $\dim G_X \geqslant \dim M$ for all $X \in V_M$. In particular this inequality holds when we take X to be $Y \in \mathcal{O} \subset V_M$.

(2) Let $f \in C_c^\infty \mathfrak{g}$ and suppose that $\mu_{\mathcal{O}}(f) = 0$ for all nilpotent orbits $\mathcal{O}$ relevant to M. Then, as in the proof of the existence of Shalika germs, there exists an open neighborhood U_f of 0 in $\mathfrak{t}$ such that $O_X(f) = 0$ for all $X \in U_f \cap V_M$. In particular $O_X(f) = 0$ for all $X \in U_f \cap \mathfrak{z}'_M$. Applying this observation to the functions f_j used to produce our provisional Shalika germs, we conclude that if $\mathcal{O}_j$ is not relevant to M, then there is a neighborhood U_j of 0 in $\mathfrak{t}$ such that the provisional Shalika germ Γ_j vanishes on $U_j \cap \mathfrak{z}'_M$. Looking back at how the true (homogeneous) Shalika germs were obtained from the provisional ones, we see that the true Shalika germ Γ_j vanishes identically on $\mathfrak{z}'_M$ when $\mathcal{O}_j$ is not relevant to M. □

A.10 $\bar{\Gamma}_i$ is a linear combination of functions $\bar{\Gamma}_j^H$ in a neighborhood of S

Again let $S \in \mathfrak{t}$ and let H be its centralizer in G. Consider one of the normalized Shalika germs $\bar{\Gamma}_i$ for G. We are interested in the behavior of $\bar{\Gamma}_i$ in a small neighborhood of S in $\mathfrak{t}$.

Lemma 13.8 *There exists a neighborhood V of S in $\mathfrak{t}$ such that the restriction of $\bar{\Gamma}_i$ to V is a linear combination of restrictions of normalized Shalika germs for H.*

Proof Same as usual. □

Corollary 13.9 *Let $M \in \mathcal{M}$. Each normalized Shalika germ $\bar{\Gamma}_i$ is locally constant on $\mathfrak{z}'_M$.*

Proof Same as usual. □

A.11 Locally bounded functions

We are going to show that the normalized Shalika germs $\bar{\Gamma}_i$ are locally bounded functions on $\mathfrak{t}$. First let's recall what this means. Let f be a complex-valued function on a topological space X. We say that f is *locally bounded* on X if every point $x \in X$ has a neighborhood U_x such that f is bounded on U_x. When X is a locally compact Hausdorff space, f is locally bounded if and only f is bounded on every compact subset of X.

A.12 Local boundedness of normalized Shalika germs

Let $\bar{\Gamma}_i$ be one of our normalized Shalika germs on $\mathfrak{t}$. We are going to show that $\bar{\Gamma}_i$ is locally bounded as a function on $\mathfrak{t}$, slightly generalizing a result of Harish-Chandra.

Theorem 13.10 *Every normalized Shalika germ $\bar{\Gamma}_i$ is locally bounded on $\mathfrak{t}$.*

Proof Same as usual once one takes into account Proposition 13.7. □

As a consequence of the local boundedness of normalized Shalika germs, we obtain a slight generalization of another result of Harish-Chandra.

Theorem 13.11 *Let $f \in C_c^\infty(\mathfrak{g})$. Then the function $X \mapsto I_X(f)$ on $\mathfrak{t}$ is bounded and compactly supported on $\mathfrak{t}$. Moreover, for each $M \in \mathcal{M}$ this function is locally constant on $\mathfrak{z}'_M$.*

Proof Same as usual. □

Appendix B: By Raf Cluckers, Julia Gordon and Immanuel Halupczok

In this appendix we use the theory of motivic integration to control bounds for orbital integrals, normalized by the discriminant, as the place varies. In Appendix A, the bound for orbital integrals is proved for a fixed local field; here we show that this bound cannot exceed a power of the cardinality of the residue field, using the tools from model theory. We emphasize that the main result of Appendix A, namely, the fact that the orbital integrals are bounded, is used in our proof. More specifically, we prove Theorems 14.1, and 14.2 which are stronger versions of, respectively, Theorem 7.3 and Proposition 7.1 with $e_G = 1$. We also prove the analogous statement for the function fields; moreover, we prove that the optimal exponents can, in some sense, be transferred between the function field and number field cases, see Theorem 14.7. We expect that the same methods could apply to weighted orbital integrals, provided that one had a statement similar to the Theorem 13.1 of Appendix A.

Let $\mathbf{F}$ be a number field with the ring of integers $\mathcal{O}$. Let G be a connected reductive algebraic group defined over $\mathbf{F}$, and $\mathfrak{g}$ its Lie algebra. Let $F = \mathbf{F}_v$ be a completion of $\mathbf{F}$. We denote the ring of integers of F by $\mathcal{O}_F$, the residue field by k_F, and let $q_F = \#k_F$ be the cardinality of k_F. For a semisimple element $\gamma \in G(F)$ and a test function $f \in C_c^\infty(G(F))$, the orbital integral at γ is denoted by $\mathbf{O}_\gamma(f)$. As in Appendix A,

$$D^G(\gamma) = \prod_{\substack{\alpha \in \Phi \\ \alpha(\gamma) \neq 1}} |1 - \alpha(\gamma)|_v,$$

where Φ is the root system of G.

We keep the set-up of Sects. 7.1 and 7.2; in particular, we first treat the case of a reductive group with a given root datum defined over a local field, and then derive the global statement from it. Thus, we start with a reductive group G defined over a local field F, and we assume that G is unramified. In order to get to this setting from the global set-up, we just have to assume that $G = \mathbf{G}_v$ where the place v is finite, and lies outside the set $\mathrm{Ram}(G)$.

Given an unramified reductive group G over a local field F as above, we recall the definition of the functions τ_λ^G from Sect. 2.2. We have a Borel subgroup $B = TU$, and let A be the maximal F-split torus in T. As in Sect. 2.2, choose a smooth reductive model $\underline{G}$ for G corresponding to a hyperspecial point in the apartment of A, and let $K = \underline{G}(\mathcal{O}_F)$ be a maximal compact subgroup. For $\lambda \in X_*(A)$, τ_λ^G is the characteristic function of the double coset $K\lambda(\varpi)K$.

We prove

Theorem 14.1 *Let G be a connected reductive algebraic group over $\mathbf{F}$, with $\mathbf{T}$ and A_v as in Sect. 7.2. There exist constants a_G and b_G that depend only on the global model of G such that for all $\lambda \in X_*(A_v)$ with $\|\lambda\| \leqslant \kappa$, for all but finitely many places v*

$$\left| \mathbf{O}_\gamma(\tau_\lambda^G) \right| \leqslant q_v^{a_G + b_G \kappa} D^G(\gamma)^{-1/2}$$

for all semisimple elements $\gamma \in G(\mathbf{F}_v)$, where q_v is the cardinality of the residue field of $\mathbf{F}_v$.

In fact, we prove a stronger and more general statement, which does not require F to have characteristic zero. By an unramified root datum we mean a root datum of an unramified reductive group over a local field F, i.e. a quintuple $\xi = (X^*, \Phi, X_*, \Phi^\vee, \theta)$, where θ is the action of the Frobenius element of F^{ur}/F on the first four components of ξ.

Theorem 14.2 *Consider an unramified root datum ξ. Then there exist constants $M > 0$, a_ξ and b_ξ that depend only on ξ, such that for each non-Archimedean local field F with residue characteristic at least M, the following holds. Let G be a connected reductive algebraic group over F with the root datum ξ. Let A be a maximal F-split torus in G, and let τ_λ^G be as above. Then for all $\lambda \in X_*(A)$ with $\|\lambda\| \leqslant \kappa$,*

$$\left| \mathbf{O}_\gamma(\tau_\lambda^G) \right| \leqslant q_F^{a_\xi + b_\xi \kappa} D^G(\gamma)^{-1/2}$$

for all semisimple elements $\gamma \in G(F)$.

The strategy of the proof is to use the theory of motivic integration developed by Cluckers and Loeser [29]. In [29], a class of functions called *con-*

structible motivic functions is defined. Here, in order to simplify the language, we are working directly with the specializations of constructible motivic functions, which we define below, and we call these "constructible functions". These functions are defined by means of formulas in a first-order language of logic, called Denef-Pas language, which we review below. The key benefit of using logic is that the formulas defining the functions are independent of the field of definition, hence this set-up is perfectly suited for proving a result that applies uniformly across almost all completions of a given number field. This method can be thought of as an extension of a geometric approach—"definable" is a less restrictive notion than "geometric", yet it provides a field-independent way of talking about orbital integrals.

The key to our proof is a general result which, roughly speaking, states that if a constructible function is bounded (which is known in our case thanks to Appendix A), then its upper bound cannot exceed a fixed power of the cardinality of the residue field (Theorem 14.6 below). In order to apply this result to orbital integrals, we need to show that they are, in some sense, constructible functions. More precisely, one would like to show that given a constructible test function $f \in C_c^\infty(G(F))$, the function $\gamma \mapsto \mathbf{O}_\gamma(f)$ is a constructible function of γ, on the set of all semisimple elements. For *regular* semisimple elements, the Lie algebra version of this statement is essentially proved by Cluckers et al. [27]. For general elements X, the Lie algebra version of this statement with a particular normalization of the measure on the orbit is proved in [26]; however, the normalization of the measures used in [26] is not the same as the canonical normalization used in Appendix A above. For non-regular semisimple elements, we show here that the canonical measure differs from a motivic measure by a constant that can be bounded by a fixed power of the cardinality of the residue field, and consequently, obtain that given f, there exists a constructible function H_f and a constant c that depends only on the root datum of the group, such that $|H_f(\gamma)| \leqslant |O_\gamma(f)| \leqslant q^c|H_f(\gamma)|$. Taking f to be the characteristic function of the maximal compact subgroup K in this argument, we obtain the special case of Theorem 14.2 with $\kappa = 0$. The full statement of Theorem 14.2 is obtained by a similar argument that allows the test functions to vary in definable families.

Much of the preliminary and introductory material is quoted freely from [25–28,43], sometimes without mentioning these ubiquitous citations.

B.1 Denef-Pas language

The Denef-Pas language is a first order language of logic designed for working with valued fields. We start by defining two sublanguages of the language of Denef-Pas: the language of rings and Presburger language.

B.1.1 The language of rings

A formula in the first-order language of rings is any syntactically correct formula built out of the following symbols:

– constants '0', '1';
– binary functions '×', '+';
– countably many symbols for variables $x_1, \ldots, x_n, \ldots$ running over a ring;
– the following logical symbols: equality '=', parentheses '(', ')', the quantifiers '∃', '∀', and the logical operations conjunction '∧', negation '¬', disjunction '∨'.

If a formula in the language of rings has n free (i.e. unquantified) variables then it defines a subset of R^n for any ring R. Note that quantifier-free formulas in the language of rings define constructible sets (in the sense of algebraic geometry).

B.1.2 Presburger language

A formula in Presburger's language is built out of variables running over $\mathbb{Z}$, the logical symbols (as above) and symbols '+', '⩽', '0', '1', and for each $d = 2, 3, 4, \ldots$, a symbol '$\equiv_d$' to denote the binary relation $x \equiv y \pmod{d}$. Note the absence of the symbol for multiplication.

Since multiplication is not allowed, sets defined by formulas in the Presburger language are in fact very basic, cf. [23] or [81]. For example, $\{(a, b) \in \mathbb{Z}^2 \mid a \equiv 1 \mod 4; a \leq b + 10\}$ is a Presburger subset of $\mathbb{Z}^2$. Since quantifiers are never needed to describe Presburger sets, they all are of a similar, simple form.

B.1.3 Denef-Pas language

The formulas in Denef-Pas language have variables of three sorts: the valued field sort, the residue field sort, and the value group sort (in our setting, the value group is always assumed to be $\mathbb{Z}$, so we will call this sort the $\mathbb{Z}$-sort). Here is the list of symbols used to denote operations and binary relations in this language:

– In the valued field sort: the language of rings.
– In the residue field sort: the language of rings.
– In the $\mathbb{Z}$-sort: the Presburger language.
– a symbol ord(·) for the valuation map from the nonzero elements of the valued field sort to the $\mathbb{Z}$-sort, and $\overline{\mathrm{ac}}(\cdot)$ for the so-called angular component, which is a function from the valued field sort to the residue field sort (more about this function below).

On top of the symbols for the constants that are already present (like 0 and 1), we will add to the Denef-Pas language all elements of $\mathcal{O}[[t]]$ as extra symbols for constants in the valued field sort. We denote this language by $\mathcal{L}_\mathcal{O}$.

Given a discretely valued field F that is an algebra over $\mathcal{O}$, together with a choice of a ring homomorphism $\iota : \mathcal{O} \to F$ and a choice of a uniformizer ϖ of the valuation, one can interpret the formulas in $\mathcal{L}_\mathcal{O}$ by letting the variables range, respectively, over F, the residue field k_F of F, and $\mathbb{Z}$ (which is the value group of F). The function symbols ord and $\overline{ac}$ are interpreted as follows. For $x \in F^\times$, $\text{ord}(x)$ denotes the valuation of x. If x is a unit (that is, $\text{ord}(x) = 0$), then $\overline{ac}(x)$ is the residue of x modulo ϖ (thus, an element of the residue field). For a general $x \neq 0$ define $\overline{ac}(x)$ as $\overline{ac}(\varpi^{-\text{ord}(x)}x)$; thus, $\overline{ac}(x)$ is the first non-zero coefficient of the ϖ-adic expansion of x. Finally we define $\overline{ac}(0) = 0$. The elements from $\mathcal{O}$ are interpreted as elements of F by using ι, the constant symbol t is interpreted as the uniformizer ϖ, and thus, by the completeness of F, elements of $\mathcal{O}[[t]]$ can be naturally interpreted in F as well.

Definition 14.3 Let $\mathcal{C}_\mathcal{O}$ be the collection of all triples (F, ι, ϖ), where F is a non-Archimedean local field which allows at least one ring homomorphism from $\mathcal{O}$ to F, the map $\iota : \mathcal{O} \to F$ is such a ring homomorphism, and ϖ is a uniformizer for F. Let $\mathcal{A}_\mathcal{O}$ be the collection of those triples (F, ι, ϖ) in $\mathcal{C}_\mathcal{O}$ in which F has characteristic zero, and let $\mathcal{B}_\mathcal{O}$ be the collection of those triples (F, ι, ϖ) where F has positive characteristic.

Given an integer M, let $\mathcal{C}_{\mathcal{O},M}$ be the collection of (F, ι, ϖ) in $\mathcal{C}_\mathcal{O}$ such that the residue field of F has characteristic larger than M, and similarly for $\mathcal{A}_{\mathcal{O},M}$ and $\mathcal{B}_{\mathcal{O},M}$.

Since our results and proofs are independent of the choices of the map ι and the uniformizer ϖ, we will often just write $F \in \mathcal{C}_\mathcal{O}$, instead of naming the whole triple. For any $F \in \mathcal{C}_\mathcal{O}$, write $\mathcal{O}_F$ for the valuation ring of F, k_F for its residue field, and q_F for the cardinality of k_F.

In summary, an $\mathcal{L}_\mathcal{O}$-formula φ with n free valued-field variables, m free residue-field variables, and r free $\mathbb{Z}$-variables defines naturally, for each $F \in \mathcal{C}_\mathcal{O}$, a subset of $F^n \times k_F^m \times \mathbb{Z}^r$ by taking the set of all tuples where φ is "true" (in the natural sense of first order logic, see e.g. [73]).

B.2 Definable sets and constructible functions

As mentioned in the introduction, to study dependence on p of various bounds we will need to have a field-independent notion of subsets of $F^n \times k_F^m \times \mathbb{Z}^r$ for $F \in \mathcal{C}_\mathcal{O}$. To achieve this, we call a collection $(X_F)_F$ of subsets $X_F \subset F^n \times k_F^m \times \mathbb{Z}^r$, where F runs over $\mathcal{C}_\mathcal{O}$, which come from an $\mathcal{L}_\mathcal{O}$-formula φ as explained at the end of Sect. B.1.3, *a definable set*. Thus, for us, a "definable set" is actually a collection of sets, namely one for each $F \in \mathcal{C}_\mathcal{O}$; in earlier work

on motivic integration, the term "specialization of a definable subassignment" was used for a similar notion. For an integer $r \geq 0$, $\mathbb{Z}^r$ will often denote the definable set $(X_F)_F$ such that $X_F = \mathbb{Z}^r$ for each F. More generally, for non-negative integers n, m, r, the notation $h[n, m, r]$ will stand for the definable set $(F^n \times k_F^m \times \mathbb{Z}^r)_F$.

For definable sets X and Y, a collection $f = (f_F)_F$ of functions $f_F \colon X_F \to Y_F$ for $F \in \mathcal{C}_\mathcal{O}$ is called a definable function and denoted by $f \colon X \to Y$ if the collection of graphs of f_F is a definable set.

Definable functions are the building blocks for constructible functions, which are defined as follows. For a definable set X, a collection $f = (f_F)_F$ of functions $f_F \colon X_F \to \mathbb{C}$ is called *a constructible function* if there exist integers N, N', and N'', such that f_F has the form, for $x \in X_F$, for all $F \in \mathcal{C}_\mathcal{O}$,

$$ f_F(x) = \sum_{i=1}^{N} q_F^{\alpha_{iF}(x)} \# \left(p_{iF}^{-1}(x) \right) \left(\prod_{j=1}^{N'} \beta_{ijF}(x) \right) \left(\prod_{\ell=1}^{N''} \frac{1}{1 - q_F^{a_{i\ell}}} \right), $$

where:

- $a_{i\ell}$ with $i = 1, \ldots, N$, $\ell = 1, \ldots, N''$ are negative integers;
- $\alpha_i \colon X \to \mathbb{Z}$ with $i = 1, \ldots N$, and $\beta_{ij} \colon X \to \mathbb{Z}$ with $i = 1 \ldots, N$, $j = 1, \ldots, N'$ are $\mathbb{Z}$-valued definable functions;
- Y_i are definable sets such that $Y_{iF} \subset k_F^{r_i} \times X_F$ for some $r_i \in \mathbb{Z}$, and $p_i \colon Y_i \to X$ is the coordinate projection.

The motivation for such a definition of a constructible function comes from the theory of integration: namely, constructible functions form a rich class of functions which is stable under integration with respect to parameters (as in Theorem 14.4 below). See [28,43] for details.

For each F in $\mathcal{C}_\mathcal{O}$, let us put the Haar measure on F so that $\mathcal{O}_F$ has measure 1, the counting measure on k_F and on $\mathbb{Z}$, and the product measure on Cartesian products. Thus, we get a natural measure on $h[n, m, r]$. Furthermore, any analytic subvariety of F^n, say, everywhere of equal dimension, together with an analytic volume form, carries a natural measure associated to the volume form, cf. [13].

The notion of a measure associated with a volume form carries over to the definable setting, roughly as follows. By the piecewise analytic nature of definable sets and definable functions, any definable subset X of $h[n, m, r]$ can be broken into finitely many pieces X_i, such that $X_i(F)$ is a subset of $V_i \times k_F^m \times \mathbb{Z}^r$ for some F-analytic subvariety V_i of F^n of the same dimension as $X_i(F)$, for each F with large residue characteristic. A definable form on $h[n, 0, 0]$ in the affine coordinates x is just a finite sum of terms of the form $f(x)dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ where f is a definable function with values in $h[1, 0, 0]$. If the functions f restrict to F-analytic functions on V_i for each such f, and if the

form is a d-form where d is the dimension of V_i, then one can use the measure associated to this analytic volume form on V_i. This construction yields natural "motivic" measures on the definable set X, associated to definable differential forms, cf. also [26, §3.5.1]. Such a construction of measures associated with differential forms behaves well in the setting of motivic integrals because there exists a natural change of variables formula for motivic integrals, see §15 of [29]. In summary, the measures that arise from definable differential forms occur naturally in the context of motivic integration and we will call such measures "motivic" below. We refer to [29, §15] for the definition of the sheaf of definable differential forms on a definable set, and other details. We note that any algebraic volume form on a variety over $\mathcal{O}_{\mathbf{F}}$, where $\mathbf{F}$ is a global field, is definable in this sense. Note, however, that in this appendix we have to deal with volume forms on orbits of elements of a group defined over a local field, and the resulting measures are not automatically motivic.

Let us recall one of the results of [25], the first part of which generalizes a result of [30], and which shows that the class of constructible functions is a natural class to work with for the purposes of integration.

Theorem 14.4 [25, Theorem 4.3.1] *Let f be a constructible function on $X \times Y$ for some definable sets X and Y. Then there exist a constructible function g on X and an integer $M > 0$ such that for each $F \in \mathcal{C}_{\mathcal{O},M}$ and for each $x \in X_F$ one has*

$$g_F(x) = \int_{y \in Y_F} f_F(x, y),$$

whenever the function $Y_F \to \mathbb{C} \colon y \mapsto f_F(x, y)$ lies in $L^1(Y_F)$, where, say $Y_F \subset F^n \times k_F^m \times \mathbb{Z}^r$.

Note that although the theorem is stated for the affine measure on F^n, it also holds for measures given by definable differential forms, by working with charts as is done in [29, §15].

Remark 14.5 In the literature on general motivic integration, one often uses a more abstract notion of "definable subassignments". Any such definable subassignment X specializes to the sets X_F discussed here for all $F \in \mathcal{C}_{\mathcal{O},M}$ for some M, and any motivic integral over X specializes to the corresponding integrals over X_F. In this paper it is sufficient and more convenient to work with the above notion of definable sets $(X_F)_F$ directly.

Let us finally fix our terminology about "families of definable sets" and "families of constructible functions". A family of definable sets X_a indexed by a parameter $a \in A$ is a definable subset X of $Y \times A$ for some definable sets Y and A, equipped with the canonical projection $p_A \colon X \to A$,

and the family members are $p_A^{-1}(a) = X_a$ for $a \in A$. Similarly, a family of constructible (respectively, definable) functions f_a on the family X_a is a constructible (respectively, definable) function on $X \subset Y \times A$. Whenever we call a specific function $f : X_{F_0} \subset F_0^n \times k_{F_0}^m \times \mathbb{Z}^r \to \mathbb{C}$ (for a specific field F_0) constructible, we mean that it appears naturally as f_{F_0} for a constructible function $(f_F)_F$ for which uniformity in F is clear from the context as soon as the residue field characteristic is large enough; we use a similar convention for calling a specific function definable, and so on.

Finally, we will occasionally need to take roots of q (in order to take the square root of the absolute value of the discriminant, for example). We adopt the same convention as in [26, §B.3.1], and call any expression of the form $H q_F^{\frac{1}{r} f}$, or a finite sum of such expressions, a motivic function, where f is a $\mathbb{Z}$-valued definable function, and H is a motivic function in the usual sense defined above. All the results about motivic functions generalize to this setting by splitting the domain into finitely many pieces according to $(f \mod r)$. We note also that the boundedness results from the next section for such functions reduce to the same results without fractional powers by considering the r-th power.

B.3 Boundedness of constructible functions

The following two theorems are the main results of this section.

Theorem 14.6 *Let H be a constructible motivic function on $W \times \mathbb{Z}^n$, where W is a definable set. Then there exist integers a, b and M such that for all $F \in \mathcal{C}_{\mathcal{O},M}$ the following holds.*

If there exists a (set-theoretical, and not necessarily uniform in F) function $\alpha^F : \mathbb{Z}^n \to \mathbb{R}$ such that

$$|H_F(w, \lambda)|_{\mathbb{R}} \leqslant \alpha^F(\lambda) \quad \text{on } W_F \times \mathbb{Z}^n,$$

then one actually has

$$|H_F(w, \lambda)|_{\mathbb{R}} \leqslant q_F^{a+b\|\lambda\|} \quad \text{on } W_F \times \mathbb{Z}^n,$$

where $\|\lambda\| = \sum_{i=1}^n \lambda_i$, and where $|\cdot|_{\mathbb{R}}$ is the usual absolute value on $\mathbb{R}$.

We observe that in the case with $n = 0$, the theorem yields that if a constructible function H on W is such that H_F is bounded on W_F for each $F \in \mathcal{C}_{\mathcal{O},M}$, then the bound for $|H_F|_{\mathbb{R}}$ can be taken to be q_F^a uniformly in F with large residue characteristics, for some $a \geq 0$.

The following statement allows one to transfer bounds, which are known for local fields of characteristic zero, to local fields of positive characteristic, and vice versa.

Theorem 14.7 *Let H be a constructible motivic function on $W \times \mathbb{Z}^n$, where W is a definable set, and let a and b be integers. Then there exists M such that, for any $F \in \mathcal{C}_{\mathcal{O},M}$, whether the statement*

$$H_F(w, \lambda) \leqslant q_F^{a+b\|\lambda\|} \quad \text{for all } (w, \lambda) \in W_F \times \mathbb{Z}^n \qquad (14.1)$$

holds or not, only depends on the isomorphism class of the residue field of F.

Informally speaking, the idea of the proof is to first eliminate all the valued-field variables, possibly at the cost of introducing more residue-field and $\mathbb{Z}$-valued variables. This step is summarized in Lemma 14.8 below, whose proof relies on the powerful cell decomposition theorem for definable sets in Denef-Pas language. Once we have a constructible function that depends only on the residue-field and value-group variables, we note that the residue-field variables can only play a very minor role in the matters of boundedness (the so-called "orthogonality of sorts" in Denef-Pas language referred to below). Finally, the question is reduced to the study of Presburger constructible functions of several $\mathbb{Z}$-variables, which are similar to constructible functions as defined above in Sect. B.2, but without the factors $\#(p_{!F}^{-1}(x))$, see [25]. Roughly, Presburger constructible functions in $x \in \mathbb{Z}^r$ are sums of products of piecewise linear functions in x and of powers of q_F, where the power also depends piecewise linearly on x. If such a function is bounded, then it is a sum of bounded terms as above, after removing possible redundancy in the sum. Each single term in x can then easily be bounded, by a power of q_F that depends linearly on x. Since the number of terms is bounded, one obtains an upper bound of the right form. The reduction to single terms instead of their sum is made precise via the Parametric Rectilinearization (see Theorem 2.1.9 of [25]) and Lemma 2.1.8 of [25]. In summary, the main tools used to obtain these rather strong results with seeming ease are the cell decomposition theorem and the understanding of Presburger constructible functions. Now we proceed with the detailed proof.

Proposition 14.8 *Let H be a constructible function on $W \times B$ for some definable sets W and B. Then there exist a definable function $f : W \times B \to h[0, m, r] \times B$ for some $m \geq 0$ and $r \geq 0$, which makes a commutative diagram with both projections to B, and a constructible function G on $h[0, m, r] \times B$ such that, for some M and all F in $\mathcal{C}_{\mathcal{O},M}$, the function H_F equals the function $G_F \circ f_F$, and such that G_F vanishes outside the range of f_F.*

Proof Let us write $W \subset h[n, a, b]$ for some integers n, a and b. It is enough to prove the lemma when $n = 1$ by a finite recursion argument. We are done since the case $n = 1$ follows from the Cell Decomposition Theorem, in the version of Theorem 7.2.1 from [29]. $\qquad\square$

Proof (*Proof of Theorem* 14.6) Let us first consider the specific case that, for each $F \in \mathcal{C}_{\mathcal{O},M}$ for some M, the set W_F is a subset of $\mathbb{Z}^r$ for some $r \geq 0$ and that H_F is of the specific form, mapping $x \in W_F \times \mathbb{Z}^n$ to

$$\sum_{i=1}^{N} s_{iF} \cdot q_F^{\alpha_{iF}(x)} \left(\prod_{j=1}^{N'} \beta_{ijF}(x) \right)$$

for some real numbers s_{iF} possibly depending on F but not on x, and some definable functions $\alpha_i \colon X \to \mathbb{Z}$ and $\beta_{ij} \colon X \to \mathbb{Z}$. Let us moreover assume that W as well as the graphs of the α_i and β_{ij} are already definable in the Presburger language (which is a sublanguage of the Denef-Pas language). Let us finally assume that there exists $a_0 \geq 0$ such that $|s_{iF}|_{\mathbb{R}} \leq q_F^{a_0}$ for each i and F. Let us call the specific situation with all these assumptions case (1). This case (1) reduces to the case that the α_i and β_{ij} are restrictions of $\mathbb{Z}$-linear functions and that $W = \Lambda_s \times \mathbb{N}^{\ell}$ for some $\ell \geq 0$ and some finite set Λ_s depending on $s \in \mathbb{Z}^n$ by Theorem 2.1.9 of [25] applied to $X = S \times W$ with $S = \mathbb{Z}^n$ in the notation of that theorem. If Λ_s is a singleton, then the result follows from Lemma 2.1.8 of [25]. For Λ_s with at least two elements, one replaces H_F by the sum of $(H_F + 1)^2$ over the elements of Λ_s and the proof is completed by Theorem 14.4 and induction on r.

The more general case where $W \subset h[0, m, r]$ for some $m \geq 0$ and some $r \geq 0$ can be reduced to case (1) by the orthogonality between the residue field sort and the value group sort. Concretely, the following form of orthogonality, see [104], is used. For any definable set $A \subset h[0, m, r]$ there exist $M > 0$ and finitely many definable sets B_i and C_i such that $B_i \subset h[0, m, 0]$ and $C_i \subset h[0, 0, r]$ for each i, and $A_F = \bigcup_i B_{iF} \times C_{iF}$ for each $F \in \mathcal{C}_{\mathcal{O},M}$, see (3.5) and (3.7) of [104]. It is this form of orthogonality that is applied to all the Denef-Pas formulas that are used to build up H (recall that constructible functions are built up from definable functions, and hence, involve finitely many formulas).

For the general case of the theorem, let us choose $f \colon W \times \mathbb{Z}^n \to h[0, m, r] \times \mathbb{Z}^n$ and G with the properties as in Lemma 14.8 with $B = \mathbb{Z}^n$. For G instead of H and $h[0, m, r]$ instead of W, we know that the theorem holds by the above discussion. But then the theorem for H follows. Indeed, by Proposition 14.8, the set $H_F(W_F \times \{\lambda\}) \cup \{0\}$ equals (as subset of $\mathbb{R}$) the set $G_F(k_F^m \times \mathbb{Z}^r \times \{\lambda\}) \cup \{0\}$ for each $\lambda \in \mathbb{Z}^n$ and each F in $\mathcal{C}_{\mathcal{O},M'}$ for some M'.

Proof (Proof of Theorem 14.7) If $W \subset h[0, m, r]$ for some $m \geq 0$ and some $r \geq 0$, then, for some M, the function

$$H_F \colon W_F \times \mathbb{Z}^n \to \mathbb{C}$$

will depend on F only via the two-sorted structure on $(k_F, \mathbb{Z})$ coming from restricting the Denef-Pas language $\mathcal{L}_\mathcal{O}$ to the sorts $(k_F, \mathbb{Z})$ (i.e., leaving out the ring language on the valued field sort and the symbols ord and $\overline{\mathrm{ac}}$).

Hence, for $W \subset h[0, m, r]$ the theorem follows. Now the general case follows from the case $W \subset h[0, m, r]$ by Proposition 14.8. □

B.4 Root data and reductive groups

B.4.1 Split reductive groups

We start out by following [27] in the treatment of the root data and definability of the group G and its Lie algebra $\mathfrak{g}$. Split reductive groups G are classified by the root data $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ consisting of the character group of a split maximal torus $\mathbf{T}$ in G, the set of roots, the cocharacter group, and the set of coroots. The set of possible root data of this form (which we will refer to as *absolute* root data) is completely field-independent. Given a root datum Ψ, the group $G(F)$ is a definable subset of $\mathbf{GL}_n(F)$, given as the image of a definable embedding $\Xi \colon G \hookrightarrow \mathbf{GL}_n$, defined over $\mathbb{Z}[1/R]$ for some large enough R (see Sect. 7.2 of the main article; we note also that in [27], such an embedding is denoted by ρ_D).

In order to show that general reductive groups are definable, we will use the fact that every reductive group splits over the separable closure of F, and the F-forms of a group are in one-to-one correspondence with the Galois cohomology set $H^1(F, \mathbf{Aut}(G))$ (see e.g. [101, §16.4.3]).

We start by giving a construction of finite separable field extensions in Denef-Pas language.

B.4.2 Field extensions

Let $[\Gamma]$ be an isomorphism class of the Galois group of a finite field extension. We can think of a representative of $[\Gamma]$ explicitly as a finite group determined by its multiplication table. Given a non-Archimedean local field F, we would like to realize all field extensions of F with Galois group in the isomorphism class $[\Gamma]$ as elements of a family of definable sets (with finitely many parameters coming from F). Let m be the order of Γ. Let $\bar{b} = (b_0, \ldots, b_{m-1}) \in F^m$. The set of tuples $\bar{b}$ such that the polynomial $P_{\bar{b}}(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ is irreducible and separable, is definable. As in [27, §3.1], one can identify the field extension $F_{\bar{b}} = F[x]/(P_{\bar{b}}(x))$ with F^m. Further, the condition that the field extension $F_{\bar{b}}/F$ is Galois is definable. Indeed, it is given by the requirement that $P_{\bar{b}}$ is irreducible over F, the degree of $F_{\bar{b}}$ over F equals m, and there exist m distinct roots of $P_{\bar{b}}(x)$ in $F_{\bar{b}}$. Note that the latter condition is expressible in Denef-Pas

language using $\bar{b}$ as parameters, and an existential quantifier. Similarly to [27], we treat the elements of the Galois group $\mathrm{Gal}(F_{\bar{b}}/F)$ as $m \times m$-matrices of variables ranging over F. More precisely, we introduce $m \times m$-matrices $\sigma_1, \ldots, \sigma_m$ of variables ranging over F, and impose the condition that $\sigma_1, \ldots, \sigma_m$ are distinct automorphisms of $F_{\bar{b}}$ over F, and there exists a bijection $\{\sigma_1, \ldots, \sigma_m\} \to \Gamma$ which is a group isomorphism. Finally, let $S_{[\Gamma]} \subset F^{m+m^3}$ be the definable set of tuples $(\bar{b}, \sigma_1, \ldots, \sigma_m)$ satisfying the conditions defined above. Note that every Galois extension of F with the Galois group of the isomorphism class $[\Gamma]$ will appear as a fibre of $S_{[\Gamma]}$ over $h[m, 0, 0]$ several times, since $\sigma_1, \ldots \sigma_m$ are not unique for each isomorphism type.

B.4.3 General connected reductive groups

Let Ψ be an absolute root datum as in Sect. B.4.1 above, and let $\mathbf{G}$ be the corresponding split group (so that we can think of $\mathbf{G}$ as a definable set). The goal is to construct the sets $G(F)$ for all connected reductive algebraic groups G with absolute root datum Ψ as members in a family of definable sets G_{zF}, indexed by a parameter z which, loosely speaking, encodes the information about the the cocycle $\mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{Aut}(G)(F^{\mathrm{sep}})$. More precisely, for every parameter $s = (\bar{b}, \sigma_1, \ldots, \sigma_m) \in S_{[\Gamma]}$ as above, we consider the groups G with the absolute root datum Ψ that split over the extension $F_{\bar{b}}$ corresponding to the parameter $\bar{b}$ (if such groups exist). Such groups are in one-to-one correspondence with the elements of the set $H^1(\mathrm{Gal}(F_{\bar{b}}/F), \mathrm{Aut}(\mathbf{G})(F_{\bar{b}}))$. Following the approach of [27, §5.1], we work with individual cocycles rather than cohomology classes. First, observe that the family of sets $Z^1(\mathrm{Gal}(F_{\bar{b}}/F), \mathrm{Aut}(\mathbf{G})(F_{\bar{b}}))$ of such cocycles is a family of definable sets, indexed by $s \in S_{[\Gamma]}$. This follows from the fact that $\mathbf{G}$ is definable: indeed, then the group $\mathrm{Aut}(\mathbf{G})(F_{\bar{b}})$ is definable as well, and we have $\mathrm{Gal}(F_{\bar{b}}/F) \simeq \{\sigma_1, \ldots \sigma_m\}$, and the cocycle condition is, clearly, definable.

Definition 14.9 We denote by $Z_{[\Gamma]}$ the definable set $Z^1(\Gamma, \mathrm{Aut}(\mathbf{G})(F_{\bar{b}}))$ equipped with the projection to the set $S_{[\Gamma]}$.

Let us now recall the construction of the group $G_z(F)$ corresponding to the cocycle z. By definition, $G_z(F)$ is the set of fixed points in $G(F_{\bar{b}})$ under the action of $\mathrm{Gal}(F_{\bar{b}}/F) \simeq \{\sigma_1, \ldots \sigma_m\}$ given by: $\sigma \cdot g = z(\sigma)(\sigma g)$, where $g \in \mathbf{G}(F_{\bar{b}})$, $\sigma \in \mathrm{Gal}(F_{\bar{b}}/F)$, and the action σg is the standard action of the Galois group, where σ acts on the coordinates of g. Such a fixed point set is definable (with parameters from $Z_{[\Gamma]}$), since $\sigma_1, \ldots, \sigma_m$ are interpreted as matrices of variables with entries in F, according to Sect. B.4.2.

B.4.4 Unramified groups

In the case G is unramified over F, i.e. when it is quasi-split and splits over an unramified extension of F, one can think of $G(F)$ as the fixed-point set of the action of the Frobenius element, which substantially simplifies the above construction, see [27, §4.2] for detail. Unramified reductive groups are determined by the root data $\xi = (\Psi, \theta)$, where Ψ is an absolute root datum as in Sect. B.4.1, and θ is the action of the Frobenius automorphism on Ψ.

Remark 14.10 The reason we are including general reductive groups here even though we can, and will, assume that G is unramified over F, is that we have to deal with the connected centralizers of semisimple elements of $G(F)$, and these can be quite general reductive groups.

When we start with a reductive group G over a global field $\mathbf{F}$, outside of the set of places $\mathrm{Ram}(G)$, the group $G \times_{\mathbf{F}} \mathbf{F}_v$ over $\mathbf{F}_v$ is unramified and there are finitely many possibilities for its root datum, as described in Sect. 5.2 of the main article. We recall the notation: the set of finite places v where $G \times_{\mathbf{F}} \mathbf{F}_v$ is unramified is partitioned into the disjoint union of sets $\mathcal{V}(\theta)$, $\theta \in \mathscr{C}(\Gamma_1)$ (see Sect. 5.2 for the definitions). Accordingly, for every conjugacy class $[\theta] \in \mathscr{C}(\Gamma_1)$, we have a definable set, which we denote by $G_{[\theta]}$, such that $G_{[\theta]\mathbf{F}_v} = G(\mathbf{F}_v)$ for all $v \in \mathcal{V}(\theta)$.

We emphasize that $G_{[\theta]F}$, by construction, is a definable subset of $\mathbf{GL}_n(F_{\bar{b}})$ for a suitable parameter $\bar{b}$, as in [27, §4.1].

B.5 Orbital integrals

Here we prove the main technical result – namely, that the orbital integrals are bounded on the both sides by constructible functions. Throughout this section, we are assuming that we are given an unramified root datum $\xi = (\Psi, \theta)$. For every local field F of sufficiently large residue characteristic, it defines an unramified reductive group G, and also gives rise to a definable set $G_{[\theta]F} = G(F)$, as in Sect. B.4.4 above. Note that we are not assuming that F has characteristic zero.

B.5.1 Two lemmas

We start with two easy technical remarks.

Lemma 14.11 *Let ξ be an unramified root datum as above, F—a local field of sufficiently large residue characteristic, and G – the corresponding reductive group over F defined by the root datum ξ. Then the set of semisimple elements in $G(F)$ is definable.*

We will denote this definable set by G_F^{ss}.

Proof The proof is, in fact, contained in the proof of [27, Lemma 7.1.1]. Indeed, the lemma follows from the fact that existence of a basis of eigenvectors is a definable condition: we can write down the conditions stating that there exists a degree $n!$ extension over which there exists a basis of eigenvectors for an element $g \in G(F) \subset \mathbf{GL}_n(F_{\bar{b}})$ for a suitable prameter $\bar{b}$. □

Next, we show that the functions τ_λ^G (see Sect. 2.2) forming the basis of the spherical Hecke algebra are constructible, and depend on λ in a definable way.

Lemma 14.12 *Let G be an unramified reductive group with the root datum ξ as above. Then there exists $M > 0$ (depending only on ξ) and a definable family of constructible functions T_λ, such that for each F in $\mathcal{C}_{\mathcal{O},M}$ one has that*

$$\tau_\lambda^G = T_{\lambda,F}.$$

Proof For unramified groups, it is proved in [26] that the hyperspecial maximal compact subgroup K is definable. One can identify the parameter λ with an r-tuple of integers $(\lambda_1, \ldots, \lambda_r)$, where r is the rank of the maximal split torus in G. We can fix an isomorphism $\chi_A \colon A \to (\mathbb{G}_m)^r$ defined over $\mathbb{Z}$. For $a \in A$, let $\phi_\lambda(a)$ be the formula stating that there exists a tuple $(t_1, \ldots t_r) \in (F^\times)^r$ with $\mathrm{ord}(t_i) = \lambda_i$ for $i = 1, \ldots, r$, such that $\chi_A(a) = (t_1, \ldots, t_r)$. Then the double coset $K\lambda K$ is defined by the condition on g:

$$\exists k_1, k_2 \in K, \quad a \in A \quad \text{such that } g = k_1 a k_2, \quad \phi_\lambda(a) = \text{'true'}.$$

Therefore, we can take $T_{\lambda,F}$ to be the characteristic function of this double coset. □

B.5.2 The measures

Recall the normalization of the measures used to define the orbital integrals in the main article and in Appendix A.

Let $\gamma \in G(F)$ be a semisimple element. Then I_γ (the connected component of the centralizer of γ) is a connected reductive group, and has a canonical measure $d\mu_{I_\gamma}^{can}$ defined by Gross [47, §4]. The G-invariant measure on the orbit O_γ is defined as the quotient measure $\frac{d\mu_G^{can}}{d\mu_{I_\gamma}^{can}}$ of the canonical measure $d\mu_G^{can}$ on G by the canonical measure on I_γ. This is the measure that appears in the statement of the main theorem. However, we do not yet know that this measure is "motivic" in general. The difficulty comes from the canonical measure on I_γ itself in the case γ is ramified. We point out that it is explained in [27,

§7.1] for split groups (and stated for unramified groups), that the canonical measure $d\mu_G^{\mathrm{can}}$ defined by Gross comes from a definable differential form, and therefore fits into the framework of motivic integration by the construction of [29, §8]. The same statement for ramified groups is still open. For now, we prove a technical lemma that allow us to circumvent this difficulty. Namely, we prove that there exists a motivic measure on the orbit, and that it differs from the canonical measure by a constant bounded on both sides by fixed powers of q.

Let $\mathbf{M}$ be a connected reductive group over F that splits over a tamely ramified extension. Let F_1 be a finite Galois extension over which $\mathbf{M}$ splits, and let $\Gamma = \mathrm{Gal}(F_1/F)$. Let x be a special point in the building of $\mathbf{M}$ over F, and let $\mathbf{M}(F)_x$ be the corresponding maximal compact subgroup of $\mathbf{M}(F)$. By definition of the canonical measure, $\mu_M^{\mathrm{can}}(\mathbf{M}(F)_x) = 1$. Our difficulty is that it is presently not known whether $\mathbf{M}(F)_x$ is definable, except in the case when the group $\mathbf{M}$ is unramified over F. For our current purposes, a weaker statement will be sufficient.

In Sect. B.4.3 above, we have constructed $\mathbf{M}(F)$ as an element of a family of definable sets (using parameters in $Z_{[\Gamma]}$, with $\mathbf{M}$ in place of G), by taking the set of Γ-fixed points of $\mathbf{M}(F_1)$, under the action determined by a cocycle z. It follows from [80] that $\mathbf{M}(F)_x \subset \mathbf{M}(F_1)_x \cap \mathbf{M}(F)$, see [1, Lemma 2.1.2] for the statement precisely in this form. Let $M_1 = \mathbf{M}(F_1)_x \cap \mathbf{M}(F)$. Then the subgroup M_1 is definable, since $\mathbf{M}(F_1)_x$ is definable because $\mathbf{M}$ is split over F_1 (see [26]).

Definition 14.13 We denote by i_M the index $[M_1 : \mathbf{M}(F)_x]$.

The proof of the next crucial lemma was provided by Sug Woo Shin. Note that this is the only place where we need to assume that the extension F_1 is tamely ramified. We observe also that a much more precise bound (which we do not need for our present purposes) could have been obtained using the results of Kushnirsky [69].

Lemma 14.14 *With the notation as above, there exists a constant c depending only on the root datum of G such that*

$$i_M = [M_1 : \mathbf{M}(F)_x] \leqslant q^c$$

when $F \in \mathcal{C}_{\mathcal{O}}$ and $\mathbf{M}$ runs over all connected centralizers of semisimple elements of $G(F)$.

Proof Let $M_2 = \mathbf{M}(F_1)_{x,0+} \cap \mathbf{M}(F) = \mathbf{M}(F)_{x,0+}$, where the equality holds by Remark 2.2.2 of [1] (note that the field is not assumed to have characteristic zero in [1]). We have $M_2 \subset \mathbf{M}(F)_x \subset M_1$, so $[M_1 : \mathbf{M}(F)_x] \leqslant [M_1 : M_2]$. Let $\bar{\mathbf{M}}_x$ be the maximal reductive quotient of the reduction mod $\mathfrak{p}$ of the $\mathcal{O}_F$-group scheme associated to the parahoric subgroup $\mathbf{M}(F)_x$ by Bruhat-Tits,

see [77, §3.2] (where the group is denoted by G and the reductive quotient – by $\mathbf{M}$). Then it follows from [77, §3.2] that M_1/M_2 can be identified with the set of k_{F_1}-points of $\bar{\mathbf{M}}_x$, where k_{F_1} is the residue field of F_1, and thus we get $i_M \leqslant \#\bar{\mathbf{M}}_x(k_{F_1})$. Since the dimension of $\bar{\mathbf{M}}_x$ is at most the dimension of G, there is a bound on $\#\bar{\mathbf{M}}_x(k_{F_1})$ given by Steinberg's formula (see [47, §3]); then we carry out the same estimate as done for the numerator in the Eq. (7.11) in the main article, to obtain

$$\#\bar{\mathbf{M}}_x(k_{F_1}) \leqslant q_1^{d_G} q_1^{r_G(d_G+1)},$$

where r_G and d_G stand for rank and dimension of G, respectively, and q_1 is the cardinality of k_{F_1}. Finally, since the degree of the extension $[F_1 : F]$ is bounded by a universal constant, we obtain the desired result. □

Now we can define a motivic measure on the orbit of γ. As above, I_γ is the set of F-points of a connected reductive algebraic group, which we will denote by $\mathbf{M}$. Let Γ be the Galois group of the finite field extension that splits $\mathbf{M}$. Then $\mathbf{M}(F) = I_\gamma$ arises in a family of definable sets (with parameters in $Z_{[\Gamma]}$) constructed in Sect. B.4.3 above. There exists a motivic measure on $\mathbf{M}(F)$ (which uses the cocycle z as a parameter, so we will denote it by $d\mu_z^{\text{mot}}$), constructed in [26, §3.5.2] (see also [44, §2.3]), and if $\mathbf{M}$ is unramified over F, this measure coincides with the canonical measure $d\mu_M^{\text{can}}$. Consider the quotient measure $\frac{d\mu_G^{\text{can}}}{d\mu_z^{\text{mot}}}$ on the orbit of γ. Since G is unramified, this is a quotient of two motivic measures.

Recall the definable open compact subgroup M_1 of $\mathbf{M}(F) = I_\gamma$ constructed above Definition 14.13. Let

$$d\mu_{I_\gamma \backslash G}^{\text{mot}} := \text{vol}_{d\mu_z^{\text{mot}}}(M_1) \frac{d\mu_G^{\text{can}}}{d\mu_z^{\text{mot}}}, \tag{14.2}$$

and let $O^{\text{mot}}(f)$ be the orbital integral with respect to this measure. We will show in Lemma 14.15 below that this is a "motivic distribution" on $C_c^\infty(G)$ in a precise sense.

For now, let us estimate the factor by which this distribution differs from the orbital integral with respect to the canonical measure.

We have:

$$\frac{d\mu_G^{\text{can}}}{d\mu_{I_\gamma}^{\text{can}}} = \frac{d\mu_G^{\text{can}}}{d\mu_z^{\text{mot}}} \frac{d\mu_z^{\text{mot}}}{d\mu_{I_\gamma}^{\text{can}}},$$

where $\frac{d\mu_z^{\text{mot}}}{d\mu_{I_\gamma}^{\text{can}}}$ is a constant, namely, the factor by which the Haar measure $d\mu_z^{\text{mot}}$ that we defined on $\mathbf{M}(F)$ differs from the canonical measure on $\mathbf{M}(F)$. Since

by definition, the volume of the compact subgroup $\mathbf{M}(F)_x$ with respect to the canonical measure is 1, we have that

$$\frac{d\mu_z^{\mathrm{mot}}}{d\mu_{I_\gamma}^{\mathrm{can}}} = \mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(\mathbf{M}(F)_x) = \mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1)/i_M,$$

where M_1 and i_M are as in Definition 14.13 above.

Combining this with (14.2), we get:

$$\frac{d\mu_G^{\mathrm{can}}}{d\mu_{I_\gamma}^{\mathrm{can}}} = \frac{1}{\mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1)} d\mu_{I_\gamma \backslash G}^{\mathrm{mot}} \frac{\mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1)}{i_M} = \frac{1}{i_M} d\mu_{I_\gamma \backslash G}^{\mathrm{mot}}. \qquad (14.3)$$

Now we are ready to prove our main theorem.

B.6 Proof of the main theorem

B.6.1 Proof of Theorem 14.2

Let $\gamma \in G_F^{\mathrm{ss}}$, and let $M = I_\gamma$ be the identity component of the centralizer of γ, as above. We assume that the residue characteristic of F is sufficiently large so that I_γ is automatically tamely ramified. As above, $f \mapsto O_\gamma^{\mathrm{mot}}(f)$ denotes the distribution on $C_c^\infty(G(F))$ defined as the orbital integral with respect to the measure $d\mu_{I_\gamma \backslash G}^{\mathrm{mot}}$ on the orbit of γ.

Let us break up the definable set G^{ss} into finitely many pieces according to the isomorphism class of the centralizer of γ (see Appendix A). Fix a Galois group Γ, and suppose $\mathbf{M}$ is an algebraic group that splits over an extension F_1 with $\mathrm{Gal}(F_1/F) \simeq \Gamma$. Let $Z_{[\Gamma]}^{\mathbf{M}}$ be the definable set of Definition 14.9 with $\mathbf{M}$ in place of G. Let $z \in Z_{[\Gamma]}^{\mathbf{M}}$ be a cocycle corresponding to $\mathbf{M}$. We observe that the set of elements γ such that I_γ is isomorphic to $\mathbf{M}$, is definable, using $\bar{b}, \sigma_1, \ldots, \sigma_m$ and z as parameters (we are using the notation of Sect. B.4.3). For brevity, we denote this definable set by $G_{\mathbf{M}}^{\mathrm{ss}}$ [precisely, we should think of it as an element in a family of definable sets indexed by $\bar{b}, \sigma_1, \ldots, \sigma_m, z$ as above; in particular, we will denote by $z(\gamma)$ the cocycle that gave rise to $\mathbf{M}$].

The following easy lemma amounts to the statement that the quotient of two motivic measures gives a motivic distribution, up to multiplication by a motivic constant (i.e., volume of a fixed definable set).

Lemma 14.15 *Let $\{f_s\}_{s \in S}$ be a family of constructible test functions on $\mathbf{G}$ indexed by some definable set S.*

Then there exists a motivic function $H^{\mathbf{M}}$ on $G_{\mathbf{M}}^{\mathrm{ss}} \times S$
such that for all fields F of sufficiently large residue characteristic,

$$\int_{I_\gamma \backslash \mathbf{G}(F)} f_s(\mathrm{Ad}(g^{-1})\gamma)\, d\mu_{I_\gamma \backslash G}^{\mathrm{mot}} = H_F^{\mathbf{M}}(\gamma, s),$$

for $\gamma \in G_{\mathbf{M}}^{\mathrm{ss}}(F)$.

Proof By definition (which we quote from [66, §2.4]), the quotient measure $\frac{d\mu_G^{\mathrm{can}}}{d\mu_z^{\mathrm{mot}}}$ is characterized by the identity

$$\int_{\mathbf{G}(F)} f(g) d\mu_G^{\mathrm{can}}(g) = \int_{I_\gamma \backslash \mathbf{G}(F)} \int_{I_\gamma} f(hg) d\mu_z^{\mathrm{mot}}(h) \frac{d\mu_G^{\mathrm{can}}}{d\mu_z^{\mathrm{mot}}}(g) \quad (14.4)$$

for all $f \in C_c^\infty(G)$. We recall that this identity characterizes the quotient measure because the map $\alpha : C_c^\infty(G) \to C_c^\infty(I_\gamma \backslash G)$ defined by

$$f \mapsto \left(g \mapsto \int_{I_\gamma} f(hg) d\mu_z^{\mathrm{mot}}(h) \right)$$

is surjective. We observe that as we think of the measures as linear functionals on the spaces $C_c^\infty(G)$, $C_c^\infty(I_\gamma \backslash G)$, etc., we can in fact replace these spaces with their respective subspaces consisting of *constructible* test functions. This follows form the fact that one can construct a family of definable balls, such that the space spanned by the characteristic functions of these balls is dense in the space $C_c^\infty(G)$, and therefore constructible test functions still distinguish between continuous distributions. We refer to [24, §3] for details of such an argument.

Using the definable open compact subgroup M_1 of I_γ, we obtain that for every constructible function $f \in C_c^\infty(I_\gamma \backslash G)$, there exists a *constructible* function $\widetilde{f} \in C_c^\infty(G)$ such that $\alpha(\widetilde{f}) = \mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1) f$. Thus, we can construct a family of constructible test functions $\widetilde{f_s}$ such that $\alpha(\widetilde{f_s})(g) = \mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1) f_s(\mathrm{Ad}(g^{-1})\gamma)$. Multiplying both sides of (14.4) by $\mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1)$, we obtain

$$\mathrm{vol}_{d\mu_z^{\mathrm{mot}}}(M_1) \int_{I_\gamma \backslash G} f_s(Ad(g^{-1})\gamma) \frac{d\mu_G^{\mathrm{can}}}{d\mu_z^{\mathrm{mot}}}(g)$$

$$= \int_{I_\gamma \backslash G} \alpha(\widetilde{f_s})(g) \frac{d\mu_G^{\mathrm{can}}}{d\mu_z^{\mathrm{mot}}} = \int_{\mathbf{G}(F)} \widetilde{f_s}(g) d\mu_G^{\mathrm{can}}(g). \quad (14.5)$$

The left-hand side of (14.5) equals $\int_{I_\gamma \backslash G} f_s(Ad(g^{-1})\gamma) d\mu_{I_\gamma \backslash G}^{\mathrm{mot}}$ by definition; the right-hand side is a motivic function of all the parameters involved (i.e., of γ, on which it depends directly and also via $z(\gamma)$, and of s) by the main theorem on motivic integrals, [29, Theorem 10.1.1] (briefly restated above as Theorem 14.4), since $\mathbf{G}$ is assumed to be unramified over F, and the canonical measure on G is motivic. $\qquad \square$

Since the test functions τ_λ^G form a definable family of constructible functions by Lemma 14.12, the above lemma can be applied to this family, and yields the existence of a constructible function $H^{\mathbf{M}}(\gamma, \lambda)$ on $G_{\mathbf{M}}^{\mathrm{ss}}$, such that

$$H_F^{\mathbf{M}}(\gamma, \lambda) = O_\gamma^{\mathrm{mot}}(\tau_\lambda^G)$$

for every $\gamma \in G_{\mathbf{M}}^{\mathrm{ss}}$, $\lambda \in \mathbb{Z}^n$.

Therefore, by the relation (14.3), we have:

$$O_\gamma\left(\tau_\lambda^G\right) = \frac{1}{i_M} O_\gamma^{\mathrm{mot}}\left(\tau_\lambda^G\right) = \frac{1}{i_M} H_F^{\mathbf{M}}(\gamma, \lambda). \qquad (14.6)$$

By Lemma 14.14, we have

$$\left| O_\gamma(\tau_\lambda^G) \right| \leqslant \left| H_F^{\mathbf{M}}(\gamma, \lambda) \right| \leqslant q^c \left| O_\gamma(\tau_\lambda^G) \right|.$$

We observe that $D^G(\gamma)$ is a constructible function of γ, and by our convention on fractional powers of q, so is $D^G(\gamma)^{1/2}$.

By the Theorem A.1, the function $O_\gamma(\tau_\lambda^G) D^G(\gamma)^{1/2}$ is bounded for every λ. Therefore, the constructible function $H_F^{\mathbf{M}}(\gamma, \lambda) D^G(\gamma)^{1/2}$ is bounded for every λ, and now our Theorem 14.2 follows from Theorem 14.6.

B.6.2 Proof of Theorem 14.1

As discussed in Sect. 5.2 of the main article, the set of all unramified finite places is partitioned into finitely many families according to the root datum of the group $G \times_{\mathbf{F}} \mathbf{F}_v$. Applying Theorem 14.2 to all these families and taking the maximum of the a_G and b_G values, we obtain Theorem 14.1.

Remark 14.16 Though our method sheds no light on the optimal values of a_G and b_G, Theorem 14.7 allows to transfer these values between positive characteristic and characteristic zero: namely, if, for example, some values a_G and b_G were obtained in the function fields case by geometric methods, Theorem 14.7 would immediately imply that the same values work for characteristic zero fields of sufficiently large residue characteristic. We also note that for good orbital integrals, it should be possible to get a bound on a_G in terms of the dimension of G, using [35].

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