STABILITY UNDER INTEGRATION OF SUMS OF PRODUCTS OF REAL GLOBALLY SUBANALYTIC FUNCTIONS AND THEIR LOGARITHMS

RAF CLUCKERS and DANIEL J. MILLER

Abstract

We study Lebesgue integration of sums of products of globally subanalytic functions and their logarithms, called constructible functions*. Our first theorem states that the class of constructible functions is stable under integration. The second theorem treats integrability conditions in Fubini-type settings, and the third result gives decay rates at infinity for constructible functions. Further, we give preparation results for constructible functions related to integrability conditions.*

1. Introduction

Consider the following question: Given an o-minimal expansion $\mathcal{R}$ of the real field, what is the smallest class of functions $\mathcal{C}$ which is stable under integration and contains all the functions definable in $\mathcal{R}$? We require that the functions in $\mathcal{C}$ have domains which are definable in $\mathcal{R}$, and by stability under integration we mean that if $f : X \times \mathbb{R}^m \to \mathbb{R}$ is in $\mathcal{C}$ and $y \mapsto f(x, y)$ is Lebesgue integrable on $\mathbb{R}^m$ for all $x \in X$, then the function $F : X \to \mathbb{R}$ defined by

$$F(x) = \int_{\mathbb{R}^m} f(x, y)\, dy \tag{1.1}$$

is also in $\mathcal{C}$. We call an integral of the form (1.1) a *parameterized integral* since it is parameterized by the variables x. In general, the functions in $\mathcal{C}$ will not be definable in $\mathcal{R}$, but is it possible to simply describe them using the functions which are definable in $\mathcal{R}$? Furthermore, if the functions definable in $\mathcal{R}$ have certain tame properties, such as having simple asymptotic behavior at ∞, do the functions in $\mathcal{C}$ inherit these properties?

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To the best of our knowledge, satisfactory answers to these questions are not currently known. This is true even in what is probably the most important special case, when $\mathcal{R}$ is the real field itself, in which case the definable functions of $\mathcal{R}$ are called the *semialgebraic functions*. This special case is closely related to the study of periods and families of periods, which are integrals and parameterized integrals, respectively, of functions which are zero-definable in the real field (meaning that the integrands are semialgebraic functions defined by systems of polynomial equations and inequalities with rational coefficients; see Kontsevich and Zagier [16] and the appendix of Kontsevich and Soibelman [15]). In order to avoid the many extreme difficulties of the semialgebraic case, in this paper we instead consider the case where $\mathcal{R}$ is the expansion of the real field by all restricted analytic functions, whose definable functions are called the *globally subanalytic functions*. Even though this approach does not give simple descriptions of families of periods in terms of semialgebraic functions, it nevertheless does provide qualitative information about, for example, the asymptotic behavior of families of periods because the asymptotic behavior of the semialgebraic functions and the larger system of globally subanalytic functions are quite similar.

In this regard we consider sums of products of globally subanalytic functions and their logarithms, which we call *constructible functions*. Any constructible function can be easily represented as a parameterized integral of a globally subanalytic function. Our main theorem states that the class of constructible functions is stable under integration, from which it follows that the class of constructible functions is the smallest class of functions which is stable under integration and contains all the globally subanalytic functions. In order to use our main theorem in conjunction with Fubini's theorem, we also prove that for constructible functions, the notion of "integrable almost everywhere" has a globally subanalytic interpretation. In addition, we bound the decay rates of constructible functions which vanish at infinity and prove a preparation theorem for constructible functions related to a certain integrability condition. Collectively, these results establish a theory of integration for constructible functions which is both simple and tame. They also naturally generalize the work of Comte, Lion, and Rolin [19], [9] on the integration of globally subanalytic functions, and they are analogues of results previously established for p-adic and motivic integrals in [12], [4], [6], and [5]. We will further discuss the historical context of our work after we have precisely formulated our main results.

1.1. The main results

A function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is called *globally subanalytic* if its graph is a globally subanalytic set, and a set $A \subset \mathbb{R}^n$ is called *globally subanalytic* if its image under the natural embedding of $\mathbb{R}^n$ into n-dimensional real projective space, namely, $\mathbb{R}^n \to \mathbb{P}^n(\mathbb{R}) : (x_1, \ldots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$, is a subanalytic subset of $\mathbb{P}^n(\mathbb{R})$ in the classical sense. This condition on A is equivalent to saying that A is definable in the expansion of the real field by all restricted analytic functions $f : \mathbb{R}^k \to \mathbb{R}$ with $k \in \mathbb{N}$, where a restricted analytic function f is, by definition, associated to an analytic function f_0 on an open set $U \subset \mathbb{R}^k$ containing $[-1, 1]^k$ and defined by

$$f : \mathbb{R}^k \to \mathbb{R} : x \mapsto \begin{cases} f_0(x) & \text{if } x \in [-1, 1]^k, \\ 0 & \text{otherwise.} \end{cases}$$

Henceforth, in the remainder of the paper, we abbreviate the terminology and use the word *subanalytic* to mean globally subanalytic for both sets and functions. It is important to note that with this convention on our terminology, the natural logarithm $\log : (0, +\infty) \to \mathbb{R}$ is not subanalytic. (For more background on subanalytic sets and functions, see Bierstone and Milman [3], Denef and Van den Dries [13, Section 4], and Van den Dries and Miller [24, 2.5(4)].)

Definition 1.1
For X a subanalytic set, $m \geq 0$ an integer, and $f : X \times \mathbb{R}^m \to \mathbb{R}$ a Lebesgue measurable function, consider the parameterized integral

$$I_X(f) : X \to \mathbb{R} : x \mapsto \begin{cases} \int_{\mathbb{R}^m} f(x, y)\, dy & \text{if } f(a, \cdot) \text{ is integrable for all } a \in X, \\ 0 & \text{otherwise,} \end{cases}$$

where $dy = dy_1 \wedge \cdots \wedge dy_m$ is the Lebesgue measure on $\mathbb{R}^m$ and $f(a, \cdot)$ is the function $y \in \mathbb{R}^m \mapsto f(a, y)$, and where integrable means Lebesgue integrable.

Note that I_X behaves very crudely with respect to integrability conditions, since we define $I_X(f)$ to be identically equal to zero if $f(a, \cdot)$ is nonintegrable for some $a \in X$. This is done merely out of convenience in order to simplify the formulation of Theorem 1.3, and Theorem 1.4 will show how to weaken this assumption.

Theorem 1.3 will show that the following question has a particularly nice answer.

> By starting with the subanalytic functions, what functions can be obtained through application of the integral operators I_X?

The answer will be in terms of the following rings of constructible functions.

Definition 1.2
For each subanalytic set X, let $\mathcal{C}(X)$ be the $\mathbb{R}$-algebra of real-valued functions on X generated by all subanalytic functions on X and the functions $x \mapsto \log f(x)$, where $f : X \to (0, +\infty)$ is subanalytic. We call $f \in \mathcal{C}(X)$ a *constructible function* on X, and $\mathcal{C}(X)$ is called the *algebra of constructible functions* on X.

(Constructible functions are also called *log-analytic functions* by some authors.) It is easy to see that every constructible function can be obtained from a subanalytic function through the application of a single integral operator I_X. Indeed, for $f \in \mathcal{C}(X)$, there are subanalytic functions $f_i : X \to \mathbb{R}$ and $f_{i,j} : X \to (0, +\infty)$ such that

$$f(x) = \sum_{i=1}^{k} f_i(x) \prod_{j=1}^{l_i} \log f_{i,j}(x).$$

By absorbing sign information inside the functions f_i, we may assume that $f_{i,j}(x) > 1$ for all x. Therefore if we write

$$f(x) = \sum_{i=1}^{k} \int_1^{f_{i,1}(x)} \cdots \int_1^{f_{i,l_i}(x)} \frac{f_i(x)}{t_{i,1} \cdots t_{i,l_i}} \, dt_{i,1} \cdots dt_{i,l_i},$$

and if we extend each integrand $(x, t_{i,1}, \ldots, t_{i,l_i}) \mapsto \frac{f_i(x)}{t_{i,1} \cdots t_{i,l_i}}$ to $X \times \mathbb{R}^{l_1+\cdots+l_k}$ by zero outside its natural domain, we see that $f = I_X(F)$ for a single subanalytic function $F : X \times \mathbb{R}^{l_1+\cdots+l_k} \to \mathbb{R}$. It follows from this observation and the following theorem that the constructible functions are, in fact, *precisely* the functions obtainable from the subanalytic functions through the integral operators I_X.

THEOREM 1.3 (Stability under integration)
Let f be in $\mathcal{C}(X \times \mathbb{R}^m)$ for some subanalytic set X. Then $I_X(f)$ is in $\mathcal{C}(X)$.

The definition of I_X deals crudely with integrability issues. So in order to apply Theorem 1.3 in combination with Fubini's theorem, we show that for constructible functions, the notion of "integrable almost everywhere" has a subanalytic interpretation. In simplest form this means the following.

THEOREM 1.4 (Integrability)
Let f be in $\mathcal{C}(\mathbb{R}^{n+1})$. Suppose that for all y in a dense subset C of $\mathbb{R}^n$, the function

$$f(y, \cdot) : \mathbb{R} \to \mathbb{R} : z \mapsto f(y, z)$$

is integrable. Then there exists a subanalytic dense subset C' of $\mathbb{R}^n$ such that $f(y, \cdot)$ is integrable over $\mathbb{R}$ for all $y \in C'$.

Note that if $\mathbb{R}^n \setminus C$ has measure zero, then C is dense in $\mathbb{R}^n$. Also note that a subanalytic set C' is dense in $\mathbb{R}^n$ if and only if $\mathbb{R}^n \setminus C'$ has measure zero, if and only if the dimension of $\mathbb{R}^n \setminus C'$ is less than n.

We do not provide a separate proof of Theorem 1.4, but instead prove Theorem 1.4′, which is a stronger, parameterized version of Theorem 1.4. The proof of Theorem 1.3 will be reduced to the case where $m = 1$ by Theorem 1.4′ (with $Y = \mathbb{R}^{m-1}$) and Fubini's theorem.

THEOREM 1.4′ (Integrability in families)
Let f be in $\mathcal{C}(X \times Y \times \mathbb{R})$ for some subanalytic sets X and Y. Suppose that for each $x \in X$ and for all y in a dense subset C_x of Y, the function

$$f(x, y, \cdot) : \mathbb{R} \to \mathbb{R} : z \mapsto f(x, y, z)$$

is integrable, where C_x may depend on x. Then there are dense subsets C'_x of Y such that the set $\{(x, y) \in X \times \mathbb{R}^n : y \in C'_x\}$ is subanalytic and such that for all $x \in X$ and all $y \in C'_x$, the function $f(x, y, \cdot)$ is integrable.

Along the way we prove the following proposition for bounding constructible functions at infinity.

PROPOSITION 1.5 (Decay rates)
Let f be in $\mathcal{C}(X \times \mathbb{R})$ for some subanalytic set X, and suppose that

$$\lim_{y \to +\infty} f(x, y) = 0$$

for all $x \in X$. Then there exist a constant $r > 0$ and a subanalytic function $g : X \to (0, +\infty)$ such that

$$|f(x, y)| \le y^{-r}$$

for all $x \in X$ and all y with $y > g(x)$.

1.2. Some context
It is classical to study asymptotic expansions for one-parameter integrals like

$$E(z) := \int_{z = f(x),\ x \in (-1,1)^n} \frac{dx_1 \wedge \cdots \wedge dx_n}{df}$$

or its Fourier transform, where $f : U \to \mathbb{R}$ is analytic on an open set U containing $[-1, 1]^n$, having some isolated critical points in $(-1, 1)^n$, and where dx/df is the Gelfand-Leray differential form and z runs over noncritical values of f. Such asymptotic expansions are interesting for z going to a critical value of f and are used, for example, in Varchenko's definition of the Hodge filtration and the spectrum of an isolated singularity of f (see [1] and [17]). Just to name some other studies of similar parameterized integrals, we mention the link made to monodromy by Malgrange [21], the work for nonisolated critical points by Barlet [2] and Loeser [20], and Varchenko's work on abelian integrals [25].

Instead of looking at such classical asymptotic expansions with only one parameter, in Theorem 1.3 we describe globally what happens for a richer class of integrands (namely, what we call *constructible functions*) and when arbitrarily many parameters are involved.

Lion and Rolin [19], in their théorème d'intégration, examined bounded constructible functions (with bounded domain and bounded range) and described the parameterized integrals of such bounded functions, but did not show that they are again constructible. Instead they only showed that such an integral is a pointwise limit of constructible functions. Combined with their théorème d'élimination (see [19]), they also showed that such an integral is piecewise the restriction of constructible functions, but possibly with pieces more general than subanalytic sets, which again misses the strength and naturality of Theorem 1.3.

Comte, Lion, and Rolin [9] continued the work of [19] by showing that the k-dimensional volume of sets in a subanalytic family is a constructible function in the parameters of the family, provided that the volumes are finite. This indeed is the special case of Theorem 1.3, where f itself is subanalytic as opposed to constructible. (The latter may involve logarithms.) They also show in [9, théorème 1′] that the set of parameters where the volume is finite is subanalytic, which compares to Theorem 1.4 but which does not generalize literally to constructible functions, as is shown in the following example.

Example 1.6
Let $g : \mathbb{R} \to \mathbb{R}$ be a function in $\mathcal{C}(\mathbb{R})$ that is not integrable over $\mathbb{R}$. Define $f : (\mathbb{R} \setminus \{0\})^3 \to \mathbb{R} : (x, y, z) \mapsto (x - \log|y|)g(z)$. Then the set of (x, y) such that $z \mapsto f(x, y, z)$ is integrable is not subanalytic since it is the graph of $\log|\cdot|$.

This example shows that the assumption in Theorem 1.4 that the set C is dense is essential and that a naive adaptation of [9, théorème 1′] to the context of constructible functions is false.

Logarithms play an important role in many studies of integrals and differential equations, but it is somehow surprising that they suffice to yield a framework closed under integration as given by Theorem 1.3, although [9] and the p-adic situation already hinted in that direction.

1.3. The p-adic analogue
Using Haar measures on the p-adic numbers, Igusa [14] gave several asymptotic expansions for p-adic one-parameter integrals, and Denef [10], [12] obtained results which are very close to the p-adic analogue of Theorem 1.3. Similar results to our two main theorems were made uniformly in big p and for motivic integrals in [6]. These steps meant a global description instead of asymptotic expansions, arbitrarily many parameters in the integrals, and a framework closed under integration. In [4] and [5],

one enriched the class of integrands to include p-adic subanalytic functions, instead of only p-adic semialgebraic functions.

In some sense, the p-adic situation is easier since many p-adic integrals (e.g., of p-adic constructible functions) can be reduced to certain sums over the residue field and the value group, and these are often more easy to handle. For the reals there is no such reduction, and one has to use very precise and specific versions of preparation theorems to prepare the integrands on nice pieces in a finite partition.

While for the p-adics integration is well understood as soon as one has a nice cell decomposition, and such cell decomposition is known for the semialgebraic case as well as for the subanalytic case (see [11], [4]), for the reals a theory of functions closed under integration for now only makes sense in the subanalytic case and is completely open in the semialgebraic case. This is so because many non-semialgebraic functions arise as parameterized integrals of semialgebraic functions, and not only their logarithms play a role. As mentioned earlier in the introduction, this relates to the difficulties of studying periods and families of periods.

However, our results do provide qualitative information about the tame behavior of parameterized integrals of semialgebraic functions since they are contained in the larger system of constructible functions. The tame behavior of constructible functions also has certain applications to analysis. In [8] we study L^p-properties of constructible functions for general p, as well as variants of Theorem 1.4′. In [7] we give an application of our work to harmonic analysis and generalized Fourier transforms in the context of Chapter VIII of Stein's book [23].

1.4. Our method

The heart of our proof of Theorems 1.3 and 1.4′ and Proposition 1.5 lies in the preparation theorem (Theorem 3.11) for constructible functions and our main proposition (Proposition 6.1) about compatibility of such preparations with integrability conditions. More specifically, in Theorem 3.11 we consider a constructible function $f : X \to \mathbb{R}$ for some subanalytic set $X \subset \mathbb{R}^n$, and we construct a subanalytic cell decomposition of X such that on each cell A in the decomposition, f can be expressed as a finite sum $f(x) = \sum_{i \in I} T_i(x)$, where each term $T_i(x)$ is a very basic term with good properties, namely, roughly a product of a rational monomial, an integral logarithmic monomial, and a subanalytic unit, with distinct powers in the logarithmic monomials. In Proposition 6.1 we then show (for simplicity, say, when $X = A$) that if the set of $x_{<n} = (x_1, \ldots, x_{n-1})$ for which $f(x_{<n}, \cdot) : x_n \mapsto f(x_{<n}, x_n)$ is integrable is dense in $\Pi_{n-1}(A) = \{x_{<n} : x \in A\}$, then for all $x_{<n} \in \Pi_{n-1}(A)$ and all $i \in I$ the function $T_i(x_{<n}, \cdot)$ is integrable. Theorem 1.4′ and Proposition 1.5 will follow. Together with Fubini's theorem, this will reduce the proof of Theorem 1.3 to the case $m = 1$, which will be proven by integrating each of the terms T_i using an adapted version of

the procedure of Lion and Rolin [19]. Note that these proofs avoid taking limits for improper integrals, which were used previously in [19] and [9].

We begin by establishing refined versions of the Lion-Rolin preparation theorem of [18] for subanalytic functions which are used to prove Theorem 3.11 (see Theorems 3.9 and 3.10 below). To prove Proposition 6.1 we use an additional, new technique: we introduce special functions, called *sliver functions*, to study the relative asymptotic behavior of the terms T_i to show that they do not cancel each other out on some thin, open "sliver."

2. Cylindrical preparation of subanalytic functions

In this section we recall the statement of the subanalytic (cylindrical) preparation theorem from [18] (see also [22]) and its supporting definitions, and we fix some notation to be used throughout the paper.

Notation 2.1

Consider integers m and n with $0 \le m \le n$ and a tuple of variables $x = (x_1, \dots, x_n)$. For any increasing map $\lambda : \{1, \dots, m\} \to \{1, \dots, n\}$, define the coordinate projection $\Pi_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ by $\Pi_\lambda(x) := (x_{\lambda(1)}, \dots, x_{\lambda(m)})$, with the understanding that when $m = 0$, one uses the conventions that $\mathbb{R}^0 := \{0\}$ and $\Pi_\emptyset(x) := 0$, where $\emptyset$ denotes the empty map. An important special case is when $\lambda(i) = i$ for all $i \in \{1, \dots, m\}$, and in this case we will write Π_m instead of Π_λ. (Thus $\Pi_0 = \Pi_\emptyset$.) We also use the notation $x_{\le m} = (x_1, \dots, x_m)$, $x_{<m} = (x_1, \dots, x_{m-1})$, and $x_{>m} = (x_{m+1}, \dots, x_n)$. This notation can be applied to components of maps as well; for example, if $f = (f_1, \dots, f_n)$ for some real-valued functions f_i, then $f_{\le m} = (f_1, \dots, f_m)$. If $A \subset \mathbb{R}^n$ and $0 \le m \le n$, we write $A_{x_{\le m}} = \{x_{>m} : (x_{\le m}, x_{>m}) \in A\}$ for the fiber of A over $x_{\le m} \in \mathbb{R}^m$. Further, we write $\mathrm{im}(g)$ for the image of a function g.

Definitions 2.2

Call a function $f : X \subset \mathbb{R}^\ell \to \mathbb{R}^k$ *analytic* if it extends to an analytic function on an open neighborhood of X. An analytic function $u : A \to \mathbb{R}$ on a set $A \subset \mathbb{R}^n$ is a *unit on* A if either $u(x) > 0$ on A or $u(x) < 0$ on A. A *restricted analytic function* is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that the restriction of f to $[-1, 1]^n$ is analytic and $f(x) = 0$ on $\mathbb{R}^n \setminus [-1, 1]^n$.

Recall from the introduction that we call a set or a function *subanalytic* if and only if it is definable in the expansion of the real field by all restricted analytic functions. Thus in this paper, *subanalytic* is an abbreviation of *globally subanalytic*, and in this meaning, the natural logarithm $\log : (0, +\infty) \to \mathbb{R}$ is not subanalytic.

A *subanalytic term* is a function which can be constructed as a finite composition of restricted analytic functions, the algebraic operations of addition and multiplication, and the rational power functions which are defined on $\mathbb{R}$ by

$$x \mapsto \begin{cases} x^r & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

for each rational number r. We shall also use this terminology for restrictions of subanalytic terms to subanalytic sets.

For the rest of the section we fix an ordered list of variables $x_1, \ldots, x_{n+1}$, where $n \geq 0$, and we write x for $(x_1, \ldots, x_n)$ and write y for x_{n+1}, since the variable x_{n+1} will play a special role.

Definitions 2.3
A set $A \subset \mathbb{R}^{n+1}$ is a *subanalytic cylinder* if

(i) $n = 0$, and A is of one of the following four forms:
Form 1 $A = \{a\}$,
Form 2 $A = (a, +\infty)$,
Form 3 $A = (-\infty, a)$,
Form 4 $A = (a, b)$,
where $a < b$ are real numbers, or

(ii) $n > 0$, and $B = \Pi_n(A)$ is a subanalytic set defined in a quantifier-free manner using subanalytic terms and the relations $=$ and $<$, and A is of one of the following four forms:
Form 1 $A = \{(x, y) \in B \times \mathbb{R} : y = a(x)\}$,
Form 2 $A = \{(x, y) \in B \times \mathbb{R} : y > a(x)\}$,
Form 3 $A = \{(x, y) \in B \times \mathbb{R} : y < a(x)\}$,
Form 4 $A = \{(x, y) \in B \times \mathbb{R} : a(x) < y < b(x)\}$,
where $a, b : B \to \mathbb{R}$ are analytic subanalytic terms and $a(x) < b(x)$ on B. We call B the *base* of A.

We say that A is *thin* (in y) if it is of Form 1 and that A is *fat* (in y) if it is of Form 2, 3, or 4.

If A is a fat subanalytic cylinder with base B, a *center for* A is an analytic subanalytic term $\theta : B \to \mathbb{R}$ whose graph is disjoint from A.

If A is a fat subanalytic cylinder and θ is a center for A, a *strong subanalytic unit on* A *with center* θ is a function $u : A \to \mathbb{R}$ of the form $u = U \circ \varphi$, where $\varphi : A \to \mathbb{R}^N$ is a bounded function for some natural number N, U is an analytic unit on the closure of the image of φ, and φ has the form

$$\varphi(x, y) = \big(a_1(x)|y - \theta(x)|^{r_1}, \ldots, a_N(x)|y - \theta(x)|^{r_N}\big),$$

where $a_1, \dots, a_N$ are analytic subanalytic terms on the base of A and $r_1, \dots, r_N$ are rational numbers (some, or all, of which may be equal to zero).

For any real-valued functions f and g on a set A, we say that f *is equivalent to* g *on* A, written $f \sim g$ on A, if there exists $\epsilon > 1$ such that for all $x \in A$,

$$\begin{cases} \epsilon^{-1} f(x) \leq g(x) \leq \epsilon f(x) & \text{if } f(x) \geq 0, \\ \epsilon f(x) \leq g(x) \leq \epsilon^{-1} f(x) & \text{if } f(x) < 0. \end{cases}$$

We write $f \sim_\epsilon g$ on A if we want to specify ϵ.

A partition $\mathcal{P}$ of a set X is called *compatible* with a set $\mathcal{S}$ of subsets of X if for all $P \in \mathcal{P}$ and all $S \in \mathcal{S}$ either $P \subset S$ or $P \cap S = \emptyset$.

THEOREM 2.4 (Subanalytic preparation theorem; see [18], [22])
Let $\mathcal{F}$ be a finite set of real-valued subanalytic functions on a subanalytic set $X \subset \mathbb{R}^{n+1}$, let $\mathcal{S}$ be a finite set of subanalytic subsets of X, and let $\epsilon > 1$. There exists a finite partition of X into subanalytic cylinders which is compatible with $\mathcal{S}$ and is such that the following hold for each cylinder A in this partition:

(i) *If A is thin, then for each $f \in \mathcal{F}$ there exists an analytic subanalytic term $t : B \to \mathbb{R}$ such that $f(x, y) = t(x)$ on A.*

(ii) *If A is fat, then there exists a center θ for A such that each $f \in \mathcal{F}$ can be written in the form*

$$f(x, y) = a(x)|y - \theta(x)|^r u(x, y)$$

on A, where a is an analytic subanalytic term on the base of A, r is a rational number, and u is a strong subanalytic unit on A with center θ. Moreover, if θ is not identically zero, then $y \sim_\epsilon \theta$ on A.

Remark 2.5
In Theorem 2.4 the requirement that $y \sim_\epsilon \theta$ on A if θ is not identically zero implies that if each of the fibers of A over $\Pi_n(A)$ are unbounded (namely, A is of Form 2 or 3 of Definitions 2.3), then $\theta = 0$.

In addition to the subanalytic preparation theorem, we will need the following two lemmas. The first is a reformulation of [22, Lemma 3.4], and the second is a strengthening of [22, Lemma 4.6].

LEMMA 2.6 (see [22])
Let $A \subset \mathbb{R}^{n+1}$ be a subanalytic cylinder with center θ, let u be a strong subanalytic unit on A with center θ, and let r be a rational number. There exists a finite partition of $\Pi_n(A)$ into subanalytic cylinders such that for each cylinder B in the partition there

exist a natural number N and functions $\varphi : A \cap (B \times \mathbb{R}) \to \mathbb{R}^{N+2}$ and $U : \mathbb{R}^{N+2} \to \mathbb{R}$ such that $u(x, y) = U \circ \varphi(x, y)$ on $A \cap (B \times \mathbb{R})$, U is an analytic unit on the closure of the image of φ, and φ is a bounded function of the form

$$\varphi(x, y) = \big(a_1(x), \ldots, a_N(x), b(x)|y - \theta(x)|^{1/p}, c(x)|y - \theta(x)|^{-1/p}\big),$$

where $a_1, \ldots, a_N, b, c$ are analytic subanalytic terms on B and p is a positive integer such that rp is an integer.

LEMMA 2.7

Let θ_1 and θ_2 be real-valued subanalytic terms on $\Pi_n(X)$ for a subanalytic set $X \subset \mathbb{R}^{n+1}$. There exists a finite partition of X into subanalytic cylinders such that for each fat cylinder A in this partition, θ_1 and θ_2 are both centers for A such that $\theta_1 - \theta_2$ has a constant sign on $\Pi_n(A)$, and when $\theta_1 \neq \theta_2$ on $\Pi_n(A)$, at least one of the following three cases holds on A, where the expressions in square brackets are strong subanalytic units on A (with center θ_2 in case (i), and center θ_1 in cases (ii) and (iii)):

(i) $$y - \theta_1(x) = \big(\theta_2(x) - \theta_1(x)\big) \cdot \Big[1 + \frac{y - \theta_2(x)}{\theta_2(x) - \theta_1(x)}\Big],$$

(ii) $$y - \theta_2(x) = \big(\theta_1(x) - \theta_2(x)\big) \cdot \Big[1 + \frac{y - \theta_1(x)}{\theta_1(x) - \theta_2(x)}\Big],$$

(iii) $$y - \theta_2(x) = \big(y - \theta_1(x)\big) \cdot \Big[1 + \frac{\theta_1(x) - \theta_2(x)}{y - \theta_1(x)}\Big].$$

Proof

By partitioning X, we may focus on a fat cylinder A such that $\theta_1 - \theta_2$ has constant sign on $\Pi_n(A)$. We are done if $\theta_1 = \theta_2$ on $\Pi_n(A)$; the cases $\theta_1 > \theta_2$ and $\theta_1 < \theta_2$ can be handled similarly, so we assume that $\theta_1 > \theta_2$ on $\Pi_n(A)$. Choose constants a and b such that $1/2 < a < 1 < b < 1 + a$, and consider the following sets, each of which is a finite union of cylinders:

$$A_1 = \Big\{(x, y) \in A : \frac{y - \theta_2(x)}{\theta_1(x) - \theta_2(x)} > b\Big\},$$

$$A_2 = \Big\{(x, y) \in A : \Big|\frac{y - \theta_1(x)}{\theta_1(x) - \theta_2(x)}\Big| < a\Big\},$$

$$A_3 = \Big\{(x, y) \in A : \Big|\frac{y - \theta_2(x)}{\theta_1(x) - \theta_2(x)}\Big| < a\Big\},$$

$$A_4 = \Big\{(x, y) \in A : \frac{y - \theta_1(x)}{\theta_1(x) - \theta_2(x)} < -b\Big\}.$$

By the choice of a and b, these sets cover A. The expression in square brackets in (i) is clearly a strong subanalytic unit on A_3, and likewise for (ii) on A_2. A little algebra shows that

$$0 < \frac{\theta_1(x) - \theta_2(x)}{y - \theta_1(x)} < \frac{1}{b-1}$$

on A_1 and that

$$-\frac{1}{b} < \frac{\theta_1(x) - \theta_2(x)}{y - \theta_1(x)} < 0$$

on A_4. Therefore the expression in square brackets in (iii) is a strong subanalytic unit on both A_1 and A_4. □

Note that the center θ_1 plays a special role in Lemma 2.7, in the sense that when $y - \theta_1(x)$ and $y - \theta_2(x)$ are equivalent up to multiplication by a strong subanalytic unit, this unit is always constructed to have center θ_1.

Remark 2.8
In case (iii) of Lemma 2.7 the definition of strong subanalytic units shows that $(\theta_1(x) - \theta_2(x))/(y - \theta_1(x))$ is bounded on A, and also, $\theta_1(x) - \theta_2(x)$ has a constant nonzero sign on the base of A, from which it follows that there exists a constant c such that $|y - \theta_1(x)| > c(\theta_1(x) - \theta_2(x)) > 0$ on A.

3. Cell preparation of subanalytic and constructible functions

This section gives three variants of the subanalytic preparation theorem. They prepare functions on cells, rather than on cylinders, in such a specific way that we call them *cell preparation theorems*. The first two prepare (finite collections of) subanalytic functions, and the third prepares constructible functions.

Definition 3.1
A *subanalytic cell* is a set $A \subset \mathbb{R}^n$ such that $\Pi_i(A)$ is a subanalytic cylinder for all $i \in \{1, \dots, n\}$. There exists a unique increasing function $\lambda : \{1, \dots, d\} \to \{1, \dots, n\}$ whose image is the set $\{i \in \{1, \dots, n\} : \Pi_i(A) \text{ is fat}\}$. We call A a *λ-cell* if we want to specify λ. For any $I \subset \operatorname{im}(\lambda)$, we say that A is *fat in* $(x_i)_{i \in I}$.

In the previous definition, A is clearly a connected analytic submanifold of $\mathbb{R}^n$ of dimension d, and the projection $\Pi_\lambda : A \to \mathbb{R}^d$ is an analytic isomorphism onto its image, which is an open subanalytic cell in $\mathbb{R}^d$.

Definition 3.2
Suppose that $A \subset \mathbb{R}^n$ is an open subanalytic cell. For each $i \in \{1, \dots, n\}$, let $\theta_i : \Pi_{i-1}(A) \to \mathbb{R}$ be an analytic subanalytic term, and write $\widetilde{x}_i := x_i - \theta_i(x_1, \dots, x_{i-1})$. If $\widetilde{x}_i \notin \{-1, 0, 1\}$ for all $i \in \{1, \dots, n\}$ and all $x \in A$, then we call $\theta = (\theta_1, \dots, \theta_n) : A \to \mathbb{R}^n$ a *center for* A, and we call $\widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_n)$ the *coordinates for* A *with center* θ. If $\theta_1 = \cdots = \theta_n = 0$, we simply say that *zero is a center for* A.

More generally, suppose that $A \subset \mathbb{R}^n$ is a d-dimensional λ-cell. A *center for* A is a center θ for $\Pi_\lambda(A)$, and the *coordinates for* A *with center* θ are the coordinates for $\widetilde{x}$ for $\Pi_\lambda(A)$ with center θ.

Notation 3.3
Suppose that $A \subset \mathbb{R}^n$ is a d-dimensional λ-cell with center θ, and suppose that $\widetilde{x}$ are the coordinates for A with center θ. We index θ and $\widetilde{x}$ by $\mathrm{im}(\lambda)$ rather than $\{1, \dots, d\}$, writing $\theta = (\theta_{\lambda(1)}, \dots, \theta_{\lambda(d)})$ and $\widetilde{x} = (\widetilde{x}_{\lambda(1)}, \dots, \widetilde{x}_{\lambda(d)})$, and considering each $\theta_{\lambda(i)}$ to be a function of $(x_{\lambda(1)}, \dots, x_{\lambda(i-1)})$. For any tuple $\alpha = (\alpha_i)_{i \in \mathrm{im}(\lambda)}$ of rational numbers, let

$$|\widetilde{x}|^\alpha = \prod_{i \in \mathrm{im}(\lambda)} |\widetilde{x}_i|^{\alpha_i}.$$

We shall write $\mathbb{Q}^{\mathrm{im}(\lambda)}$ for tuples of rational numbers indexed by $\mathrm{im}(\lambda)$, and likewise for $\mathbb{R}^{\mathrm{im}(\lambda)}$. If $J \subset \mathrm{im}(\lambda)$ and $\alpha \in \mathbb{Q}^{\mathrm{im}(\lambda)}$ are such that $\alpha_i = 0$ for all $i \in \mathrm{im}(\lambda) \setminus J$, we say that α *has support in* J.

Definitions 3.4
Let $A \subset \mathbb{R}^n$ be a d-dimensional subanalytic λ-cell with center θ. Then for all $i \in \mathrm{im}(\lambda)$, the set $\{\widetilde{x}_i : x \in A\}$ is contained in either $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, or $(1, +\infty)$, so there exist unique $\varepsilon_i, \zeta_i \in \{-1, 1\}$ such that $0 < \varepsilon_i \widetilde{x}_i^{\zeta_i} < 1$ for all $x \in A$. Define

$$A_{\mathrm{bd}} = \left\{ (\varepsilon_i \widetilde{x}_i^{\zeta_i})_{i \in \mathrm{im}(\lambda)} : x \in A \right\}.$$

Define the isomorphism $F_A^{[\theta]} : A \to A_{\mathrm{bd}}$ by $F_A^{[\theta]}(x) = (\varepsilon_i \widetilde{x}_i^{\zeta_i})_{i \in \mathrm{im}(\lambda)}$, and define $G_A^{[\theta]} : A_{\mathrm{bd}} \to A$ to be the inverse of $F_A^{[\theta]}$. We consider A_{bd} to be a subset of $\mathbb{R}^d$ (rather than of $\mathbb{R}^{\mathrm{im}(\lambda)}$), and we write $y = (y_1, \dots, y_d) = F_A^{[\theta]}(x)$ for $x \in A$. So

$$y_i = \varepsilon_{\lambda(i)} \widetilde{x}_{\lambda(i)}^{\zeta_{\lambda(i)}}$$

for each $i \in \{1, \dots, d\}$. Clearly A_{bd} is an open subanalytic cell in $(0, 1)^d$ with center zero. Write

$$\Pi_i(A_{\mathrm{bd}}) = \left\{ y_{\le i} : y_{<i} \in \Pi_{i-1}(A_{\mathrm{bd}}), a_i(y_{<i}) < y_i < b_i(y_{<i}) \right\} \tag{3.1}$$

for each $i \in \{1, \dots, d\}$.

(1) A *strong subanalytic unit on A with center θ* is a function of the form $u = U \circ \varphi$, where $\varphi : A \to \mathbb{R}^N$ is a bounded function of the form

$$\varphi(x) = (|\widetilde{x}|^{\beta_1}, \ldots, |\widetilde{x}|^{\beta_N})$$

for some natural number N and $\beta_1, \ldots, \beta_N \in \mathbb{Q}^{\mathrm{im}(\lambda)}$, and U is an analytic unit on the closure of the image of φ.

(2) For $i \in \{1, \ldots, d\}$, call $\widetilde{x}_{\lambda(i)}$ *asymptotically determined on A* if there exists $C > 0$ such that $b_i(y_{<i}) < Ca_i(y_{<i})$ on $\Pi_{i-1}(A_{\mathrm{bd}})$. Otherwise, call $\widetilde{x}_{\lambda(i)}$ *asymptotically undetermined*.

(3) Let $\mathcal{F}$ be a finite family of subanalytic functions on A, and consider a set

$$J \subset \left\{ i \in \mathrm{im}(\lambda) : \widetilde{x}_i \text{ is asymptotically undetermined on } A \right\}.$$

By a joint induction on d, we define what it means for A to be *J-prepared with center θ* and what it means for $\mathcal{F}$ to be *J-prepared on A with center θ*.

If $d = 0$, then A is a singleton and $J = \theta = \emptyset$, and A and $\mathcal{F}$ are called *$\emptyset$-prepared with center $\emptyset$*. For $d \geq 1$, say that A is *J-prepared with center θ* if the collection of functions $\{a_d, b_d, b_d - a_d\}$ is $(\{1, \ldots, d-1\} \cap \lambda^{-1}(J))$-prepared on $\Pi_{d-1}(A_{\mathrm{bd}})$ with center zero (or center $\emptyset$ when $d = 1$), and if the closure of the image of a_d contains zero. We say that $\mathcal{F}$ is *J-prepared on A with center θ* if A is J-prepared with center θ, and if for each $f \in \mathcal{F}$ either $f = 0$ on A or

$$f(x) = |\widetilde{x}|^{\alpha} u(x)$$

on A for some strong subanalytic unit u on A with center θ and some $\alpha \in \mathbb{Q}^{\mathrm{im}(\lambda)}$ with support in J.

When $J = \{i \in \mathrm{im}(\lambda) : \widetilde{x}_i \text{ is asymptotically undetermined on } A\}$, we simply say that A (or $\mathcal{F}$) is *prepared with center θ* (on A).

(4) For $i \in \{1, \ldots, d\}$, call $\widetilde{x}_{\lambda(i)}$ *constrained on A* if $a_i > 0$ on $\Pi_{i-1}(A_{\mathrm{bd}})$, and call $\widetilde{x}_{\lambda(i)}$ *unconstrained on A* if $a_i = 0$ on $\Pi_{i-1}(A_{\mathrm{bd}})$.

Note that if A is a subanalytic λ-cell in $\mathbb{R}^n$ with center $\theta = (\theta_i)_{i \in \mathrm{im}(\lambda)}$, where $n \in \mathrm{im}(\lambda)$, and if u is a strong subanalytic unit on A with center θ, then A is also a subanalytic cylinder in $\mathbb{R}^n$ with center θ_n and u is a strong subanalytic unit on A with center θ_n (in the cylindrical senses of Definitions 2.3).

LEMMA 3.5
Let $A \subset (0, 1)^n$ be an open cell $A \subset (0, 1)^n$ which is prepared with center zero, and let $J = \{i : \widetilde{x}_i$ is asymptotically undetermined on A $\}$. Then for any nonzero $\gamma \in \mathbb{Q}^n$

with support in J, the image of the function $x \in A \mapsto x^\gamma$ is not contained in a compact subset of $(0, +\infty)$.

Proof

The proof is by induction on n. For $n = 1$ this is clear. Suppose that the statement is known for $n - 1$. Write

$$A = \left\{x : x_{<n} \in \Pi_{n-1}(A),\ a(x_{<n}) < x_n < b(x_{<n})\right\},$$

where either $a = 0$ or $a(x_{<n}) = x_{<n}^{\alpha} u(x)$, and $b(x_{<n}) = x_{<n}^{\beta} v(x)$, for some $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and $\beta = (\beta_1, \ldots, \beta_{n-1})$ in $\mathbb{Q}^{n-1}$ with support in $J \cap \{1, \ldots, n-1\}$. We may suppose that $n \in J$ and that $\gamma_n \neq 0$, since otherwise we are done. If $a = 0$, then by fixing $x_{<n} \in \Pi_{n-1}(A)$ and letting $x_n \to 0$, x^γ tends to either zero or $+\infty$, and we are done. So assume that $a > 0$. Since $n \in J$, it follows that $\alpha \neq \beta$. Define, on $\Pi_{n-1}(A)$, the functions

$$h_1(x_{<n}) := x_{<n}^{\gamma_{<n}} a(x_{<n})^{\gamma_n} \qquad \text{and} \qquad h_2(x_{<n}) := x_{<n}^{\gamma_{<n}} b(x_{<n})^{\gamma_n}.$$

Clearly g_γ takes all values between $h_1(x_{<n})$ and $h_2(x_{<n})$ for any $x_{<n} \in \Pi_{n-1}(A)$. Since $\alpha \neq \beta$, at least one of the tuples $\gamma_{<n} + \gamma_n\alpha$ or $\gamma_{<n} + \gamma_n\beta$ is nonzero. Hence, by induction, at least one of the h_i has an image which is not contained in a compact subset of $(0, +\infty)$, and so does g_γ. □

The following are immediate consequences of Lemma 3.5.

COROLLARIES 3.6

Using the notation from Definitions 3.4, suppose that A is prepared with center θ, and let $J = \{i \in \mathrm{im}(\lambda) : \widetilde{x}_i$ is asymptotically undetermined on $A\}$. For the functions a_i and b_i given in (3.1), write $a_i(y_{<i}) = y_{<i}^{\alpha_i} u_i(y_{\le i})$ provided that $a_i > 0$ on $\Pi_{i-1}(A_{\mathrm{bd}})$, and write $b_i(y_{\le i}) = y_{<i}^{\beta_i} v_i(y_{\le i})$, where $\alpha_i, \beta_i \in \mathbb{Q}^{i-1}$ have support in $\{1, \ldots, i-1\} \cap \lambda^{-1}(J)$, and u_i and v_i are strong subanalytic units on $\Pi_i(A_{\mathrm{bd}})$ with center zero.

(1) *For each $i \in \{1, \ldots, d\}$, $\widetilde{x}_{\lambda(i)}$ is asymptotically determined on A if and only if $a_i > 0$ and $\alpha_i = \beta_i$, if and only if a_i/b_i is a strong subanalytic unit.*

(2) *The prepared form of a subanalytic function $f : A \to \mathbb{R}$ is unique in the following sense: if $f(x) = |\widetilde{x}|^\alpha u(x) = |\widetilde{x}|^\beta v(x)$ on A, where $\alpha, \beta \in \mathbb{Q}^n$ have support in J and u and v are strong subanalytic units on A with center θ, then $\alpha = \beta$.*

Definition 3.7

Inductively define a *subanalytic cell decomposition* of a subanalytic set $X \subset \mathbb{R}^n$ to be a finite partition $\mathcal{A}$ of X by subanalytic cells such that, when $n > 0$, $\{\Pi_{n-1}(A) : A \in \mathcal{A}\}$ is a subanalytic cell decomposition of $\Pi_{n-1}(X)$.

Definition 3.8

Let $\mathcal{F}$ be a finite collection of real-valued subanalytic functions on a subanalytic set X. A *cell preparation* of $\mathcal{F}$ is a finite subanalytic cell decomposition of X, say, $\mathcal{A}$, such that for each $A \in \mathcal{A}$ there exists a center θ for A such that $\mathcal{F}$ is prepared on A with center θ. We call θ the center *associated* with A by the preparation.

THEOREM 3.9

Suppose that $\mathcal{F}$ is a finite collection of real-valued subanalytic functions on a subanalytic set X and that $\mathcal{S}$ is a finite set of subanalytic subsets of X. There exists a cell preparation of $\mathcal{F}$ compatible with $\mathcal{S}$.

Proof

Apply Theorem 2.4 to $\mathcal{F}$ and $\mathcal{S}$, and let $\mathcal{A}$ be the finite partition of X into subanalytic cylinders, compatible with $\mathcal{S}$, which is given by the preparation. The proof now proceeds by induction on n, where $X \subset \mathbb{R}^n$. The case $n = 0$, with $\mathbb{R}^0 = \{0\}$, being trivial, suppose that $n \geq 1$. By further partitioning the cylinders in $\mathcal{A}$, we may assume that for each fat cylinder $A \in \mathcal{A}$ with associated center $\theta_n : \Pi_{n-1}(A) \to \mathbb{R}$,

$$A = \left\{x : x_{<n} \in \Pi_{n-1}(A), a(x_{<n}) < \varepsilon_i \widetilde{x}_n^{\zeta_n} < b(x_{<n})\right\} \tag{3.2}$$

for some $\varepsilon_n, \zeta_n \in \{-1, 1\}$ and subanalytic functions $a, b : \Pi_{n-1}(A) \to [0, 1]$ such that $a < b$ and either $a = 0$ or $a > 0$, where $\widetilde{x}_n = x_n - \theta_n(x_{<n})$. By partitioning the cylinders in $\mathcal{A}$ even further, we may also assume that $\mathcal{A}$ is a cylindrical decomposition of X over $\Pi_{n-1}(X)$, namely, that there exists a finite partition $\mathcal{B}$ of $\Pi_{n-1}(X)$ into subanalytic sets such that $\mathcal{A} = \bigcup_{B \in \mathcal{B}} \mathcal{A}_B$, where $\mathcal{A}_B$ is a set of disjoint cylinders with base B. Fix $B \in \mathcal{B}$. For each fat cylinder $A \in \mathcal{A}_B$ there exists a center $\theta_n : B \to \mathbb{R}$ such that each $f \in \mathcal{F}$ can be written in the form

$$f(x) = c(x_{<n})|\widetilde{x}_n|^r u\left(a_1(x_{<n})|\widetilde{x}_n|^{r_1}, \ldots, a_N(x_{<n})|\widetilde{x}_n|^{r_n}\right) \tag{3.3}$$

on A. Now apply the induction hypothesis to $\mathcal{B}$ and the following set of functions on each $B \in \mathcal{B}$ (extended by zero outside B):

- for each $f \in \mathcal{F}$ and each fat cylinder $A \in \mathcal{A}_B$, the functions $c, a_1, \ldots, a_N$ from (3.3);
- for each fat cylinder $A \in \mathcal{A}_B$, the functions a, b, and $b - a$ from (3.2);

- for each thin cylinder $A \in \mathcal{A}_B$, say of the form $A = \{x \in B \times \mathbb{R} : x_n = d(x_{<n})\}$, and each $f \in \mathcal{F}$, the function $f \circ d$.

This gives a decomposition of X into subanalytic cells which associates to each cell A in the decomposition a center $\theta = (\theta_1, \dots, \theta_n)$ such that $\Pi_{n-1}(A)$ is prepared with center $\theta_{<n}$, and each $f \in \mathcal{F}$ is either identically zero on A or can be written in the form

$$f(x) = |\widetilde{x}|^{\alpha} u(x)$$

on A for a strong subanalytic unit $u(x)$ with center θ, where for each $i < n$, $\alpha_i = 0$ if $\widetilde{x}_i$ is asymptotically determined on A. If A is thin in x_n, then $\mathcal{F}$ is prepared on A with center θ. If A is fat in x_n and $a = 0$, then we are also done on A. So let A be fat in x_n with $a > 0$. There are two possible reasons why $\mathcal{F}$ might not be prepared on A with center θ:

(1) It might be that $\alpha_n \neq 0$, but $\widetilde{x}_n$ is asymptotically determined on A; that is, the image of a/b is contained in a compact subset of $(0, +\infty)$.

(2) It might be that a and b are both strong subanalytic units on $\Pi_{n-1}(A)$ with center $\theta_{<n}$, which violates the requirement that zero be in the closure of the image of a.

Suppose that we are in case (1). Then

$$f(x) = |\widetilde{x}_{<n}|^{\alpha_{<n}} |a(\widetilde{x}_{<n})|^{\alpha_n} \cdot \left[\frac{|\widetilde{x}_n|^{\alpha_n}}{|a(\widetilde{x}_{<n})|^{\alpha_n}} u(x) \right].$$

The expression in square brackets is a strong subanalytic unit, and we are done.

Finally, we treat case (2), where we suppose that $\varepsilon_n = \zeta_n = 1$, the other cases being similar. Since clearly $\widetilde{x}_n$ is asymptotically determined on A, our treatment of case (1) shows that we can assume that $\alpha_n = 0$. We will change the center so that any strong subanalytic unit u with the old center remains a strong subanalytic unit with the new center. Write

$$u(x) = U(|\widetilde{x}|^{\gamma_1}, \dots, |\widetilde{x}|^{\gamma_k})$$

for some natural number k, tuples $\gamma_1, \dots, \gamma_k \in \mathbb{Q}^n$, and a function U which is analytic on the closure of the image of $x \in A \mapsto (|\widetilde{x}|^{\gamma_1}, \dots, |\widetilde{x}|^{\gamma_k})$, which is bounded in $\mathbb{R}^k$. For any i, the closure of the image of the map $x \in A \mapsto (|\widetilde{x}_{<n}|^{\gamma_{<n}(i)}, \widetilde{x}_n)$ is a compact subset of $\mathbb{R} \times (0, +\infty)$, and the map $(s, t) \mapsto s \cdot t^{\gamma_n(i)}$ is analytic on this set. So we may assume that $u(x)$ is of the form

$$u(x) = U(|\widetilde{x}_{<n}|^{\gamma_1}, \dots, |\widetilde{x}_{<n}|^{\gamma_{\ell-1}}, \widetilde{x}_n)$$

for an analytic function $U(t_1, \ldots, t_\ell)$ and $\gamma_1, \ldots, \gamma_{\ell-1} \in \mathbb{Q}^{n-1}$, for some natural number ℓ. But then we may write A as

$$A = \{x \in \Pi_{n-1}(A) \times \mathbb{R} : 0 < x_n - (\theta_n + a)(x_{<n}) < (b - a)(x_{<n})\}$$

and write u as

$$u(x) = U\big(\widetilde{x}_{<n}^{\gamma_1}, \ldots, \widetilde{x}_{<n}^{\gamma_{\ell-1}}, (x_n - (\theta_n + a)(x_{<n})) + a(x_{<n})\big).$$

The function $b - a$ is prepared on $\Pi_{n-1}(A)$ with center $\theta_{<n}$, so A is prepared with center $(\theta_{<n}, \theta_n + a)$. Also, the function a is a strong subanalytic unit on $\Pi_{n-1}(A)$ with center $\theta_{<n}$, so $u(x)$ is a strong subanalytic unit on A with center $(\theta_{<n}, \theta_n + a)$. Thus $\mathcal{F}$ is prepared on A with center $(\theta_{<n}, \theta_n + a)$. □

In order to apply the cell preparation theorem multiple times we will need the following variant of Theorem 3.9.

THEOREM 3.10

Suppose that $\mathcal{F}$ is a finite set of real-valued subanalytic functions on a subanalytic set X, suppose that $\mathcal{S}$ is a subanalytic cell decomposition of X, and suppose that for each $S \in \mathcal{S}$ we have an associated center θ_S for S. There exists a cell preparation of $\mathcal{F}$ compatible with $\mathcal{S}$ such that for each cell A in this preparation, say, a λ-cell with center $\theta = (\theta_{\lambda(1)}, \ldots, \theta_{\lambda(d)})$, if S is the unique cell in $\mathcal{S}$ containing A, then for each $i \in \mathrm{im}(\lambda_S)$ exactly one of the following holds:

(i) *The cell A is thin in x_i, and*

$$x_i - \theta_{S,i}(x_1, \ldots, x_{i-1}) = a(x_1, \ldots, x_{i-1})$$

on A for a subanalytic term $a(x_1, \ldots, x_{i-1})$ which is prepared on $\Pi_{i-1}(A)$ with center $(\theta_j)_{j \in \mathrm{im}(\lambda), j<i}$.

(ii) *The cell A is fat in x_i, and*

$$x_i - \theta_{S,i}(x_1, \ldots, x_{i-1}) = a(x_1, \ldots, x_{i-1}) u(x_1, \ldots, x_i)$$

on A for a subanalytic term $a(x_1, \ldots, x_{i-1})$ which is prepared on $\Pi_{i-1}(A)$ with center $(\theta_j)_{j \in \mathrm{im}(\lambda), j<i}$ and a strong subanalytic unit $u(x_1, \ldots, x_i)$ on $\Pi_i(A)$ with center $(\theta_j)_{j \in \mathrm{im}(\lambda), j \le i}$.

(iii) *The cell A is fat in x_i, $\theta_i = \theta_{S,i}$, and $x_i - \theta_i(x_1, \ldots, x_{i-1})$ is asymptotically undetermined on A.*

Proof
The proof of Theorem 3.10 is a simple modification of the proof of Theorem 3.9 using Lemma 2.7. Indeed, for the cases $n \geq 1$, begin as before by applying Theorem 2.4 to get a cylindrical preparation of $\mathcal{F}$ compatible with $\mathcal{S}$. Call it $\mathcal{A}$ as before. Consider a fat cylinder $A \in \mathcal{A}$, and fix the unique $S \in \mathcal{S}$ such that $A \subset S$. Let $\theta_{A,n}$ be the center for A given by the cylindrical preparation $\mathcal{A}$. Thus each $f \in \mathcal{F}$ can be written in the form

$$f(x) = a(x_{<n})|x_n - \theta_{A,n}(x_{<n})|^r u\big(x_{<n}, x_n - \theta_{A,n}(x_{<n})\big) \tag{3.4}$$

on A, where $u(x_{<n}, x_n)$ is a strong subanalytic unit with center zero on

$$\big\{(x_{<n}, x_n - \theta_{A,n}(x_{<n})) : x \in A\big\}.$$

By applying Lemma 2.7 to $\theta_{S,n}$ and $\theta_{A,n}$ on A, we get a partition $\mathcal{A}'_A$ of A by subanalytic cylinders such that for each fat cylinder $A' \in \mathcal{A}'_A$, either $\theta_{A,n} = \theta_{S,n}$ on A', or at least one of the following holds on A':

(1) $$x_n - \theta_{S,n}(x_{<n}) = \big(\theta_{A,n}(x_{<n}) - \theta_{S,n}(x_{<n})\big) \cdot \left[1 + \frac{x_n - \theta_{A,n}(x_{<n})}{\theta_{A,n}(x_{<n}) - \theta_{S,n}(x_{<n})}\right],$$

(2) $$x_n - \theta_{A,n}(x_{<n}) = \big(\theta_{S,n}(x_{<n}) - \theta_{A,n}(x_{<n})\big) \cdot \left[1 + \frac{x_n - \theta_{S,n}(x_{<n})}{\theta_{S,n}(x_{<n}) - \theta_{A,n}(x_{<n})}\right],$$

(3) $$x_n - \theta_{A,n}(x_{<n}) = \big(x_n - \theta_{S,n}(x_{<n})\big) \cdot \left[1 + \frac{\theta_{S,n}(x_{<n}) - \theta_{A,n}(x_{<n})}{x_n - \theta_{S,n}(x_{<n})}\right],$$

where the expressions in square brackets are strong subanalytic units. We now assign a center θ_n to A' as follows: In case 1 we let $\theta_n = \theta_{A,n}$. In cases (2) and (3) we let $\theta_n = \theta_{S,n}$, and for each $f \in \mathcal{F}$ we use the equation from either (2) or (3) to express each instance of $x_n - \theta_{A,n}(x_{<n})$ in (3.4) in terms of $x_n - \theta_n(x_{<n})$ so that f is prepared on A' with center θ_n.

Doing this for each fat cell A in $\mathcal{A}$ gives a cylindrical preparation $\mathcal{A}' = \bigcup_{A \in \mathcal{A}} \mathcal{A}'_A$ of $\mathcal{F}$ on X. We now proceed as before in the proof of Theorem 3.9, except that we use $\mathcal{A}'$ in place of $\mathcal{A}$; we include the functions $\theta_{A,n} - \theta_{S,n}$ (extended by zero), and on thin cylinders A we include the functions $x_n - \theta_{S,n}$ (extended by zero) in the set of functions to which the induction hypothesis is applied, and we strengthen our induction hypothesis to assume that cases (i), (ii), or (iii) of Theorem 3.10 hold in the variables $x_1, \ldots,$ x_{n-1}. □

THEOREM 3.11 (Cell preparation of constructible functions)
Suppose that $\mathcal{F}$ is a finite set of functions in $\mathcal{C}(X)$ for a subanalytic set $X \subset \mathbb{R}^n$, and suppose that $\mathcal{S}$ is a finite set of subanalytic subsets of X. There exists a finite

subanalytic cell decomposition of X, compatible with $\mathcal{S}$, such that for each cell A in this decomposition, there exists a center θ such that A is prepared with center θ and the following hold.

Suppose that A is a d-dimensional λ-cell, let $\widetilde{x} = (\widetilde{x}_{\lambda(1)}, \ldots, \widetilde{x}_{\lambda(d)})$ be the coordinates for A with center θ, and let

$$J = \big\{ j \in \mathrm{im}(\lambda) : \widetilde{x}_j \textit{ is asymptotically undetermined on } A \big\}.$$

Each $f \in \mathcal{F}$ can be written as a finite sum

$$f(x) = \sum_{i \in I} T_i(x) \tag{3.5}$$

on A, where each term $T_i(x)$ is of the form

$$u_i(\widetilde{x}) \Big(\prod_{j \in J} |\widetilde{x}_j|^{\alpha_j(i)} (\log |\widetilde{x}_j|)^{\ell_j(i)} \Big) \tag{3.6}$$

for rational numbers $\alpha_j(i)$, integers $\ell_j(i) \geq 0$, and strong subanalytic units u_i with center θ. Moreover, the tuples $(\ell_j(i))_{j \in J}$ are distinct for different i in I.

Proof
Each $f \in \mathcal{F}$ may be written in the form

$$f(x) = \sum_i g_i(x) \prod_k \log g_{ik}(x),$$

where the $g_i : X \to \mathbb{R}$ and $g_{ik} : X \to (0, +\infty)$ are subanalytic and i and k run over finite index sets. Apply Theorem 3.9 to all the g_{ik} for all $f \in \mathcal{F}$. Fix a cell A' from this preparation, with associated center θ'. Let x' be the coordinates for A' with center θ', and let J' be the set of indices j in $\{1, \ldots, n\}$ for which x_j is asymptotically undetermined on A'. Thus we may write each $f \in \mathcal{F}$ in the form

$$f(x) = \sum_i g_i(x) \prod_k \log \Big(\Big(\prod_{j \in J'} |x'_j|^{\gamma_j(i,k)} \Big) u_{i,k}(x') \Big)$$

on A' for rational numbers $\gamma_j(i, k)$ and strong subanalytic units $u_{i,k}$. By expanding the logarithmic expressions and by using the facts that logarithms of strong subanalytic units are subanalytic and that sums of subanalytic functions are subanalytic, we may write each $f \in \mathcal{F}$ in the form

$$f(x) = \sum_{i \in I_f} f_i(x) \Big(\prod_{j \in J'} (\log |x'_j|)^{\ell_{f,j}(i)} \Big) \tag{3.7}$$

on A', where the $(\ell_{f,j}(i))_{j\in J'}$ are tuples of natural numbers which are distinct for different i in the finite index set I_f, and the f_i are subanalytic functions.

Consider the following "lexicographical" ordering of the power set of $\{1,\ldots,n\}$: for any subsets M and N of $\{1,\ldots,n\}$, define $M < N$ if and only if there exists $m \in \{1,\ldots,n\}$ such that $M \cap \{m+1,\ldots,n\} = N \cap \{m+1,\ldots,n\}$ and $\max(M \cap \{1,\ldots,m\}) < \max(N \cap \{1,\ldots,m\})$, with the understanding that $\max(\emptyset)$ is defined to be zero. Note that the smallest member of this ordering is the empty set. With respect to this ordering, we induct on the set

$$L' := \big\{ j \in J' : \ell_{f,j}(i) > 0 \text{ for some } f \in \mathcal{F} \text{ and } i \in I_f \big\}. \tag{3.8}$$

If L' is empty, then each $f \in \mathcal{F}$ is subanalytic on A', in which case we are done by applying Theorem 3.9 to $\mathcal{F}$ on A'. So assume that L' is nonempty. Apply Theorem 3.10 to $\{f_i : f \in \mathcal{F}, i \in I_f\}$. Fix an open cell $A'' \subset A'$ from this preparation, with associated center θ''. Let x'' be the coordinates for A'' with center θ'', and let J'' be the set of indices j in $\{1,\ldots,n\}$ for which x''_j is asymptotically undetermined on A''. Thus each $f \in \mathcal{F}$ can be written as

$$f(x) = \sum_{i\in I_f} u_{f,i}(x'')\Big(\prod_{j\in J''} |x''_j|^{\alpha_{f,j}(i)}\Big)\Big(\prod_{j\in J'} (\log|x'_j|)^{\ell_{f,j}(i)}\Big) \tag{3.9}$$

on A'' for some rational numbers $\alpha_{f,j}(i)$ and strong subanalytic units $u_{f,i}$, and each x'_j can be expressed in terms of x'' in the form (i), (ii), or (iii) from Theorem 3.10. We are done if case (iii) holds for every j in L', so assume otherwise. Let m be the maximum j in L' such that x'_j is of the form (i) or (ii). By expanding each of the logarithmic terms $(\log(x'_m))^{\ell_{f,m}(i)}$ in (3.9), where x'_m is expressed in terms of $x''_{\leq m}$ in accordance with the form (i) or (ii), we may write each $f \in \mathcal{F}$ in the form (3.7) on A'' but with a new set (3.8), which we call L'', which is lexicographically less than L', since $L'' \cap \{m+1,\ldots,n\} = L' \cap \{m+1,\ldots,n\}$ and $\max(L'' \cap \{1,\ldots,m\}) < m = \max(L' \cap \{1,\ldots,m\})$. We are done by the induction hypothesis. □

Remark 3.12

Remark 2.5 and the proof of Theorem 3.9 imply that the cell preparation constructed by Theorem 3.9 has the following property:

> For every cell A in the preparation with associated center $\theta = (\theta_1,\ldots,\theta_n)$, and every $i \in \{1,\ldots,n\}$, if each of the fibers of $\Pi_i(A)$ over $\Pi_{i-1}(A)$ are unbounded, then $\theta_i = 0$.

Similarly, if in Theorem 3.10 we assume that the given cell decomposition $\mathcal{S}$, with associated centers θ_S, has this property, then the cell preparation constructed by Theorem

3.10 has this property. Therefore the cell preparation constructed in Theorem 3.11 also has this property, because its proof applies Theorems 3.9 and 3.10 in succession.

4. Sliver functions

Slivers will be used to study relative asymptotic classes of basic terms in the proof of Proposition 6.1.

Definition 4.1

Let $\epsilon \in (0, 1)$, and for each $i \in \{2, \dots, n\}$ let (p_i, q_i) be an interval with $0 < p_i < q_i < +\infty$. Write $(p, q) = (p_2, q_2) \times \cdots \times (p_n, q_n)$. Define $\psi : (0, \epsilon) \times (p, q) \to \mathbb{R}^n$ by

$$\psi(t) = (t_1, t_1^{t_2}, t_1^{t_3}, \dots, t_1^{t_n}),$$

where $t = (t_1, \dots, t_n)$. We call ψ a *sliver function for* a set $A \subset \mathbb{R}^n$ if $\operatorname{im}(\psi) \subset A$, and in this case we also call $\operatorname{im}(\psi)$ a *sliver in* A.

Remark 4.2

A sliver function $\psi : (0, \epsilon) \times (p, q) \to A$ is an analytic isomorphism onto its image, since it is clearly an injective analytic map and its Jacobian matrix is lower triangular with diagonal entries which are nonzero on $(0, \epsilon) \times (p, q)$. In particular, the image of ψ is an open subset of A.

If $\psi : (0, \epsilon) \times (p, q) \to \mathbb{R}^n$ is a sliver function and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$, then

$$\psi(t)^\alpha = t_1^{\alpha_1 + \sum_{j=2}^n \alpha_j t_j},$$

so the asymptotic behavior of $\psi(t)^\alpha$ as $t_1 \to 0$ is determined by the values of the affine function $t_{>1} \mapsto \alpha_1 + \sum_{j=2}^n \alpha_j t_j$ on (p, q). By shrinking (p, q) we may gain more control over the asymptotics of $\psi(t)^\alpha$ as $t_1 \to 0$. The following lemma is therefore useful when studying the relative asymptotics of finite sets of rational monomial functions along slivers and, through extension via the preparation theorems, of prepared subanalytic and constructible functions along slivers.

LEMMA 4.3

Let $f_1, \dots, f_k$ be distinct affine functions on an open set $U \subset \mathbb{R}^n$. Then there exist $c > 0$, $i \in \{1, \dots, k\}$, and an open set V in U such that $f_i(x) + c < f_j(x)$ on V for all $j \neq i$.

Proof

The set $C = \{x \in U : f_i(x) = f_j(x) \text{ for distinct } i, j \in \{1, \dots, k\}\}$ is closed and nowhere dense in U, so $U \setminus C$ is open and dense in U. Choose $a \in U \setminus C$, and note that the values of $f_1(a), \ldots, f_k(a)$ are distinct. Let V be any sufficiently small neighborhood of a in $U \setminus C$. □

LEMMA 4.4
Fix a sliver function $\psi : (0, \epsilon) \times (p, q) \to \mathbb{R}^n$*, and fix* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Q}^n \setminus \{0\}$ *such that* $\psi(t)^\beta$ *is bounded on* $(0, \epsilon) \times (p, q)$*. Then, up to shrinking* (p, q)*, we can ensure that* $\psi(t)^\beta \to 0$ *as* $t_1 \to 0$ *uniformly on* (p, q)*.*

Proof
The function $\psi(t)^\beta = t_1^{\beta_1 + \sum_{i=2}^n \beta_i t_i}$ is bounded on $(0, \epsilon) \times (p, q)$; thus $\beta_1 + \sum_{i=2}^n \beta_i t_i \geq 0$ on (p, q). Since $\beta \neq 0$, up to shrinking the intervals (p_i, q_i), we can ensure that $\beta_1 + \sum_{i=2}^n \beta_i t_i \geq c$ on (p, q) for some $c > 0$. Then $0 < \psi(t)^\beta \leq t_1^c$, so $\psi(t)^\beta$ tends uniformly to zero as $t_1 \to 0$. □

PROPOSITION 4.5
Let $A \subset (0, 1)^n$ *be an open, subanalytic cell which is prepared with center zero and is such that* $x_1, \ldots, x_n$ *are all asymptotically undetermined on* A*. Then there exists a sliver function for* A*.*

Proof
We proceed by induction on n. If $n = 1$, the statement is easy since A is then of the form $(0, b)$ with $b \leq 1$. Suppose next that $n > 1$ and that we have chosen a sliver function $\psi' : (0, \epsilon) \times (p', q') \to \Pi_{n-1}(A)$ for $\Pi_{n-1}(A)$, where $(p', q') = (p_2, q_2) \times \cdots \times (p_{n-1}, q_{n-1})$. Write

$$A = \big\{x : x_{<n} \in \Pi_{n-1}(A),\ a(x_{<n}) < x_n < b(x_{<n})\big\}.$$

We suppose that $a > 0$ on $\Pi_{n-1}(A)$, for the case $a = 0$ is proven by simply omitting a in the following argument. Hence we can write

$$a(x_{<n}) = u(x_{<n}) x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$$

and

$$b(x_{<n}) = v(x_{<n}) x_1^{\beta_1} \cdots x_{n-1}^{\beta_{n-1}},$$

for unique $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{Q}^{n-1}$. Consider the inequalities

$$a\big(\psi'(t_{<n})\big) = u\big(\psi'(t_{<n})\big) t_1^{\alpha_1 + \sum_{j=2}^{n-1} \alpha_j t_j} < t_1^{t_n} < v\big(\psi'(t_{<n})\big) t_1^{\beta_1 + \sum_{j=2}^{n-1} \beta_j t_j} = b\big(\psi_0(t_{<n})\big) \tag{4.1}$$

on $(0, \epsilon) \times (p', q')$. Note that $0 < a < b$ on $\Pi_{n-1}(A)$, so for each $t_{<n}$ there exists t_n satisfying the above inequalities. Taking the logarithm with base t_1 of these inequalities, one gets

$$\log_{t_1}\bigl(u(\psi_0(t_{<n}))\bigr) + \alpha_1 + \sum_{j=2}^{n-1} \alpha_j t_j > t_n > \log_{t_1}\bigl(v(\psi_0(t_{<n}))\bigr) + \beta_1 + \sum_{j=2}^{n-1} \beta_j t_j. \tag{4.2}$$

Note that $\log_{t_1}\bigl(u(\psi_0(t_{<n}))\bigr)$ and $\log_{t_1}\bigl(v(\psi_0(t_{<n}))\bigr)$ go to zero uniformly when $t_1 \to 0$, so $\alpha_1 + \sum_{j=2}^{n-1} \alpha_j t_j \geq \beta_1 + \sum_{j=2}^{n-1} \beta_j t_j$ on (p', q'). Since x_n is asymptotically undetermined, one has $\alpha \neq \beta$, so up to shrinking (p', q'), we may fix positive constants $p_n < q_n$ and δ such that

$$\alpha_1 + \sum_{j=2}^{n-1} \alpha_j t_j \geq q_n + \delta \qquad \text{and} \qquad p_n - \delta \geq \beta_1 + \sum_{j=2}^{n-1} \beta_j t_j$$

on (p', q'). Put $(p, q) = (p_2, q_2) \times \cdots \times (p_n, q_n)$. Up to shrinking ϵ, we may ensure that $|\log_{t_1}(u(\psi_0(t_{<n})))|$ and $|\log_{t_1}(v(\psi_0(t_{<n})))|$ are bounded above by δ on $(0, \epsilon) \times (p, q)$. Thus (4.2) holds on $(0, \epsilon) \times (p, q)$, so (4.1) does as well. □

5. Coordinate transforms

Let $A \subset (0, 1)^n$ be an open subanalytic cell which is prepared with center zero. In order to use the existence of slivers, we will perform a coordinate change $H : B \to A$ such that the transformed cell B satisfies the conditions of Proposition 4.5. Let

$$J = \bigl\{ j \in \{1, \ldots, n\} : x_j \text{ is asymptotically undetermined on } A \bigr\},$$

and write $\{1, \ldots, n\} \setminus J = \{d_1 < \cdots < d_k\}$ for some $k \geq 0$. Define $H := H_{d_1} \circ \cdots \circ H_{d_k} : B \to A$, where the H_{d_i} are defined as follows.

Writing $d = d_1$, one has $d > 1$, as follows from Definitions 3.4. By Definitions 3.4 and Corollary 3.6.1, we can write

$$\Pi_d(A) = \bigl\{ x_{\leq d} : x_{<d} \in \Pi_{d-1}(A),\ x_{<d}^{\alpha} u(x_{<d}) < x_d < x_{<d}^{\alpha} v(x_{<d}) \bigr\}$$

for some $\alpha \in \mathbb{Q}^{d-1}$ with support in $\{1, \ldots, d-1\} \cap J$ and strong subanalytic units u and v such that $x_{<d}^{\alpha} v(x_{<d}) - x_{<d}^{\alpha} u(x_{<d})$ is J-prepared on $\Pi_{d-1}(A)$. It follows that $v - u$ is also J-prepared on $\Pi_{d-1}(A)$. Let $R_d > 0$ be a constant such that $v - u \leq R_d$ on $\Pi_{d-1}(A)$. Define $H_d : \Pi_{d-1}(A) \times (0, 1)^{n-d+1} \to \mathbb{R}^n$ by

$$H_d(y) = \bigl(y_1, \ldots, y_{d-1},\ \ y_{<d}^{\alpha}(R_d y_d + u(y_{<d})),\ \ y_{d+1}, \ldots, y_n \bigr),$$

and define

$$D := H_d^{-1}(A).$$

By restricting to D, this defines a map $H_d : D \to A$, where we let y range over D and write $x = H_d(y)$. Clearly D is an open cell contained in $(0, 1)^n$,

$$\Pi_d(D) = \left\{ y_{\le d} : y_{<d} \in \Pi_{d-1}(D), 0 < y_d < \frac{v(y_{<d}) - u(y_{<d})}{R_d} \right\},$$

and thus y_d is asymptotically undetermined on D. Note also that $R_d y_d + u(y_{<d})$ is a strong subanalytic unit on $\Pi_d(D)$ with center zero. Now construct H_{d_2} as H_d, but by starting with D instead of with A, and so on, up to the map H_{d_k}.

LEMMA 5.1

With notation from the above construction and writing $x = H(y)$ for $y = (y_1, \ldots, y_n)$ in B, the set B is an open, subanalytic cell in $(0, 1)^n$ which is J-prepared with center zero, and for all i,

(i) *y_i is asymptotically undetermined on B;*

(ii) *y_i is constrained on B if and only if x_i is constrained on A and $i \in J$.*

Moreover, for any subanalytic function $g : A \to \mathbb{R}$ which is prepared on A with center zero, $g \circ H$ is J-prepared on B with center zero, and for every strong subanalytic unit w on A with center zero, $w \circ H$ is a strong subanalytic unit on B with center zero.

Proof

The last sentence follows an argument similar to case (2) of the proof of Theorem 3.9, since there a similar transformation is performed. The other statements follow from the construction. □

6. Integrability and preparation of constructible functions

In this section we prove our key technical result, Proposition 6.1, which gives a strong connection between integrability conditions and the preparation result for constructible functions given by Theorem 3.11. Slivers and the transformation H from Section 5 are used to prove Proposition 6.1.

PROPOSITION 6.1

Consider the situation and notation of Theorem 3.11. Fix $f \in \mathcal{F}$ and a cell A from the decomposition of X, and write $f(x) = \sum_{i \in I} T_i(x)$, as in (3.5). Then the following statements are equivalent:

(i) *For all $x_{<n} \in \Pi_{n-1}(A)$, the function $f(x_{<n}, \cdot)$ is integrable over the fiber $A_{x_{<n}}$.*

(ii) *The set of the $x_{<n}$ in $\Pi_{n-1}(A)$ for which the function $f(x_{<n}, \cdot)$ is integrable over $A_{x_{<n}}$ is dense in $\Pi_{n-1}(A)$.*

(iii) *For all* $x_{<n} \in \Pi_{n-1}(A)$ *and all* $i \in I$*, the function* $T_i(x_{<n}, \cdot)$ *is integrable over* $A_{x_{<n}}$.

Proof

Statement (i) clearly implies (ii), and (iii) clearly implies (i); thus the only nontrivial implication is that (ii) implies (iii).

So assume (ii). We may assume that $\widetilde{x}_n$ is unconstrained on A, since otherwise there is nothing to prove. Let $F = G_A^{[\theta]} \circ H : B \to A$ be the map constructed by composing the map $G_A^{[\theta]} : A_{\mathrm{bd}} \to A$ from Definitions 3.4 with the map $H : B \to A_{\mathrm{bd}}$ from Lemma 5.1, and write $y = (y_1, \ldots, y_d)$ for the coordinates on B. For each $i \in I$ and $y \in B$, let

$$S_i(y) = T_i\big(F(y)\big) \cdot \frac{\partial F_n}{\partial y_d}(y). \tag{6.1}$$

Thus $f \circ F(y)\frac{\partial F_n}{\partial y_d}(y) = \sum_{i \in I} S_i(y)$ on B. It follows from the construction of F and from the situation given by Theorem 3.11 that for each $i \in I$,

$$S_i(y) = u_i(y) \prod_{k \in K} y_k^{r_k(i)} (\log y_k)^{\ell_k(i)} \tag{6.2}$$

for $K = \lambda^{-1}(J)$, a strong subanalytic unit u_i with center zero, rational numbers $r_k(i)$, and nonnegative integers $\ell_k(i)$ such that the tuples $(\ell_k(i))_{k \in K}$ are distinct for different i in I. It is enough to show that $\overline{r} > -1$, where

$$\overline{r} = \min\big\{r_d(i) : i \in I\big\}.$$

Define

$$I_1 = \big\{i \in I : r_d(i) = \overline{r}\big\},$$
$$\overline{\ell} = \max\big\{\ell_d(i) : i \in I_1\big\},$$
$$I_2 = \big\{i \in I_1 : \ell_d(i) = \overline{\ell}\big\}.$$

By Proposition 4.5, we may fix a sliver function $\psi' : (0, \epsilon) \times (p, q) \to B'$ for $B' = \Pi_{d-1}(B)$, where $(p, q) = (p_2, q_2) \times \cdots \times (p_{d-1}, q_{d-1})$. Write $t = (t_1, \ldots, t_{d-1})$ for a tuple of variables ranging over $(0, \epsilon) \times (p, q)$. By Lemma 4.3, by shrinking (p, q), and up to reordering the terms S_i, we may assume that there is a positive constant c such that for all $i \in I_2$ with $(r_k(i))_{k \in K} \neq (r_k(1))_{k \in K}$ one has

$$c + r_1(1) + \sum_{k \in K \setminus \{1\}} r_k(1) t_k \leq r_1(i) + \sum_{k \in K \setminus \{1\}} r_k(i) t_k \tag{6.3}$$

on (p, q). Define

$$I_3 = \{i \in I_2 : (r_k(i))_{k\in K} = (r_k(1))_{k\in K}\}.$$

For each $i \in I$, define

$$\ell'(i) = \sum_{k\in K\setminus\{d\}} \ell_k(i),$$

and define

$$\overline{\ell}' = \max\{\ell'(i) : i \in I_3\},$$
$$I_4 = \{i \in I_3 : \ell'(i) = \overline{\ell}'\}.$$

Define a map $\psi_0 : (0, \epsilon) \times (p, q) \times \mathbb{R} \to \mathbb{R}^d$ by

$$\psi_0(t, y_d) = (\psi'(t), y_d),$$

and put $B_0 := \psi_0^{-1}(B)$ and $\psi : B_0 \to B : (t, y_d) \mapsto \psi_0(t, y_d)$.

Consider the function $W : B \to \mathbb{R}$ defined by

$$W(y) = \Big(\prod_{k\in K} y_k^{r_k(1)}\Big)(\log y_1)^{\overline{\ell}'}(\log y_d)^{\overline{\ell}}.$$

For any function $h : B \to \mathbb{R}$ and for $(t, y_d) \in B_0$, write $h(t, y_d)$ for $h(\psi(t, y_d))$. One then has for $(t, y_d) \in B_0$,

$$W(t, y_d) = t_1^{r_1(1)+\sum_{k\in K\setminus\{1,d\}} r_k(1)t_k}(\log t_1)^{\overline{\ell}'}\, y_d^{\overline{r}}(\log y_d)^{\overline{\ell}}, \tag{6.4}$$

and for $i \in I$, one has

$$S_i(t, y_d) = t_1^{r_1(i)+\sum_{k\in K\setminus\{1,d\}} r_k(i)t_k}(\log t_1)^{\ell'(i)} y_d^{r_d(i)}(\log y_d)^{\ell_d(i)} \cdot \Big(\prod_{k\in\{2,\dots,d-1\}\cap J} t_k^{\ell_k(i)}\Big) u_i(t, y_d)$$

Thus, for $i \in I$ and $(t, y_d) \in B_0$,

$$\frac{S_i(t, y_d)}{W(t, y_d)} = t_1^{r_1(i)-r_1(1)+\sum_{k\in K\setminus\{1,d\}}(r_k(i)-r_k(1))t_k}(\log t_1)^{\ell'(i)-\overline{\ell}'} \cdot y_d^{r_d(i)-\overline{r}}(\log y_d)^{\ell_d(i)-\overline{\ell}}\Big(\prod_{k\in\{2,\dots,d-1\}\cap J} t_k^{\ell_k(i)}\Big) u_i(t, y_d). \tag{6.5}$$

It follows from the definitions of the sets $I_1, \dots, I_4$ and from (6.3) and (6.5) that for all $i \in I \setminus I_4$,

$$\lim_{t_1 \to 0} \Big(\lim_{y_d \to 0} \frac{S_i(t, y_d)}{W(t, y_d)} \Big) = 0, \tag{6.6}$$

where the limit $t_1 \to 0$ is uniform on (p, q) and that for all $i \in I_4$,

$$\frac{S_i(t, y_d)}{W(t, y_d)} = \Big(\prod_{k \in \{2, \dots, d-1\} \cap J} t_k^{\ell_k(i)} \Big) u_i(t, y_d). \tag{6.7}$$

Write $u_i(y) = U_i \circ \varphi_i(y)$, where φ_i is a bounded rational monomial map on B and U_i is an analytic unit on the closure of the image of φ_i. Since φ_i is bounded and y_d is unconstrained on B and can approach zero, the powers of y_d that occur in each of the components of φ_i must all be nonnegative. So $\lim_{y_d \to 0} u_i(y) = U_i \circ \varphi_i(y_{<d}, 0)$, which is a strong subanalytic unit on B', where we have extended φ_i naturally on the closure of B in $B' \times \mathbb{R}$. Lemma 4.4 implies that by shrinking (p, q), we can ensure that $\lim_{t_1 \to 0} u_i \circ \psi(t, 0) = U_i(0)$ uniformly on (p, q). In particular, zero is in the domain of U. In summary,

$$\lim_{t_1 \to 0} \Big(\lim_{y_d \to 0} u_i \circ \psi(t, y_d) \Big) = U_i(0), \tag{6.8}$$

where the limit $t_1 \to 0$ is uniform on (p, q) and where by the definition of strong subanalytic units, $U_i(0) \neq 0$. By construction, the tuples

$$\big(\ell_k(i)\big)_{k \in \{2, \dots, d-1\} \cap K}$$

are distinct for different i in I_4, so

$$\sum_{i \in I_4} U_i(0) \Big(\prod_{k \in \{2, \dots, d-1\} \cap K} t_k^{\ell_k(i)} \Big) \tag{6.9}$$

is a nonzero polynomial. Thus by shrinking (p, q), we can ensure that (6.9) is bounded away from zero on (p, q).

Equations (6.6) – (6.8) and the fact that (6.9) is bounded away from zero on (p, q) show that by shrinking ϵ we may ensure that for all $t \in (p, q)$, the limit

$$\lim_{y_d \to 0} \frac{\sum_{i \in I} S_i(t, y_d)}{W(t, y_d)} \tag{6.10}$$

exists and is nonzero. Since the sliver $\mathrm{im}(\psi)$ is open in B', statement (ii) of Proposition 6.1 implies that the function $y_d \mapsto \sum_{i \in I} S_i(t, y_d)$ is integrable for all t in a dense subset of $(0, \epsilon) \times (p, q)$, and thus by (6.10), the same is true for $y_d \mapsto W(t, y_d)$. Therefore $\overline{r} > -1$ by (6.4), which finishes the proof. □

7. Proofs of Theorem 1.4′ and Proposition 1.5

In this section we prove Theorem 1.4′ and Proposition 1.5.

Proof of Theorem 1.4′

Let $f : X \times Y \times \mathbb{R} \to \mathbb{R}$, the set C, and the variables x, y, z be as in the statement of the theorem, and assume that $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^n$. We first prove the theorem assuming $Y = \mathbb{R}^n$. Apply Theorem 3.11 to f. This gives a subanalytic cell decomposition $\mathcal{A}$ of $X \times \mathbb{R}^n \times \mathbb{R}$, and $\mathcal{A}$ induces a subanalytic cell decomposition of $X \times \mathbb{R}^n$, namely, $\mathcal{B} = \{\Pi_{k+n}(A) : A \in \mathcal{A}\}$. Consider a cell $A \in \mathcal{A}$ which is fat in y, let $B = \Pi_{k+n}(A)$, and fix $x \in \Pi_k(A)$. Since B_x is open in $\mathbb{R}^n$ and C_x is dense in $\mathbb{R}^n$, the set $(C \cap B)_x$ is dense in B_x. The function $z \mapsto f(x, y, z)$ is integrable on $A_{(x,y)}$ for all $y \in (C \cap B)_x$ and hence, by Proposition 6.1, for all $y \in B_x$. Therefore if we define

$$C' = \bigcup \{D \in \mathcal{B} : D \text{ is fat in } y\},$$

the set C' is a subanalytic, C'_x is dense in $\mathbb{R}^n$ for all $x \in X$, and the linearity of integration implies that for all $x \in X = \Pi_k(C')$ and all $y \in C'_x$, the map $z \mapsto f(x, y, z)$ is integrable on $\mathbb{R}$. This completes the proof when $Y = \mathbb{R}^n$.

To finish we need to show how to reduce to the case $Y = \mathbb{R}^n$. Let $\mathcal{S}$ be a stratification of Y into subanalytic cells, and let $\mathcal{H}$ be the highest-level strata of $\mathcal{S}$, namely, the set of all $S \in \mathcal{S}$ which are not contained in the boundary of any other member of $\mathcal{S}$. Thus $\bigcup \mathcal{H}$ is dense and open in Y, and for all $x \in X$ and all $S \in \mathcal{H}$, the set $C_x \cap S$ is dense in S. It therefore suffices to fix $S \in \mathcal{H}$, assume that C_x is a dense subset of S for all $x \in X$, and study the restriction of f to $X \times S \times \mathbb{R}$. By projecting into a lower-dimensional space, we may assume that S is open in $\mathbb{R}^n$. Now extend f by zero on $X \times (\mathbb{R}^n \setminus S) \times \mathbb{R}$. Fix $x \in X$. Because S is open in $\mathbb{R}^n$ and $f(x, y, z) = 0$ for all $y \notin S$, the function $z \mapsto f(x, y, z)$ is integrable on $\mathbb{R}$ for all y in a dense subset of $\mathbb{R}^n$ if and only if it is integrable on $\mathbb{R}$ for all y in a dense subset of S. So we are done by the case $Y = \mathbb{R}^n$. □

LEMMA 7.1

For any $f \in \mathcal{C}(X)$, where $X \subset \mathbb{R}^n$ is subanalytic, there exists a subanalytic function $h : X \to (0, +\infty)$ such that $|f(x)| \leq h(x)$ for all $x \in X$.

Proof

Let $f \in \mathcal{C}(X)$. Write $f(x) = \sum_i f_i(x) \prod_j \log f_{i,j}(x)$ for subanalytic functions $f_i : X \to \mathbb{R}$ and $f_{i,j} : X \to (0, +\infty)$, where the indices i and j run over finite index sets. The function $L_+ : (0, +\infty) \to (0, +\infty)$ defined by

$$L_+(t) = \begin{cases} \frac{1}{t} & \text{if } 0 < t \leq 1, \\ t & \text{if } t > 1, \end{cases}$$

satisfies $|\log t| < L_+(t)$ for all $t > 0$. Therefore, for all $x \in X$,

$$|f(x)| \leq \sum_i |f_i(x)| \prod_j L_+\big(f_{i,j}(x)\big) \leq h(x)$$

for the positively valued subanalytic function

$$h(x) = \max\Big\{1, \sum_i |f_i(x)| \prod_j L_+\big(f_{i,j}(x)\big)\Big\}. \qquad \square$$

Proof of Proposition 1.5
Let $f \in \mathcal{C}(X \times \mathbb{R})$ for a subanalytic set $X \subset \mathbb{R}^{n-1}$, and suppose that $\lim_{x_n \to +\infty} f(x_{<n}, x_n) = 0$ for all $x_{<n} \in X$. Our goal is to show that there exist a constant $r > 0$ and a subanalytic function $g : X \to (0, +\infty)$ such that

$$|f(x)| \leq x_n^{-r}$$

for all $x_{<n} \in X$ and all $x_n > g(x_{<n})$. Let $\mathcal{A}$ be the subanalytic cell decomposition of $X \times \mathbb{R}$ given by applying Theorem 3.11 to f. Fix $A' \in \{\Pi_{n-1}(A) : A \in \mathcal{A}\}$, and fix the unique cell $A \in \mathcal{A}$ of the form $A = \{x : x_{<n} \in A', x_n > a(x_{<n})\}$. The function $f\big|_A$ is of the form given by Theorem 3.11, and we shall henceforth use the notation in the statement of Theorem 3.11 without redefinition. Remark 3.12 implies that $\widetilde{x}_n = x_n$. As in the proof of Proposition 6.1, let $F = G_A^{[\theta]} \circ H : B \to A$, write $y = (y_1, \ldots, y_d)$ for a tuple of variables ranging over B, and write $y' = (y_1, \ldots, y_{d-1})$ and $B' = \Pi_{d-1}(B)$. Note that $F_n(y) = 1/y_d$. Therefore $\lim_{y_d \to 0^+} f \circ F(y', y_d) = 0$ for all $y' \in B'$, and it suffices to find an $r > 0$ and a subanalytic function $g : B' \to (0, +\infty)$ such that $|f \circ F(y)| \leq y_d^r$ for all $y \in B$ with $y_d < g(y')$.

For each $i \in I$, let $S_i(y) = T_i \circ F(y)$, so that $f \circ F(y) = \sum_{i \in I} S_i(y)$ on B. Note that the function $S_i(y)$ differs from the notation in the proof of Proposition 6.1 because we do not multiply by $\frac{\partial F_n}{\partial y_d}$, but it is of the same form as given by (6.2). We therefore use the notation of the proof of Proposition 6.1 without redefinition, and we proceed as in the proof of the proposition until we get to (6.10). At this point, note that since $\lim_{y_d \to 0^+} f \circ F(y', y_d) = 0$ for all $y' \in B'$, and since the limit (6.10) exists and is nonzero, $\lim_{y_d \to 0^+} W(t, y_d) = 0$ for all $t \in (0, \epsilon) \times (p, q)$. Therefore $\overline{r} > 0$ by (6.4).

Fix $\epsilon > 0$ such that $\overline{r} - \epsilon\,\overline{\ell} > 0$, and fix r such that $0 < r < \overline{r} - \epsilon\,\overline{\ell}$. The number r may be chosen so that $\overline{r} - \epsilon\,\overline{\ell} - r$ is rational. Note that $|\log y_d| < y_d^{-\epsilon}$ for all sufficiently small $y_d > 0$. Therefore (6.2) shows that there exists a $\delta > 0$ such that for all $y \in B$ with $y_d < \delta$,

$$|f \circ F(y)| \leq \sum_{i \in I} |u_i(y)| \Big(\prod_{k \in K \setminus \{d\}} y_k^{r_k(i)} |\log y_k|^{\ell_k(i)} \Big) y_d^{\overline{r} - \epsilon \overline{\ell}}.$$

Since the functions $|u_i(y)|$ are units, it follows from Lemma 7.1 that there is a subanalytic function $h : B' \to (0, +\infty)$ such that

$$|f \circ F(y)| \leq h(y') y_d^{\overline{r} - \epsilon \overline{\ell}}$$

on $B \cap (\mathbb{R}^{d-1} \times (0, \delta))$. Define a subanalytic function $g : B' \to (0, +\infty)$ by

$$g(y') = \min\Big\{\delta,\ h(y')^{-1/(\overline{r} - \epsilon \overline{\ell} - r)}\Big\}.$$

Then $|f \circ F(y)| \leq y_d^r$ for all $y \in B$ with $y_d < g(y')$. □

8. Proof of Theorem 1.3

The presented proof of Theorem 1.3 uses an adapted version of the integration procedure of [19]. We will describe this procedure in detail since the setup and generality is different from the one of [19]. We proceed by induction on m, where the integration is performed over $\mathbb{R}^m$. The base case of $m = 1$ is nontrivial and relies on Theorem 3.11 and Proposition 6.1. In contrast, the induction step will follow rather immediately from the base case by Fubini's theorem and Theorem 1.4′.

Proof of Theorem 1.3 for $m = 1$

Let $f \in \mathcal{C}(X \times \mathbb{R})$ for a subanalytic set $X \subset \mathbb{R}^k$. Our goal is to show that $I_X(f)$ is in $\mathcal{C}(X)$. We may assume that $\mathbb{R} \to \mathbb{R} : y \mapsto f(x, y)$ is integrable on $\mathbb{R}$ for all $x \in X$. Apply Theorem 3.11 to get a subanalytic cell decomposition of $X \times \mathbb{R}$ such that on each cell A in the decomposition, $f(x, y)$ can be expressed as a finite sum of terms of the form (3.6) which, by Proposition 6.1, are integrable over the fiber A_x for all $x \in \Pi_k(A)$. Because of the linearity of integration, we focus on a single cell A in the decomposition of $X \times \mathbb{R}$ which is fat in y with base B and on a single term $S(x, y) = T_i(x, y)$ on A of the form (3.6). It suffices to show that $B \to \mathbb{R} : x \mapsto \int_{A_x} S(x, y)\, dy$ is in $\mathcal{C}(B)$.

CLAIM

By partitioning A into smaller subanalytic cylinders (not cells) and performing subanalytic coordinate transformations, adjusting S by a Jacobian accordingly, and by the linearity of the integral, we can reduce to the case where A is a cylinder of the form

$$A = \left\{(x, y) \in B \times \mathbb{R} : a(x) < y < b(x)\right\} \tag{8.1}$$

for subanalytic functions $a, b : B \to \mathbb{R}$ such that either $0 = a(x) < b(x) \le \epsilon$ on B or $0 < a(x) < b(x) \le \epsilon$ on B for some $\epsilon > 0$, and S is of the form

$$S(x, y) = \Big(\sum_{i=1}^{l} S_i(x) y^{-i} + F(S_0(x), y)\Big)(\log y)^s \tag{8.2}$$

on A for some $l, s \in \mathbb{N}$, subanalytic functions $S_1, \ldots, S_l : B \to \mathbb{R}$ and $S_0 : B \to \mathbb{R}^N$, where $N \in \mathbb{N}$ and S_0 is bounded, and an analytic function $F(X, Y)$ on a neighborhood V of $\mathrm{cl}(S_0(B)) \times [-\epsilon, \epsilon]$ that is represented by a power series in Y, $\sum_{i=0}^{\infty} F_i(X) Y^i$, which converges on V, where $X = (X_1, \ldots, X_N)$ and where $\mathrm{cl}(\cdot)$ denotes the topological closure.

Note

Because $y \mapsto S(x, y)$ is integrable on A_x for all $x \in B$, necessarily $l = 0$ when $a(x) = 0$ on B.

We first use the claim to prove the theorem. Note that

$$\begin{aligned}\int_{a(x)}^{b(x)} S(x, y)\, dy = \sum_{i=2}^{l} S_i(x) \int_{a(x)}^{b(x)} y^{-i} (\log y)^s\, dy \\ + S_1(x) \int_{a(x)}^{b(x)} \frac{(\log y)^s}{y}\, dy \\ + \int_{a(x)}^{b(x)} F\big(S_0(x), y\big)(\log y)^s\, dy\end{aligned}$$

on B. Clearly,

$$\int_{a(x)}^{b(x)} \frac{(\log y)^s}{y}\, dy = \frac{1}{s+1} (\log y)^{s+1} \Big|_{a(x)}^{b(x)}.$$

To integrate the other terms, define analytic functions

$$G(X, Y) = \sum_{i=0}^{\infty} \frac{F_i(X)}{i+1} Y^{i+1},$$

$$H(X, Y) = \sum_{i=0}^{\infty} \frac{F_i(X)}{i+1} Y^{i},$$

on V, and note that $\frac{\partial G}{\partial Y}(X, Y) = F(X, Y)$ and $G(X, Y) = YH(X, Y)$. The theorem now follows by inducting on s, using the fact that when $s = 0$,

$$\sum_{i=2}^{l} S_i(x) \int_{a(x)}^{b(x)} y^{-i}\, dy = \sum_{i=2}^{l} \frac{S_i(x)}{-i+1} y^{-i+1} \Big|_{a(x)}^{b(x)},$$

$$\int_{a(x)}^{b(x)} F\big(S_0(x), y\big)\, dy = G\big(S_0(x), y\big)\Big|_{a(x)}^{b(x)},$$

and when $s > 0$, integration by parts gives

$$\begin{aligned}\int_{a(x)}^{b(x)} y^{-i}(\log y)^s\, dy = {}& \frac{1}{-i+1} y^{-i+1}(\log y)^s \Big|_{a(x)}^{b(x)} \\ & - \frac{s}{-i+1} \int_{a(x)}^{b(x)} y^{-i}(\log y)^{s-1}\, dy,\end{aligned}$$

$$\begin{aligned}\int_{a(x)}^{b(x)} F\big(S_0(x), y\big)(\log y)^s\, dy = {}& G\big(S_0(x), y\big)(\log y)^s \Big|_{a(x)}^{b(x)} \\ & - s \int_{a(x)}^{b(x)} H\big(S_0(x), y\big)(\log y)^{s-1}\, dy.\end{aligned}$$

We now prove the claim. We may suppose that A is an $(n+1)$-dimensional λ-cell with center θ. By applying the map $G_A^{[\theta]}$ of Definitions 3.4 and adjusting S by a Jacobian, we may assume that A is an open subanalytic cell in $(0, 1)^{n+1}$ with center zero which is of the form

$$A = \big\{(x, y) \in B \times \mathbb{R} : a(x) < y < b(x)\big\} \tag{8.3}$$

for subanalytic terms a and b on B, where $a(x) < b(x)$ on B and either $a(x) = 0$ on B or $a(x) > 0$ on B, and that

$$S(x, y) = g(x) y^r (\log y)^s u(x, y) \tag{8.4}$$

on A, where $g \in \mathcal{C}(B)$, $r \in \mathbb{Q}$, $s \in \mathbb{N}$, and $u(x, y)$ is a strong subanalytic unit on A with center zero. We are done if $g(x)$ is identically equal to zero on B, so assume otherwise. By the linearity of the integral, we may pull $g(x)$ out of the integral and thus suppose that $g(x) = 1$. By applying Lemma 2.6, which partitions B into smaller subanalytic sets and thereby partitions A into smaller subanalytic cylinders (not cells), we may assume that A is a subanalytic cylinder of the form (8.3), and that $S(x, y)$ is of the form (8.4) but with $u(x, y) = U \circ \varphi(x, y)$, where $\varphi : A \to \mathbb{R}^{N+2}$ is a bounded function of the form

$$\varphi(x, y) = \big(c_1(x), \ldots, c_N(x), c_{N+1}(x)y^{1/p}, c_{N+2}(x)y^{-1/p}\big)$$

for analytic subanalytic terms $c_1, \ldots, c_{N+2}$ on B and a positive integer p such that pr is an integer, and $U : \mathbb{R}^{N+2} \to \mathbb{R}$ is an analytic unit on the closure of the image of φ. Let $(X, Y, Z) = (X_1, \ldots, X_N, Y, Z)$ denote a tuple of variables ranging over the domain of U, and define $c : B \to \mathbb{R}^N$ by $c(x) = (c_1(x), \ldots, c_N(x))$. By further partitioning B, we may assume that $c_{N+1}(x) = 0$ on B or that $c_{N+1}(x) \neq 0$ on B. If $c_{N+1}(x) = 0$ on B, then apply the change of variables $y \mapsto y^p$, and if $c_{N+1}(x) \neq 0$ on B, then apply the change of variables $y \mapsto (y/c_{N+1}(x))^p$. In either case we may adjust the definition of S by a Jacobian to assume that r is an integer and that

$$\varphi(x, y) = \Big(c(x), y, \frac{d(x)}{y}\Big) \tag{8.5}$$

for a subanalytic term d on B. By further partitioning B and absorbing sign information of $d(x)$ into U, we may assume that $d(x) = 0$ on B or $d(x) > 0$ on B.

Let K be the closure of $\{(y, d(x)/y) : (x, y) \in A\}$. The set K is compact, and for each $(Y_0, Z_0) \in K$ and $\epsilon > 0$, the set $\{(x, y) \in A : |y - Y_0| < \epsilon, |d(x)/y - Z_0| < \epsilon\}$ can be partitioned into finitely many subanalytic cylinders. So it suffices to fix $(Y_0, Z_0) \in K$, and we may assume that for all $(x, y) \in A$,

$$|y - Y_0| < \epsilon, \tag{8.6}$$

$$\Big|\frac{d(x)}{y} - Z_0\Big| < \epsilon, \tag{8.7}$$

where $\epsilon > 0$ can be chosen to be as small as we wish. Note that if, after performing a change of variable of the form $(x, y) \mapsto (x, h(x)(y + y_0))$ for some subanalytic function $h(x)$ and constant y_0, we can express $u(x, y)$ in the form $U(c(x), y)$ with $|y|$ small (for a new choice of $U(X, Y)$ and $c(x)$), then $S(x, y)$ will be of the form (8.2) in these new variables.

If $d(x) = 0$, then we are in this form by performing the change of variables $y \mapsto y + Y_0$. If $Y_0 \neq 0$, then $d(x)/y$ is analytic in y, and we are in this form by the same change of variables $y \mapsto y + Y_0$. So we may assume that $Y_0 = 0$ and $d(x) > 0$ on B.

Suppose that $Z_0 \neq 0$. Perform the change of variables $y \mapsto (d(x)(y+1))/Z_0$. Then $d(x)/y$ in the old variables becomes $Z_0/(1+y)$ in the new variables. Thus (8.7) becomes $|(Z_0/(y+1)) - Z_0| < \epsilon$, which means that $|y|$ is small if ϵ is small. So $Z_0/(y+1)$ is analytic in y, and so also $U(c(x), (d(x)/Z_0)(y+1), (Z_0/(y+1)))$, and we are done.

Now suppose that $Z_0 = 0$. Define

$$F(X, Y, Z) = \begin{cases} Y^r U(X, Y, Z) & \text{if } r \geq 0, \\ Z^{-r} U(X, Y, Z) & \text{if } r < 0. \end{cases}$$

By pulling $d(x)^r$ out of the integral if $r < 0$, we may assume that

$$S(x, y) = (\log y)^s F\Big(c(x), y, \frac{d(x)}{y}\Big).$$

Note that F is analytic on the domain of U. We now use Lion and Rolin's "splitting lemma," which can be proven by writing F as a doubly indexed power series in Y and Z, say, $F(X, Y, Z) = \sum_{(i,j)\in\mathbb{N}^2} F_{i,j}(X) Y^i Z^j$, and then splitting this sum into the three sums $\sum_{i-j\leq -2} F_{i,j}(X) Y^i Z^j$, $\sum_{i-j=-1} F_{i,j}(X) Y^i Z^j$, and $\sum_{i-j\geq 0} F_{i,j}(X) Y^i Z^j$.

THE SPLITTING LEMMA

There exist $\epsilon \in (0, 1)$ *and functions* $F_{\leq -2}$, F_{-1}, *and* $F_{\geq 0}$ *which are analytic on* $\mathrm{cl}(c(B)) \times [-\epsilon, \epsilon]^2$, $\mathrm{cl}(c(B)) \times [-\epsilon, \epsilon]$, *and* $\mathrm{cl}(c(B)) \times [-\epsilon, \epsilon]^2$, *respectively, such that*

$$F\Big(c(x), y, \frac{d(x)}{y}\Big) = \Big(\frac{d(x)}{y}\Big)^2 F_{\leq -2}\Big(c(x), d(x), \frac{d(x)}{y}\Big) + \Big(\frac{d(x)}{y}\Big) F_{-1}\big(c(x), d(x)\big) + F_{\geq 0}\big(c(x), d(x), y\big)$$

on $\{(x, y) \in A : |y| < \epsilon$ *and* $|d(x)/y| < \epsilon\}$.

To finish, use the splitting lemma to express $S(x, y)$ as the sum of the function

$$\bigg(F_{-1}\big(c(x), d(x)\big)\Big(\frac{d(x)}{y}\Big) + F_{\geq 0}\big(c(x), d(x), y\big)\bigg)(\log y)^s,$$

which is in the form (8.2), and the function

$$(\log y)^s \Big(\frac{d(x)}{y}\Big)^2 F_{\leq -2}\Big(c(x), d(x), \frac{d(x)}{y}\Big),$$

which can be reduced to the form (8.2) using the change of variables $y \mapsto d(x)/y$ and adjusting by a Jacobian. $\square$

Proof of Theorem 1.3 for a general value of m
Let $m \geq 1$, and let $f \in \mathcal{C}(X \times \mathbb{R}^m)$ for a subanalytic set $X \subset \mathbb{R}^k$. Our goal is to show that $I_X(f)$ is in $\mathcal{C}(X)$. We may assume that $\mathbb{R}^m \to \mathbb{R} : y \mapsto f(x, y)$ is integrable on $\mathbb{R}^m$ for all $x \in X$. The case $m = 1$ has been proven, so let $m > 1$. By Fubini's theorem and Theorem 1.4′, there exists a subanalytic family $\{C_x\}_{x \in X}$ of dense subsets of $\mathbb{R}^{m-1}$ such that for all $x \in X$ and all $y_{<m} \in C_x$, the function $y_m \mapsto f(x, y_{<m}, y_m)$ is integrable over $\mathbb{R}$. Replacing $f(x, y)$ with the product of f and the characteristic function of $\{(x, y_{<m}) : y_{<m} \in C_x\}$ does not affect the function $I_X(f)$. We may therefore assume that $y_m \mapsto f(x, y)$ is integrable on $\mathbb{R}$ for all $(x, y_{<m}) \in X \times \mathbb{R}^{m-1}$. The base case of our induction shows that $g = I_{X \times \mathbb{R}^{m-1}}(f)$ is in $\mathcal{C}(X \times \mathbb{R}^{m-1})$. By Fubini's theorem, $y_{<m} \mapsto g(x, y_{<m})$ is integrable over $\mathbb{R}^{m-1}$ for all $x \in X$ and, moreover, $I_X(f) = I_X(g)$. The induction hypothesis then shows that $I_X(g)$ is in $\mathcal{C}(X)$, which completes the proof of Theorem 1.3. $\square$

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Cluckers
Laboratoire Painlevé, CNRS-UMR 8524, Université Lille 1, 59655 Villeneuve d'Ascq CEDEX, France
Departement Wiskunde, Katholieke Universiteit Leuven, B-3001 Leuven, Belgium; raf.cluckers@wis.kuleuven.be

Miller
Department of Mathematics, Computer Science, and Economics, Emporia State University, Emporia, Kansas 66801, USA; dmille10@emporia.edu