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PRESBURGER SETS AND P-MINIMAL FIELDS

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Abstract. We prove a cell decomposition theorem for Presburger sets and introduce a dimension theory for Z-groups with the Presburger structure. Using the cell decomposition theorem we obtain a full classification of Presburger sets up to definable bijection. We also exhibit a tight connection between the definable sets in an arbitrary p-minimal field and Presburger sets in its value group. We give a negative result about expansions of Presburger structures and prove uniform elimination of imaginaries for Presburger structures within the Presburger language.

§1. Introduction. At the "Algèbre, Logique et Cave Particulière" meeting in Lyon (1995), A. Pillay posed the question of whether there exists some dimension theory for Z-groups with the Presburger structure which would give rise to a classification of all Presburger sets up to definable bijection, possibly using other invariants as well. In this paper we answer this question of Pillay: we classify the Presburger sets up to definable bijection (Thm. 4), using as only classifying invariant the (logical) algebraic dimension. In order to prove this classification, we first formulate a cell decomposition theorem for Presburger groups (Thm. 1) and a rectilinearisation theorem for the definable sets (Thm. 2). Also a rectilinearisation theorem depending on parameters is proven (Thm. 3).

Expansions of Presburger groups have recently been studied intensively. One could say that on the one hand one looks for (concrete) expansions which remain decidable and have bounded complexity, and on the other hand different kinds of minimality conditions (like coset-minimality, etc.) are used to characterize general classes of expansions (see e.g., [1], [12]). In section 5 we examine expansions of Presburger groups satisfying natural kinds of minimality conditions.

In [7], D. Haskell and D. Macpherson defined the notion of p-minimal fields, as a p-adic counterpart of o-minimal fields. A p-minimal field always is a p-adically closed field, and its value group is a Z-group. Interactions between definable sets in a given p-adically closed field and Presburger sets in its value group have been studied in the context of p-adic integration for several p-minimal structures (see [5], [6]). In Theorem 6, we exhibit a close connection between definable sets in arbitrary p-minimal fields and Presburger sets in the corresponding value groups.

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In the last section, we use the cell decomposition theorem in an elementary way to obtain uniform elimination of imaginaries for Z-groups without introducing extra sorts.

1.1. Terminology and notation. In this paper G always denotes a Z-group, i.e., a group which is elementary equivalent to the integers $\mathbb{Z}$ in the Presburger language $\mathscr{L}_{Pres} = \langle +, \leq, \{\equiv (\mathrm{mod} n)\}_{n>0}, 0, 1\rangle$ where $\equiv (\mathrm{mod} n)$ is the equivalence relation in two variables modulo the integer $n > 0$. We call $(G, \mathscr{L}_{Pres})$ a Presburger structure and we write H for the nonnegative elements in G. By a Presburger set, function, etc., we mean a $\mathscr{L}_{Pres}$-definable set, function, etc., and by *definable* we always mean definable with parameters (otherwise we say $\emptyset$-definable, S-definable, etc.). We call a set $X \subset G^m$ *bounded* if there is a tuple $z \in H^m$ such that $-z_i \leq x_i \leq z_i$ for each $x \in X$ and $i = 1, \ldots, m$. For $k \leq m$ we write $\pi_k : G^m \to G^k$ for the projection on the first k coordinates and for $X \subset G^{k+n}$ and $x \in \pi_k(X)$ we write X_x for the fiber $\{y \in G^n \mid (x, y) \in X\}$. We recall that the theory $\mathrm{Th}(\mathbb{Z}, \mathscr{L}_{Pres})$ has definable Skolem functions, quantifier elimination in $\mathscr{L}_{Pres}$ and is decidable [13].

§2. Cell decomposition theorem. We prove a cell decomposition theorem for Presburger structures, by first proving it in dimension 1 and subsequently using a compactness argument. An elementary arithmetical proof can also be given, using techniques like in the proof of Lemma 3.2 in [3], but our proof has the advantage that it goes through in other contexts as well (see section 5 and 6). As always, G denotes a Z-group.

Definition 1. We call a function $f : X \subset G^m \to G$ *linear* if there is a constant $\gamma \in G$ and integers a_i, $0 \leq c_i < n_i$ for $i = 1, \ldots, m$ such that $x_i - c_i \equiv 0 \pmod{n_i}$ and

$$f(x) = \sum_{i=1}^{m} a_i\left(\frac{x_i - c_i}{n_i}\right) + \gamma.$$

for all $x = (x_1, \ldots, x_m) \in X$. We call f *piecewise linear* if there is a finite partition $\mathscr{P}$ of X such that all restrictions $f|_A$, $A \in \mathscr{P}$ are linear. We speak analogously of linear and piecewise linear maps $g : X \to G^n$.

The following definition fixes the notion of (Presburger) cells.

Definition 2. A cell of type (0) (also called a (0)-cell) is a point $\{a\} \subset G$. A (1)-cell is a set with infinite cardinality of the form

$$(1) \qquad \{x \in G \mid \alpha \,\square_1\, x \,\square_2\, \beta,\ x \equiv c \pmod{n}\},$$

with α, β in G, integers $0 \leq c < n$ and $\square_i$ either $\leq$ or no condition. Let $i_j \in \{0, 1\}$ for $j = 1, \ldots, m$ and $x = (x_1, \ldots, x_m)$. A $(i_1, \ldots, i_m, 1)$-cell is a set A of the form

$$(2) \qquad A = \{(x, t) \in G^{m+1} \mid x \in D,\ \alpha(x) \,\square_1\, t \,\square_2\, \beta(x),\ t \equiv c \pmod{n}\},$$

with $D = \pi_m(A)$ a $(i_1, \ldots, i_m)$-cell, α, $\beta : D \to G$ linear functions, $\square_i$ either $\leq$ or no condition and integers $0 \leq c < n$ such that the cardinality of the fibers $A_x = \{t \in G \mid (x, t) \in A\}$ can not be bounded uniformly in $x \in D$ by an integer. A $(i_1, \ldots, i_m, 0)$-cell is a set of the form

$$\{(x, t) \in G^{m+1} \mid x \in D,\ \alpha(x) = t\},$$

with $\alpha : D \to G$ a linear function and $D \subset G^m$ a $(i_1, \ldots, i_m)$-cell.

REMARKS. (i) Although we consider in Definition 2 a condition on the cardinality of fibers, the type of a cell does not alter if one takes elementary extensions.
(ii) To an infinite $(i_1, \dots, i_m)$-cell $A \subset G^m$ we can associate (as in [14]) a projection $\pi_A : G^m \to G^k$ such that the restriction of π_A to A gives a bijection from A onto a $(1, \dots, 1)$-cell $A' \subset G^k$. Also, a $(i_1, \dots, i_m)$-cell is finite if and only if $i_1 = \dots = i_m = 0$, and then it is a singleton.
(iii) Let A be a $(i_1, \dots, i_m, 1)$-cell as in Eq. (2), then it is clear that a linear function $f : A \to G$ can be written as

$$f(x, t) = a\left(\frac{t-c}{n}\right) + \gamma(x), \quad (x, t) \in A, \tag{3}$$

with a an integer, $\gamma : D \to G$ a linear function and c, n, D as in Eq. (2).

THEOREM 1 (Cell Decomposition). *Let $X \subset G^m$ and $f : X \to G$ be $\mathcal{L}_{Pres}$-definable. Then there exists a finite partition $\mathcal{P}$ of X into cells, such that the restriction $f|_A : A \to G$ is linear for each cell $A \in \mathcal{P}$. Moreover, if X and f are S-definable, then also the parts A can be taken S-definable.*

PROOF BY INDUCTION ON m. If $X \subset G$, $f : X \to G$ are $\mathcal{L}_{Pres}$-definable, then Theorem 1 follows easily by using quantifier elimination and elementary properties of linear congruences. Alternatively, the more general Thm. 4.8 of [12] can be used to prove this one dimensional version (see also Proposition 2 below). Let $X \subset G^{m+1}$ and $f : X \to G$ be $\mathcal{L}_{Pres}$-definable, $m > 0$. We write $(\square_1, \square_2) \in \{\leq, \emptyset\}^2$ to say that $\square_1$, resp. $\square_2$, represents either the symbol $\leq$ or no condition. Let $\mathcal{S}$ be the set $\mathbb{Z} \times \{(n, c) \in \mathbb{Z}^2 \mid 0 \leq c < n\} \times \{\leq, \emptyset\}^2$. For any $d = (a, n, c, \square_1, \square_2) \in \mathcal{S}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in G^3$ we define a Presburger function

$$F_{(d,\xi)} : \{t \in G \mid \xi_1 \square_1 t \square_2 \xi_2,\ t \equiv c \pmod{n}\} \to G : t \mapsto a\left(\frac{t-c}{n}\right) + \xi_3.$$

The domain $Dom(F_{(d,\xi)})$ of such a function $F_{(d,\xi)}$ is either empty, a (1)-cell or a finite union of (0)-cells. For fixed $k > 0$ and $d \in \mathcal{S}^k$, let $\varphi_{(d,k)}(x, \xi)$ be a Presburger formula in the free variables $x = (x_1, \dots, x_m)$ and $\xi = (\xi_1, \dots, \xi_k)$, with $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3})$, such that $G \models \varphi_{(d,k)}(x, \xi)$ if and only if the following are true:
(i) $x \in \pi_m(X)$,
(ii) the collection of the domains $Dom(F_{(d_i,\xi_i)})$ for $i = 1, \dots, k$ forms a partition of the fiber $X_x \subset G$,
(iii) $F_{(d_i,\xi_i)}(t) = f(x, t)$ for each $t \in Dom(F_{(d_i,\xi_i)})$ and $i = 1, \dots, k$.
Now we define for each k and $d \in \mathcal{S}^k$ the set

$$B_{(d,k)} = \{x \in G^m \mid \exists \xi \quad \varphi_{(d,k)}(x, \xi)\}.$$

Each set $B_{(d,k)}$ is $\mathcal{L}_{Pres}$-definable and the (countable) collection $\{B_{(d,k)}\}_{k,d}$ covers $\pi_m(X)$ since each $x \in \pi_m(X)$ is in some $B_{(d,k)}$ by the induction basis. We can do this construction in any elementary extension of G, so by logical compactness we must have that finitely many sets of the form $B_{(d,k)}$ already cover $\pi_m(X)$. Consequently, we can take Presburger sets $D_1, \dots, D_s$ such that $\{D_i\}$ forms a partition of $\pi_m(X)$ and each D_i is contained in a set $B_{(d,k)}$ for some k and k-tuple d. For each $i = 1, \dots, s$, fix a k and k-tuple d with $D_i \subset B_{(d,k)}$, then we can define the Presburger set

$$\Gamma_i = \{(x, \xi) \in D_i \times G^{3k} \mid \varphi_{(d,k)}(x, \xi)\}$$

satisfying $\pi_m(\Gamma_i) = D_i$ by construction. Since the theory $\mathrm{Th}(G, \mathscr{L}_{Pres})$ has definable Skolem functions, we can choose definably for each $x \in D_i$ tuples $\xi \in G^{3k}$ such that $(x, \xi) \in \Gamma_i$. Combining it all, it follows that there exists a finite partition $\mathscr{P}$ of X consisting of Presburger sets of the form

$$A = \{(x,t) \in G^{m+1} \mid x \in C,\ \alpha(x) \,\square_1\, t \,\square_2\, \beta(x),\ t \equiv c \pmod{n}\},$$

such that $f|_A$ maps $(x,t) \in A$ to $a(\frac{t-c}{n}) + \gamma(x)$; with $\alpha, \beta, \gamma : C \to G$ and $C \subset G^m$ $\mathscr{L}_{Pres}$-definable, $\square_i$ either $\leq$ or no condition, integers a, $0 \leq c < n$ and $\pi_m(A) = C$. The theorem now follows after applying the induction hypothesis to C and $\alpha, \beta, \gamma : C \to G$ and partitioning further. $\dashv$

§3. Dimension theory for Presburger arithmetic. Any Presburger structure satisfies the exchange property for algebraic closure. This is a corollary of a more general result in [1] but can also be proven using the cell decomposition theorem elementarily. In particular this yields an algebraic dimension function on the Presburger sets in the following (standard) way.

DEFINITION 3. Let $X \subset G^m$ be A-definable for some finite set A by a formula $\varphi(x, a)$ where $a = (a_1, \ldots, a_s)$ enumerates A, then the (algebraic) dimension of X, written $\dim(X)$, is the greatest integer k such that in some elementary extension $\bar{G}$ of G there exists $x = (x_1, \ldots, x_m) \in \bar{G}^m$ with $\bar{G} \models \varphi(x, a)$ and $\mathrm{rk}(x_1, \ldots, x_m, a_1, \ldots, a_s) - \mathrm{rk}(a_1, \ldots, a_s) = k$, where $\mathrm{rk}(B)$ of a set $B \subset \bar{G}$ is the cardinality of a maximal algebraically independent subset of B (in the sense of model theory, see [9]).

This dimension function is independent of the choice of a set of defining parameters A and the following properties of algebraic dimension are standard.

PROPOSITION 1. (i) *For Presburger sets $X, Y \subset G^m$ we have* $\dim(X \cup Y) = \max(\dim X, \dim Y)$.
(ii) *Let $f : X \to G^m$ be $\mathscr{L}_{Pres}$-definable, then* $\dim(X) \geq \dim(f(X))$.

The dimension of a cell C is directly related to the type of C (see Lemma 1). Also, if we have a Presburger set X and a finite partition $\mathscr{P}$ of X into cells, the dimension of X is directly related to the types of the cells in $\mathscr{P}$ (see Cor. 1).

LEMMA 1. *Let $C \subset G^m$ be a $(i_1, \ldots, i_m)$-cell, then* $\dim(C) = i_1 + \ldots + i_m$.

PROOF. For a (0)- and a (1)-cell this is clear. Possibly after projecting, we may suppose that $C \subset G^m$ is a $(1, \ldots, 1)$-cell. The Lemma follows now from the definition of the type of a cell using induction on m and a compactness argument. $\dashv$

COROLLARY 1. *For any Presburger set $X \subset G^m$ and any finite partition $\mathscr{P}$ of X into cells we have*

$$\begin{aligned}\dim(X) &= \max\{i_1 + \ldots + i_m \mid C \in \mathscr{P},\ C \textit{ is a } (i_1, \ldots, i_m)\textit{-cell}\} \\ &= \max\{i_1 + \ldots + i_m \mid X \textit{ contains a } (i_1, \ldots, i_m)\textit{-cell}\}.\end{aligned} \tag{4}$$

PROOF. The first equality is a consequence of Lemma 1 and Proposition 1. To prove (4) we take a $(i_1, \ldots, i_m)$-cell $C \subset X$ such that $i_1 + \ldots + i_m$ is maximal. By the cell decomposition we can obtain a partition $\mathscr{P}$ of X into cells such that $C \in \mathscr{P}$. Now use the previous equality to finish the proof. $\dashv$

REMARK. It is also possible to take Eq. (4) as the definition for the dimension of a Presburger set and to proceed similarly as in [14] by van den Dries to develop a dimension theory for Presburger structures.

§4. Classification of Presburger sets.

The cell decomposition theorem provides us with the technical tools to classify the $\emptyset$-definable Presburger sets up to $\mathscr{L}_{Pres}$-definable bijection. The key step to this classification is a rectilinearisation theorem, which also has a parametric formulation. We recall that G denotes a Z-group and $H = \{x \in G \mid x \geq 0\}$, we also write $H^0 = \{0\}$. Also notice that a set A is $\emptyset$-definable if and only if A is $\mathbb{Z}$-definable, to be precise, definable over $\mathbb{Z} \cdot 1 \subset G$.

THEOREM 2 (Rectilinearisation). *Let X be a $\emptyset$-definable Presburger set, then there exists a finite partition $\mathscr{P}$ of X into $\emptyset$-definable Presburger sets, such that for each $A \in \mathscr{P}$ there is an integer $l \geq 0$ and a $\emptyset$-definable linear bijection $f : A \to H^l$.*

PROOF. We give a proof by induction on $\dim X$. If $\dim X = 0$ then X is finite and the theorem follows, so we choose a Presburger set X with $\dim X = m+1$, $m \geq 0$. Any $\mathscr{L}_{Pres}$-definable object occurring in this proof will be $\emptyset$-definable; we will alternately apply $\emptyset$-definable linear bijections and partition further. By the cell decomposition theorem and possibly after projecting (see the remark after Definition 2), we may suppose that X is a $(1, \dots, 1)$-cell contained in G^{m+1}, so we can write

$$X = \{(x, t) \in G^{m+1} \mid x \in D,\ \alpha(x) \,\square_1\, t \,\square_2\, \beta(x),\ t \equiv c \pmod{n}\},$$

with $x = (x_1, \dots, x_m)$, $\pi_m(X) = D \subset G^m$ a $(1, \dots, 1)$-cell, integers $0 \leq c < n$, $\alpha, \beta : D \to G$ $\emptyset$-definable linear functions and $\square_i$ either $\leq$ or no condition. By induction we may suppose that $D = H^m$. If both $\square_1$ and $\square_2$ are no condition, the theorem follows easily, so we may suppose that one of the $\square_i$, say $\square_1$, is $\leq$. Moreover, after a linear transformation $(x, t) \mapsto (x, \frac{t-c}{n})$ we may assume that $c = 0$ and $n = 1$, then we can apply the following linear bijection

$$f : X \to A : (x, t) \mapsto (x_1, \dots, x_m, t - \alpha(x)),$$

onto

$$A = \{(x, t) \in H^{m+1} \mid t \,\square_2\, \beta(x) - \alpha(x)\}.$$

Because $\beta(x) - \alpha(x)$ is a linear function from H^m to G there are integers k_i such that

$$A = \{(x, t) \in H^{m+1} \mid t \,\square_2\, k_0 + \sum_{i=1}^{m} k_i x_i\}. \tag{5}$$

Moreover, since $\pi_m(A) = H^m$, all integers k_i must be nonnegative. We proceed by induction on $k_1 \geq 0$. If $k_1 = 0$ then $A = H \times \{(x_2, \dots, x_m, t) \in H^m \mid t \leq k_0 + \sum_{i=2}^{m} k_i x_i\}$ and the theorem follows by induction on the dimension. Now suppose $k_1 > 0$, then we partition A into two parts

$$A_1 = \{(x, t) \in H^{m+1} \mid t \leq x_1 - 1\},$$

$$A_2 = \{(x, t) \in H^{m+1} \mid x_1 \leq t \leq k_0 + \sum_{i=1}^{m} k_i x_i\},$$

where $\pi_m(A_2) = H^m$ and $\pi_m(A_1) = \{x \in H^m \mid 1 \leq x_1\}$. We apply the linear bijection

$$A_2 \to B : (x, t) \mapsto (x_1, \dots, x_m, t - x_1)$$

with

$$B = \{(x, t) \in H^{m+1} \mid t \leq k_0 + (k_1 - 1)x_1 + \sum_{i=2}^{m} k_i x_i\}$$

and the theorem for B follows by induction on k_1. We conclude the proof by the following linear bijection:

$$A_1 \to H^{m+1} : (x, t) \mapsto (x_1 - 1 - t, x_2, \dots, x_m, t). \qquad \dashv$$

Theorem 3 (Parametric Rectilinearisation). *Let $X \subset G^{m+n}$ be a $\emptyset$-definable Presburger set, then there exists a finite partition $\mathcal{P}$ of X into $\emptyset$-definable Presburger sets, such that for each $A \in \mathcal{P}$ there is a set $B \subset G^{m+n}$ with $\pi_m(A) = \pi_m(B)$ and a $\emptyset$-definable family $\{f_\lambda\}_{\lambda \in \pi_m(A)}$ of linear bijections $f_\lambda : A_\lambda \subset G^n \to B_\lambda \subset G^n$ with B_λ a set of the form $H^l \times \Lambda_\lambda$ where Λ_λ is a bounded λ-definable set and the integer l only depends on $A \in \mathcal{P}$.*

Proof. We give a proof by induction on n, following the lines of the proof of Theorem 2. So we assume that X is a cell

$$X = \{(\lambda, x, t) \in G^{m+(n+1)} \mid (\lambda, x) \in D,\ \alpha(\lambda, x) \square_1 t \square_2 \beta(\lambda, x),\ t \equiv c \pmod n\},$$

with $\lambda = (\lambda_1, \dots, \lambda_m)$, $x = (x_1, \dots, x_n)$, $D \subset G^{m+n}$ a cell, integers $0 \leq c < n$, $\alpha, \beta : D \to G$ $\emptyset$-definable linear functions and $\square_i$ either $\leq$ or no condition. By subsequently applying the induction hypothesis to D, partitioning further and applying linear bijections (similar as to obtain Eq. (5) in the proof of Theorem 2, keeping the parameters λ fixed now), we may assume that X has the form

$$X = \{(\lambda, x, t) \in G^{m+n+1} \mid (\lambda, x) \in D',\ 0 \leq t \leq \gamma(\lambda, x)\},$$

with $\pi_{m+n}(X) = D' \subset G^{m+n}$ a Presburger set such that for each $\lambda \in \pi_m(D')$ $D'_\lambda = H^l \times \Gamma_\lambda$ where Γ_λ is a λ-definable bounded set, l a fixed positive integer and $\gamma : D' \to G$ a $\emptyset$-definable linear function. If $l = 0$, X_λ is a bounded set for each λ and the theorem follows immediately. Let thus $l \geq 1$, i.e., the projection of X on the x_1-coordinate is H, then the function γ can be written as $(\lambda, x) \mapsto k_1 x_1 + \gamma'(\lambda, x_2, \dots, x_m)$ with k_1 an integer, necessarily nonnegative because the projection of X on the x_1-coordinate is H and γ' is a linear function. The reader can finish the proof by induction on $k_1 \geq 0$, similar as in the proof of Theorem 2. $\dashv$

Theorem 4 (Classification). *Let X be a $\emptyset$-definable Presburger set with* $\dim X = m > 0$, *then there exists a $\emptyset$-definable Presburger bijection $f : X \to G^m$. In other words, there exists a $\emptyset$-definable Presburger bijection between two infinite $\emptyset$-definable Presburger sets X, Y if and only if* $\dim X = \dim Y$.

Proof. Let X be $\emptyset$-definable and infinite. We use induction on $\dim X = m$. We say for short that two Presburger sets X, Y are isomorphic if there exists a $\emptyset$-definable Presburger bijection between them and write $X \cong Y$. If $m = 1$, then Theorem 2 yields a partition $\mathscr{P}$ of X such that each part is either a point or isomorphic to H. Consider the bijections

$$f_1 : H \to G : \begin{cases} 2x & \mapsto & x, \\ 2x+1 & \mapsto & -x, \end{cases}$$
$$f_2 : H \cup \{-1\} \to H : x \mapsto x+1,$$
$$f_3 : (\{0\} \times H) \cup (\{1\} \times H) \to H : \begin{cases} (0,x) & \mapsto & 2x, \\ (1,x) & \mapsto & 2x+1; \end{cases}$$

the bijections f_1, f_2, applied repeatedly to (isomorphic copies of) parts in $\mathscr{P}$ yield a definable bijection from X onto H and thus $G \cong X$ by applying f_1 (in the obvious way). Now let $\dim X = m > 1$. Using Theorem 2 we find a partition $\mathscr{P}$ of X such that each part is isomorphic to H^l and thus to G^l since $H \cong G$ by f_1. Since $\dim X = m$, at least one part is isomorphic to G^m. Take $A, B \in \mathscr{P}$ with $A \cong G^m$ and $B \cong G^l$, then it suffices to prove that $A \cup B \cong G^m$. If $l = 0$ this is clear and if $l > 0$ then $A \cup B \cong G \times (A' \cup B')$ for some disjoint and $\emptyset$-definable sets A', B' with $A' \cong G^{m-1}$ and $B' \cong G^{l-1}$. The induction hypothesis applied to $A' \cup B'$ finishes the proof. ⊣

§5. Expansions of Z-groups. We define the notion of Presburger minimality ($\mathscr{L}_{Pres}$-minimality) for expansions of Presburger structures $(G, \mathscr{L}_{Pres})$. This notion of $\mathscr{L}_{Pres}$-minimality is a concrete instance of the general notion of $\mathscr{L}$-minimality as in [10] and has already been studied in [12].

Definition 4. Let G be a Z-group and $\mathscr{L}$ an expansion of the language $\mathscr{L}_{Pres}$, then we say that $(G, \mathscr{L})$ is $\mathscr{L}_{Pres}$-minimal if every $\mathscr{L}$-definable subset of G is already $\mathscr{L}_{Pres}$-definable (allowing parameters as always). We say that $\mathrm{Th}(G, \mathscr{L})$ is $\mathscr{L}_{Pres}$-minimal if every model of this theory is $\mathscr{L}_{Pres}$-minimal.

Comparing this notion with the terminology of [12], a structure $(G, \mathscr{L})$ is $\mathscr{L}_{Pres}$-minimal if and only if it is a *discrete coset-minimal group without definable proper convex subgroups* (see [12]). Theorem 4.8 of [12] says that a definable function in one variable between such groups is piecewise linear. We reformulate this result with our terminology.

Proposition 2 ([12], Thm. 4.8). *Let $(G, \mathscr{L})$ be $\mathscr{L}_{Pres}$-minimal, then any definable function $f : G \to G$ is piecewise linear.*

Proposition 2 allows us to repeat without any change the compactness argument of the proof of the cell decomposition theorem for any model of a $\mathscr{L}_{Pres}$-minimal theory. This leads to the following description of $\mathscr{L}_{Pres}$-minimal theories.

Theorem 5. *Let $(G, \mathscr{L})$ be an expansion of a Presburger structure $(G, \mathscr{L}_{Pres})$, then the following are equivalent*:

(i) $\mathrm{Th}(G, \mathscr{L})$ *is $\mathscr{L}_{Pres}$-minimal*;

(ii) $(G, \mathscr{L})$ *is a definitional expansion of $(G, \mathscr{L}_{Pres})$; precisely, any $\mathscr{L}$-definable set $X \subset G^m$ is already $\mathscr{L}_{Pres}$-definable.*

Thus, the theory $\mathrm{Th}(G, \mathscr{L}_{Pres})$ *does not admit any proper $\mathscr{L}_{Pres}$-minimal expansion.*

Proof. Any Presburger minimal theory has definable Skolem functions. For if $X \subset G^{m+1}$ is a definable set in some model G, we can choose definably for any $x \in \pi_m(X)$ the smallest nonnegative element in X_x if there is any, and the largest negative element otherwise. This implies the definability of Skolem functions by induction. Now replace in the statement of the cell decomposition Theorem (theorem 1) the word $\mathscr{L}_{Pres}$-*definable* by $\mathscr{L}$-*definable*. Then repeat the case $m = 1$ of the proof of Theorem 1, using now the $\mathscr{L}_{Pres}$-minimality and Proposition 2. Using the same compactness argument as in the proof of Theorem 1 we find that any $\mathscr{L}$-definable set $X \subset G^m$ is a finite union of Presburger cells, thus a fortiori, X is $\mathscr{L}_{Pres}$-definable. ⊣

REMARK. For an arbitrary expansion $(G, \mathscr{L})$ of $(G, \mathscr{L}_{Pres})$ it is, as far as I know, an open problem whether the statements (i) and (ii) of Thm. 5 are equivalent with the following:

(iii) $(G, \mathscr{L})$ is $\mathscr{L}_{Pres}$-minimal.

In the special case $G = \mathbb{Z}$, statements (i), (ii) and (iii) are indeed equivalent, proven by C. Michaux and R. Villemaire in [11].

§6. Application to p-minimal fields. In this section, we let K be a p-adically closed field with value group G. Recall that a p-adically closed field is a field K which is elementary equivalent to a finite field extension of the field $\mathbb{Q}_p$ of p-adic numbers; in particular, the value group G is a Z-group and K has quantifier elimination in the Macintyre language $\mathscr{L}_{Mac} = \langle +, -, ., 0, 1, \{P_n\}_{n \geq 1} \rangle$ where P_n denotes the set of n-th powers in $K^\times$. We write $v : K \to G \cup \{\infty\}$ for the valuation map and for any $m > 0$ we write $\bar{v}$ for the map $\bar{v} : (K^\times)^m \to G^m : x \mapsto (v(x_1), \ldots, v(x_m))$. We give a definition of p-minimality, extending the original definition of [7] slightly.

DEFINITION 5. Let K be a p-adically closed field and let $(K, \mathscr{L})$ be an expansion of $(K, \mathscr{L}_{Mac})$. We say that the structure $(K, \mathscr{L})$ is p-minimal if any $\mathscr{L}$-definable subset of K is already $\mathscr{L}_{Mac}$-definable (allowing parameters). The theory $\mathrm{Th}(K, \mathscr{L})$ is called p-minimal if every model of this theory is p-minimal.

Examples of p-minimal fields known at this moment are p-adically closed fields with the semi-algebraic structure and with subanalytic structure with restricted power series (see [8]). Theorem 6 exhibits a close connection between definable sets in a p-minimal field K and Presburger sets in the value group G of K; to prove it, we use Lemma 2, which is a reformulation of the interpretability of $(G, \mathscr{L}_{Pres})$ in $(K, \mathscr{L}_{Mac})$.

LEMMA 2. *Let K be a p-adically closed field with value group G, then for any $\mathscr{L}_{Pres}$-definable set $S \subset G^m$ the set $\bar{v}^{-1}(S) = \{(x_1, \ldots, x_m) \in (K^\times)^m \mid \bar{v}(x) \in S\}$ is $\mathscr{L}_{Mac}$-definable.*

PROOF. Let $S \subset G^m$ be $\mathscr{L}_{Pres}$-definable. By Theorem 1 we may suppose that S is a Presburger cell. The Lemma follows now inductively from the fact that conditions imposed on $(x_1, \ldots, x_{m-1}, t) \in (K^\times)^m$ of the form $\pm v(t) \leq \frac{1}{e}(\sum_{i=1}^{m-1} a_i v(x_i)) + d$ or $v(t) \equiv c \pmod{n}$ are $\mathscr{L}_{Mac}$-definable for any integers a_i, $e \neq 0$, $0 \leq c < n$ and $d \in G$ (see e.g., [4, Lemma 2.1]). ⊣

THEOREM 6. *Let $(K, \mathscr{L})$ be a p-minimal field with p-minimal theory and let G be the value group of K. Then for any $\mathscr{L}$-definable set $X \subset (K^\times)^m$ the set*

$$\bar{v}(X) = \{(v(x_1), \ldots, v(x_m)) \in G^m \mid (x_1, \ldots, x_m) \in X\} \subset G^m$$

is $\mathscr{L}_{Pres}$-definable.

PROOF. Put $S_m = \{\bar{v}(X) \subset G^m \mid X \subset (K^\times)^m,\ X \text{ is } \mathscr{L}\text{-definable}\}$, then it is easy to see that the collection $(S_m)_{m\geq 0}$ determines a structure on G (i.e., the collection $\cup_m S_m$ is precisely the collection of $\mathscr{L}'$-definable sets for some language $\mathscr{L}'$). We first argument that this structure is in fact $\mathscr{L}_{Pres}$-minimal. Choose a $\mathscr{L}$-definable set $X \subset K^\times$, then, by p-minimality, X is $\mathscr{L}_{Mac}$-definable. We can thus apply the p-adic semi-algebraic cell decomposition ([4], in the formulation of [2, Lemma 4]) to the set X to obtain that X is a finite union of p-adic cells, i.e., sets of the form

$$\{x \in K \mid v(a_1) \square_1 v(x-c) \square_2 v(a_2),\ x - c \in \lambda P_n\} \subset K^\times,$$

with $a_1, a_2, c, \lambda \in K$ and $\square_i$ either $\leq, <$ or no condition. The image under v of such a cell is either a finite union of (0)-cells or a (1)-cell and thus a $\mathscr{L}_{Pres}$-definable subset of G. By consequence, the structure $(S_m)_{m\geq 0}$ is $\mathscr{L}_{Pres}$-minimal. By the Presburger minimality of $(S_m)_{m\geq 0}$, the p-minimality of $\mathrm{Th}(K, \mathscr{L})$, and Lemma 2 to interprete G into K, we can repeat the compactness argument of the proof of the cell decomposition theorem 1 for the structure $(S_m)_m$ on G to find that each $A \in \cup_m S_m$ is a finite union of Presburger cells. This proves the theorem. ⊣

REMARK. For a p-adically closed field K it is proven by the author in [2] that there exists a $\mathscr{L}_{Mac}$-definable bijection $X \to Y$ between two infinite parameter free $\mathscr{L}_{Mac}$-definable sets X, Y if and only if $\dim X = \dim Y$. This is proven by reducing to a Presburger problem similar to the classification theorem in section 4. An analogous classification for $\emptyset$-definable sets in arbitrary p-minimal structures is not known up to now.

§7. Elimination of imaginaries. As a last application of the cell decomposition theorem we prove uniform elimination of imaginaries for Presburger structures. We say that a structure $(M, \mathscr{L})$ has uniform elimination of imaginaries if for any $\emptyset$-definable equivalence relation on M^k there exists a $\emptyset$-definable function $F : M^k \to M^r$ for some r such that two tuples $x, y \in M^k$ are equivalent if and only if $F(x) = F(y)$.

THEOREM 7. *The theory $Th(\mathbb{Z}, \mathscr{L}_{Pres})$ has uniform elimination of imaginaries, precisely, any Presburger structure $(G, \mathscr{L}_{Pres})$ eliminates imaginaries uniformly.*

PROOF. Since $\mathrm{Th}(\mathbb{Z}, \mathscr{L}_{Pres})$ has definable Skolem functions, we only have to prove the following statement for an arbitrary Z-group G (see e.g., [9, Lemma 4.4.3]). For any $\emptyset$-definable Presburger set $X \subset G^{m+1}$ there exists a $\emptyset$-definable Presburger function $F : G^m \to G^n$ for some n, such that $F(x) = F(x')$ if and only if $X_x = X_{x'}$ (if $x \notin \pi_m(X)$ then we put $X_x = \emptyset$). So let $X \subset G^{m+1}$ be a $\emptyset$-definable Presburger set. Apply the cell decomposition theorem to obtain a partition $\mathscr{P}$ of X into cells. For each cell $A \in \mathscr{P}$ of the form $A = \{(x,t) \in G^{m+1} \mid x \in D,\ \alpha(x) \square_{1A} t \square_{2A} \beta(x),\ t \equiv c \pmod{n}\}$ (as in Eq. 2) and each $\xi = (\xi_1, \xi_2) \in G^2$ we define a set

$$C_A(\xi) = \{t \in G \mid \xi_1 \square_{1A} t \square_{2A} \xi_2,\ t \equiv c \pmod{n}\}.$$

Notice that for each $x \in \pi_m(X)$ we have at least one partition of X_x into sets of the form $C_A(\xi)$ with $A \in \mathscr{P}$ and $\xi \in G^2$. For $x, y \in G$ we write $x \lhd y$ if and only if one of the following conditions is satisfied

(i) $0 \leq x < y$,

(ii) $0 < x \leq -y$,
(iii) $0 < -x < y$,
(iv) $0 < -x < -y$.

This gives a new ordering $0 \lhd 1 \lhd -1 \lhd 2 \lhd -2 \lhd \dots$ on G with zero as its smallest element. For each $k > 0$ we also write $\lhd$ for the lexicographical order on G^k built up with $\lhd$. The order $\lhd$ is $\mathscr{L}_{Pres}$-definable and each Presburger set has a unique $\lhd$-smallest element. For each $x \in G^m$ and each $I \subset \mathscr{P}$ with cardinality $|I| = s \geq 0$ we let $y_I(x) = (\xi_A)_{A \in I}$, $\xi_A \in G^2$, be the $\lhd$-smallest tuple in G^{2s} such that $\cup_{A \in I} C_A(\xi_A) = X_x$ if there exists at least one such tuple and we put $y_I(x) = (0, \dots, 0) \in G^{2s}$ otherwise. One can reconstruct the set X_x given all tuples $y_I(x)$, $I \subset \mathscr{P}$. Let F be the function mapping $x \in \pi_m(X)$ to $y = (y_I(x))_{I \subset \mathscr{P}}$. Since the lexicographical order $\lhd$ is $\mathscr{L}_{Pres}$-definable it is clear that F is $\mathscr{L}_{Pres}$-definable and that $F(x) = F(x')$ if and only if $X_x = X_{x'}$ for each $x, x' \in G^m$. ⊣

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