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Igusa's Conjecture on Exponential Sums Modulo p and p^2 and the Motivic Oscillation Index

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We prove the modulo p and modulo p^2 cases of Igusa's conjecture on exponential sums. This conjecture predicts specific uniform bounds in the homogeneous polynomial case of exponential sums modulo p^m when p and m vary. We introduce the motivic oscillation index of a polynomial f and prove the stronger, analogue bounds for $m = 1, 2$ using this index instead of the original bounds. The modulo p^2 case of our bounds holds for all polynomials; the modulo p case holds for homogeneous polynomials and under extra conditions also for nonhomogeneous polynomials. We obtain natural lower bounds for the motivic oscillation index by using results of Segers. We also show that, for p big enough, Igusa's local zeta function has a nontrivial pole when there are $\mathbf{F}_p$-rational singular points on $f = 0$. We introduce a new invariant of f, the flaw of f.

1 Introduction

Let f be a polynomial over $\mathbb{Q}$ in n variables. Consider the "global" exponential sums

$$E_f(N) := \frac{1}{N^n} \sum_{x \in \{0,\dots,N-1\}^n} \exp\left(2\pi i \frac{f(x)}{N}\right),$$

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where N varies over the positive integers. Up to prime factorization of N, to study the dependence of $E_f(N)$ on N, it suffices to study $E_f(p^m)$ in terms of integers $m>0$ and primes p, which requires a mixture of finite field and p-adic techniques.

Igusa's conjecture predicts sharp bounds for $|E_f(p^m)|$ in terms of integers $m>0$ and primes p when f is homogeneous. Analogue conjectures for analogue sums over finite field extensions of $\mathbf{Q}$ also make sense. Fixing $m=1$ or $m=2$, the conjectured uniformity in p gives a link between finite field and p-adic exponential sums.

In any case, to find sharp bounds for $|E_f(p^m)|$ for fixed p uniformly in m is often much more easy than bounding uniformly in p and m. Using a resolution of singularities π as explained later, Igusa defines a rational number $\alpha=\alpha(\pi)\leq 0$, such that

$$|E_f(p^m)| < c_p m^{n-1} p^{m\alpha} \tag{1.0.1}$$

for each m, each prime p, and some constants c_p depending on p. When f is homogeneous, he conjectures in [14] that c_p can be taken independently of p.

Let us define the number $\alpha(\pi)$ that is used in (1.0.1) for any polynomial f over $\mathbf{Q}$, following Igusa [14], or [13], [10]. If f is a constant, put $\alpha(\pi):=0$. Otherwise, let

$$\pi_f : Y \to \mathbf{A}^n_{\mathbf{C}}$$

be an embedded resolution of singularities with normal crossings of $f=0$. Let (N_i,ν_i), $i\in I$, be the numerical data of π_f, that is, for each irreducible component E_i of $f\circ\pi_f=0$, $i\in I$, let N_i be the multiplicity of E_i in $\operatorname{div}(f\circ\pi_f)$, and ν_i-1 the multiplicity of E_i in the divisor associated to $\pi_f^*(dx)$ with $dx=dx_1\wedge\ldots\wedge dx_n$. The *essential numerical data* of π_f are the pairs (N_i,ν_i) for $i\in J$ with $J=I\setminus I'$ and where I' is the set of indices i in I, such that $(N_i,\nu_i)=(1,1)$ and such that E_i does not intersect another E_j with $(N_j,\nu_j)=(1,1)$. Define $\alpha(\pi_f)$ as

$$\alpha(\pi_f) = -\min_{i\in J}\frac{\nu_i}{N_i}$$

when J is nonempty and define $\alpha(\pi_f)$ as $-2n$ otherwise. Take a similar resolution π_{f-c} of $f-c=0$ for each critical value $c\in\mathbf{C}$ of $f:\mathbf{C}^n\to\mathbf{C}$ and put $\pi:=(\pi_{f-c})_c$, where c runs over $\{0\}$ and the critical values of f. Define

$$\alpha(\pi) := \max_c(\alpha(\pi_{f-c})),$$

where c runs over $\{0\}$ and the critical values of f. Note that the number $\alpha(\pi)$ often depends on π, [17]. Namely, $\alpha(\pi)$ is independent of the choice of π when $\alpha(\pi)\geq -1$ by properties of the log canonical threshold of $f-c$ for all c, but on the other hand, when $\alpha(\pi) < -1$, then $\alpha(\pi)$ always depends on the choice of π, see Remark 1.2.

Now we are ready to state (a slightly generalized form of) what Igusa conjectured in the introduction of his book of 1978.

Conjecture 1.1. ([14]) Suppose that f is a homogeneous polynomial over $\mathbb{Q}$ in n variables. Let π and $\alpha(\pi)$ be as above.[1] Then there exists a constant c, such that for each prime p and each positive integer m, one has

$$|E_f(p^m)| < cm^{n-1}p^{m\alpha(\pi)}. \qquad \square$$

This conjecture might hold for a much wider class of polynomials than homogeneous ones, for example, for quasi-homogeneous polynomials. It is an open question how to generalize the conjecture optimally to the case of general f, cf. Example 7.2 and Section 8.

In certain cases, Conjecture 1.1 is proved. Igusa proved the case that the projective hypersurface defined by f is smooth [14], by using Deligne's bounds in [7] for finite field exponential sums. The case that π is a toric resolution and f is nondegenerate with respect to its Newton polyhedron at the origin is proven under special conditions by Denef and Sperber in [12], and in full generality by the author in [2].

In this paper, we prove Igusa's Conjecture 1.1, when we fix $m=1$, resp. $m=2$, see Corollary 6.2. Actually, we prove a bit more:

(1) Instead of using $\alpha(\pi)$, we define and use an exponent α_f, the *motivic oscillation index of f*, which is intrinsically associated to f and at least as sharp, and sometimes sharper, than $\alpha(\pi)$, since $\alpha_f \le \alpha(\pi)$.

(2) For $m=2$, the conjecture holds for arbitrary f.

(3) when we restrict to big enough primes p, we prove the analogue bounds for all finite field extensions of the fields $\mathbb{Q}_p$ and all fields $\mathbf{F}_q(t)$ of characteristic p.

The motivic oscillation index α_f is defined and studied in Section 3; this notion is based on a suggestion made to us by Jan Denef. In particular, we show that (1.0.1) holds with $\alpha = \alpha_f$, instead of $\alpha = \alpha(\pi)$.

For arbitrary polynomials f in n variables over some number field, we establish in Theorem 5.1 the lower bound

$$\frac{-n+\delta_f}{2} \le \alpha_f,$$

[1]For homogeneous f, one has $\pi = (\pi_f)$, since 0 is the only possible critical value of f by Euler's identity.

where δ_f is the dimension of the locus of grad f, where we say that the empty scheme has dimension $-\infty$. It is proven by showing in Theorem 2.2, that for big enough p, Igusa's local zeta function has a nontrivial pole, if and only if there is a $\mathbf{F}_p$-rational point on grad $f = f = 0$ (Theorem 2.2) and by using the lower bounds of Segers [18] for the isolated singularity case in combination with a Cartesian product argument on integrals.

Having this lower bound for α_f, Conjecture 1.1 with $m = 1$ fixed follows from upper bounds given by Katz [15] for exponential sums over finite fields of large characteristic. The case $m = 2$ follows elementarily from Hensel's Lemma and bounds on the number of $\mathbf{F}_q$-rational points on grad $f = 0$.

The analogue results over the fields $\mathbf{F}_q(t)$ of big enough characteristic just follow from a comparison principle coming from the theory of motivic integration, named transfer principle, cf. [4], [6], or with some work from [10].

In Theorem 7.1, we generalize our results for homogeneous polynomials to arbitrary polynomials satisfying Condition (7.0.1), which relates the singularities at infinity in $\mathbf{P}^n$ with the singularities in affine space.

In Section 8, we slightly sharpen Igusa's conjecture by using α_f instead of $\alpha(\pi)$. In general, α_f might not be the only ingredient for sharp bounds, so we introduce a new invariant of a polynomial f, the flaw of f, for the study of uniform bounds for $|E_f(p^m)|$.

Remark 1.2. When $\alpha(\pi) < -1$, then $\alpha(\pi)$ depends on the choice of π, as is communicated to the author by Segers and Veys [17]. Namely, let E be an exceptional irreducible component (for π) with numerical data (N, ν), which intersects an irreducible component S of the strict transform. The numerical data of S is automatically of the form $(N_S, 1)$, since it is a component of the strict transform. Since $\alpha(\pi) < -1$, we find that $N_S = 1$ and that $-\nu/N < -1$. Performing one more blowing up with center, the intersection of E and S yields a new exceptional component with numerical data $(N+1, \nu+1)$, which still intersects an irreducible component of the strict transform. Repeating this process with finitely many steps yields an alternative tuple of resolutions (π') with $\alpha(\pi) < \alpha(\pi') < -1$, and moreover, with $\alpha(\pi')$ as close to -1 as one wants. □

Some notation. Let k be a number field and let f be a polynomial over k in n variables. Define $\alpha(\pi)$ for a tuple π of embedded resolutions of polynomials $f - c$, as in the introduction. Let $\mathcal{A}_k$ be the collection of all finite field extensions of p-adic completions of k. For $K \in \mathcal{A}_k$, let $\mathcal{O}_K$ be its valuation ring with maximal ideal $\mathcal{M}_K$ and residue field k_K

with q_K elements, and let ψ_K be any additive character[2] from K to $\mathbf{C}^\times$ which is trivial on $\mathcal{O}_K$ and nontrivial on some element of order -1. Write $|\cdot|_K$ for the norm on K and $v_K : K^\times \to \mathbf{Z}$ for the valuation.

A Schwartz–Bruhat function $K^n \to \mathbf{C}$ is a locally constant function with compact support. For K in $\mathcal{A}_k$, φ a Schwartz–Bruhat function on K^n, and $y \in K$, consider the exponential integral

$$E_{f,K,\varphi}(y) := \int_{K^n} \varphi(x)\psi_K(yf(x))|dx|_K,$$

with $|dx|_K$ the normalized Haar measure on K^n, and write $E_{f,K}$ for $E_{f,K,\varphi_{\mathrm{triv}}}$, with φ_{triv} the characteristic function of $\mathcal{O}_K^n$.[3]

Let $\mathcal{B}_k$ be the collection of all fields K of the form $\mathbf{F}_q(t)$, such that K is an algebra over $\mathcal{O}_k$. Let $\mathcal{C}_k$ be the union of $\mathcal{A}_k$ and $\mathcal{B}_k$. Write $\mathcal{C}_{k,M}$ for the set of fields in $\mathcal{C}_k$ with residue field of characteristic $> M$ and write similarly for $\mathcal{A}_{k,M}$ and $\mathcal{B}_{k,M}$. For K in $\mathcal{B}_k$, use the notations $\mathcal{O}_K$, $\mathcal{M}_K$, ψ_K, etc., similarly as for K in $\mathcal{A}_k$. We can consider f as a polynomial over $\mathcal{O}_k[1/N]$ for some large enough integer N, and hence, for $K \in \mathcal{B}_{k,N}$, the polynomial f makes sense as a polynomial over K and $E_{f,K,\varphi}$ is defined as the corresponding integral for any Schwartz–Bruhat function φ on K^n; for $K \in \mathcal{B}_k$ of characteristic $\leq N$, $E_{f,K,\varphi}$ is defined to be zero.

The Vinogradov symbol $\ll$ has its usual meaning, namely that for complex valued functions f and g with g taking non-negative real values $f \ll g$ means $|f| \leq cg$ for some constant c.

By [13], Proposition 2.7, one has for any K in $\mathcal{A}_k \cup \mathcal{B}_{k,M}$ for big enough M, that

$$|E_{f,K,\varphi}(y)| \ll |v_K(y)|^{n-1}|y|^{\alpha(\pi)} \tag{1.2.1}$$

for any π as above, where both sides are considered as functions in $y \in K^\times$ and $|v_K(y)|$ denotes the real absolute value of $v_K(y)$.

Let C_f be the closed subscheme of $\mathbf{A}_k^n$ given by $0 = \mathrm{grad}\, f$ and let δ_f be the dimension of C_f, where we say that the empty scheme has dimension $-\infty$. Let S_f be the closed subscheme of $\mathbf{A}_k^n$ given by $f = 0 = \mathrm{grad}\, f$ and let ε_f be the dimension of S_f. If f is a polynomial over $\mathcal{O}_k[1/N]$ for some N, then for any ring R which is an algebra

[2]For example, one can take the standard nontrivial additive character

$$\psi_K : K \to \mathbf{C} : x \mapsto \exp(2\pi i(\mathrm{Trace}_{K,\mathbf{Q}_p}(x) \bmod \mathbf{Z}_p))$$

for K a finite field extension of $\mathbf{Q}_p$, and where $\mathrm{Trace}_{K,\mathbf{Q}_p}$ is the trace of K over $\mathbf{Q}_p$.

[3]For $k = \mathbf{Q}$, $K = \mathbf{Q}_p$, and ψ_K the standard nontrivial additive character, one thus has $E_{f,K}(p^{-m}) = E_f(p^m)$.

over $\mathcal{O}_k[1/N]$, denote the set of R-rational points on C_f by $C_f(R)$, and likewise on S_f. Use similar notation for polynomials over other fields than k.

2 Igusa's Local Zeta Function has Nontrivial Poles

Following Shavarevitch, Borevitch, Igusa, and others, for f a nonconstant polynomial in n variables over $\mathcal{O}_k$ and for K in $\mathcal{C}_k$, consider the Poincaré series

$$P_{f,K}(T) := \sum_{m\in\mathbf{N}} N_{f,K,m} T^m$$

with

$$N_{f,K,m} := \frac{1}{q_K^{mn}} \sharp\{x \in (\mathcal{O}_K \bmod \mathcal{M}_K^m)^n \mid f(x) \equiv 0 \bmod \mathcal{M}_K^m\}.$$

For $K \in \mathcal{A}_k$, it is known that $P_{f,K}(T)$ is rational in T, see [14] and [8]. Also, for $K \in \mathcal{B}_{k,M}$ for M big enough, $P_{f,K}(T)$ is rational in T, shown using resolutions with good reduction [9], [10], or by insights of motivic integration, cf. [11] or [4].[4]

Definition 2.1. For K in $\mathcal{C}_{\mathbf{Q}}$ and $Q_K(T)$ a rational function over $\mathbf{Q}$ in T, define the *nontrivial poles* of $Q_K(T)$ as the poles of $Q_K(T)$ without the pole $T = q_K$, if this is a pole of multiplicity exactly 1. □

Theorem 2.2. Let f be a nonconstant polynomial in n variables over k. There exists a number M, such that for all $K \in \mathcal{C}_{k,M}$, the rational function $P_{f,K}(T)$ has a nontrivial pole, if and only if $S_f(k_K)$ is nonempty, with $S_f(k_K)$ as in the section on notation. □

Proof. For M big enough and for K in $\mathcal{A}_{k,M}$, if $S_f(k_K)$ is nonempty, then $S_f(\mathcal{O}_K)$ is nonempty and thus, $N_{f,K,m} \neq 0$ for all m. Hence, $P_{f,K}(T)$ has at least one pole.

In [9], it is shown that for M big enough and $K \in \mathcal{A}_{k,M}$, the degree of $P_{f,K}(T)$ in T is ≤ 0. Hence, the same holds for $K \in \mathcal{B}_{k,M}$ with M big enough. So, in the case that $P_{f,K}(T)$ has only $T = q_K$ as pole, if this pole has multiplicity 1, and if K lies in $\mathcal{C}_{k,M}$ with M big enough, we can write

$$P_{f,K}(T) = (A_K + B_K T)/(1 - q_K^{-1} T),$$

[4]It is conjectured that $P_{f,K}(T)$ is rational for all $K \in \mathcal{B}_k$.

for some rational numbers A_K and B_K depending on K. By developing the right hand side into series, we get

$$N_{f,K,1} = q_K^{-1} A_K + B_K$$

and

$$N_{f,K,2} = q_K^{-2} A_K + q_K^{-1} B_K = q_K^{-1} N_{f,K,1}.$$

We will only use that $q_K^{-1} N_{f,K,1} = N_{f,K,2}$. Let $\bar{f}$ be the reduction of f modulo $\mathcal{M}_K$. Put

$$Y_{K,f} := \left\{ x \in k_K^n \,\middle|\, \bar{f}(x) = 0 \neq (\mathrm{grad}\, \bar{f})\,(x) \right\}.$$

Suppose now, that $S_f(k_K)$ is nonempty. Count points to get

$$N_{f,K,1} = \frac{1}{q_K^n} (\sharp Y_{K,f} + \sharp S_f(k_K))$$

and

$$N_{f,K,2} = \frac{1}{q_K^{n+1}} \sharp Y_{K,f} + \frac{1}{q_K^n} \sharp S_f(k_K),$$

by Taylor expansion of f around points with residue in $S_f(k_K)$ resp. in $Y_{K,f}$, and since the projection

$$\left\{ x \in \left(\mathcal{O}_k / \mathcal{M}_K^2 \mathcal{O}_k\right)^n \,\middle|\, f(x) \equiv 0 \bmod \mathcal{M}_K^2 \right\} \to \left\{ x \in (\mathcal{O}_k / \mathcal{M}_K \mathcal{O}_k)^n \,\middle|\, f(x) \equiv 0 \bmod \mathcal{M}_K \right\}$$

is surjective for K in $\mathcal{A}_{k,M}$ and M big enough.[5] This gives a contradiction with $q_K^{-1} N_{f,K,1} = N_{f,K,2}$. ■

Remark 2.3. By an observation of Igusa, cf. [10], Section 1.2, one has

$$P_{f,K}(T) = \frac{1 - T Z_{f,K}(s)}{1 - T}, \tag{2.3.1}$$

[5] This is so, because the projection $\{x \in \left(k[t]/(t^2)\right)^n \mid f(x) \equiv 0 \bmod t^2\} \to \{x \in k^n \mid f(x) = 0\}$ is surjective for k any field of characteristic zero. Alternatively, this follows from Greenleaf's theorem, cf. [1].

with $T = q_K^{-s}$ and $Z_{f,K}(s)$ Igusa's local zeta function defined for real $s > 0$ by

$$Z_{f,K}(s) := \int_{\mathcal{O}_K^n} |f(x)|^s |dx|,$$

with $|dx|$ the normalized Haar measure on $\mathcal{O}_K^n$. Since $Z_{f,K}(0) = 1$, $Z_{f,K}(s)$ and $P_{f,K}(T)$ have exactly the same poles in $T = q_K^{-s}$ and thus, Theorem 2.2 also applies to $Z_{f,K}(s)$. □

3 Definition and Basic Properties of the Oscillation Indices

3.1

Let f be a polynomial in n variables over a number field k, let K be in $\mathcal{C}_k$, and let $\varphi : K^n \to \mathbf{C}$ be a Schwartz–Bruhat function. Define $\alpha_{f,K,\varphi}$ as the infimum in $\mathbf{R} \cup \{-\infty\}$ of all real numbers α satisfying

$$E_{f,K,\varphi}(y) \ll |y|_K^{\alpha},$$

where both sides are considered as functions in $y \in K^\times$. Note that $\alpha_{f,K,\varphi} \leq 0$ because $\alpha = 0$ always satisfies the previous Vinogradov inequality. Define $\alpha_{f,K}$ as the supremum of the $\alpha_{f,K,\varphi}$ when φ varies over all Schwartz–Bruhat functions on K^n. Call $\alpha_{f,K}$ the *K-oscillation index of f*.

Definition 3.2. Define the *motivic oscillation index of f* (over k) as

$$\alpha_f := \lim_{M \to +\infty} \sup_{K \in \mathcal{A}_{k,M}} \alpha_{f,K}.$$

□

Lemma 3.3. For K in $\mathcal{A}_k \cup \mathcal{B}_{k,M}$ with M big enough and φ a Schwartz–Bruhat function on K^n, the number $\alpha_{f,K,\varphi}$ equals $-\infty$ or one of the finitely many quotients $-\nu_i/N_i$, with (N_i, ν_i) the essential numerical data of the tuple of resolutions $\pi = (\pi_{f-c})_c$ chosen as in the introduction. □

Proof. This follows from Proposition 2.7 of [13], cf. (3.5.1). ■

Corollary 3.4. For all K in $\mathcal{A}_k \cup \mathcal{B}_{k,M}$ with M big enough, there exists φ, such that

$$\alpha_{f,K,\varphi} = \alpha_{f,K},$$

and for all N, there exists K in $\mathcal{A}_{k,N}$ and L in $\mathcal{B}_{k,N}$, such that

$$\alpha_f = \alpha_{f,K} = \alpha_{f,L}.$$

Moreover, the following list of a priori inequalities holds

$$-\infty \le \alpha_{f,K,\varphi} \le \alpha_{f,K} \le \alpha_f \le \alpha(\pi) \le 0 \tag{3.4.1}$$

for big M, any K in $\mathcal{C}_{k,M}$, and any Schwartz–Bruhat function φ on K^n. □

Proof. The existence of L follows from the motivic understanding of exponential integrals over K^n for K varying in $\mathcal{C}_{k,M}$ for M big enough, as established in [4] (equivalently, one can use [10]). The inequality $\alpha_f \le \alpha(\pi)$ follows from (1.2.1). ■

Although $\alpha_{f,K,\varphi}$ are defined as infima, they yield actual bounds, analogous to (1.0.1) and (1.2.1).

Lemma 3.5. For K in $\mathcal{A}_k \cup \mathcal{B}_{k,M}$ for sufficiently big M and $\varphi : K^n \to \mathbf{C}$ a Schwartz–Bruhat function, either

$$\alpha_{f,K,\varphi} = -\infty \ \text{ and } \ E_{f,K,\varphi}(y) = 0 \ \text{ for all } y \text{ with } |y|_K \text{ sufficiently big,}$$

or,

$$\alpha_{f,K,\varphi} > -\infty \ \text{ and } \ E_{f,K,\varphi}(y) \ll |v_K(y)|^{n-1} |y|_K^{\alpha_{f,K,\varphi}} \ \text{ as functions in } y \in K^\times,$$

where $|v_K(y)|$ denotes the real absolute value of $v_K(y)$. □

Proof. The statements for K in $\mathcal{A}_k$ are immediate corollaries of the asymptotic expansions of $E_{f,K,\varphi}(y)$ given by Igusa [14] and generalized by Denef and Veys in [13] for $|y|$ going to ∞. Namely, by [13], Proposition 2.7, for $K \in \mathcal{A}_k$ and for y with $|y|$ big enough, one can write $E_{f,K,\varphi}(y)$ as a finite $\mathbf{C}$-linear combination of terms of the form

$$\chi(y)|y|^{\lambda} v_K(y)^{\beta} \psi(cy), \tag{3.5.1}$$

with $c \in K$ a critical value of f, χ a character with finite image of $K^\times$ into the complex unit circle, λ a complex number with negative real part, and β an integer between 0 and

$n-1$, where the real parts of the occurring λ are among the quotients $-\nu_i/N_i$, with (N_i, ν_i) the essential numerical data of some π, as above. From this, the statements follow for $K \in \mathcal{A}_k$.

For K in $\mathcal{B}_{k,M}$, one then uses the transfer principle of [4]. Namely, by this transfer principle, it follows that for M big enough and for K in $\mathcal{B}_{k,M}$, $E_{f,K,\varphi}(y)$ can be written as a similar linear combination, as given higher up in the proof for K in $\mathcal{A}_k$. (The statement about $\mathcal{B}_{k,M}$ also follows from [10] and the existence of embedded resolutions of $f=0$ with good reduction modulo p for p any prime bigger than some M, see also [9].) ■

The following Proposition essentially follows from Theorem 2.2 and [13], with C_f as in the section on notation.

Proposition 3.6. (**Nontriviality of α_f**)

(i) For K in $\mathcal{A}_k$ or in $\mathcal{B}_{k,M}$ for M big enough, $\alpha_{f,K}=0$, if and only if f is a constant.

(ii) If f is not a constant, then there exists $M>0$, such that, for K in $\mathcal{C}_{k,M}$, $\alpha_{f,K,\varphi_{\text{triv}}}=-\infty$, if and only if $C_f(k_K)$ is empty.

By consequence, $-\infty<\alpha_f<0$, if and only if f is nonconstant and grad $f=0$ has a solution in $\mathbf{C}^n$. □

Proof. The first statement for K in $\mathcal{A}_k$ follows from (1.2.1), (3.4.1), and the definition of $\alpha(\pi)$ in the introduction. The second statement follows from Theorem 2.2 applied to some K and one of the polynomials $f-c$, with $c\in\mathbf{C}$ a critical value of f with $c\in K$ and from the fact, that any nontrivial pole of $P_{f,K}(T)$ effectively appears as λ in one of the terms (3.5.1) when writing $E_{f,K}(y)$ as a linear combination of terms as (3.5.1), by Proposition 2.7 of [13]. ■

4 A Lemma on Parameterized Integrals

Let $\mathcal{L}_{DP}$ be the language of Denef–Pas, consisting of three sorts: valued field, residue field, and value group, and the language of fields for the residue field, the language of fields for the valued field, the language of ordered additive groups for the value group, and the extra symbols $\overline{\text{ac}}$ and ord.

Let k be a number field. Consider the language $\mathcal{L}_{DP}(k)$, which is the language $\mathcal{L}_{DP}$ together with coefficients for k in the valued and residue field sort. On a field $K\in\mathcal{A}_k$, the symbols of $\mathcal{L}_{DP}(k)$ have their natural meaning, where $\overline{\text{ac}}(x)$ for nonzero $x\in K$ is interpreted as $x\pi_K^{-\text{ord}x}$ mod $\mathcal{M}_K$ in the residue field k_K, for some (consequent) choice of uniformizer π_K of $\mathcal{O}_K$ and where $\overline{\mathrm{ac}}(0) = 0$. Note that the language $\mathcal{L}_{DP}$ can be naturally interpreted in $L((t))$ for any field L in a similar way.

Then, by quantifier elimination results by Pas [16], for any given $\mathcal{L}_{DP}(k)$-formula φ, there exists a $\mathcal{L}_{DP}(k)$-formula φ' without valued field quantifiers, such that for all $K \in \mathcal{A}_{k,M}$ with M big enough, φ and φ' determine the same set over K.

By a *basic step function* on K^n with K in $\mathcal{A}_k$, we mean a characteristic function of a Cartesian product of factors of the form $\mathcal{O}_K$ and $a + \mathcal{M}_K$ for $a \in \mathcal{O}_K$. For example, the characteristic function of $\mathcal{O}_K \times \mathcal{M}_K \times (1 + \mathcal{M}_K)^{n-2}$ is a basic step function on K^n.

The goal of this section is to prove the following lemma.

Lemma 4.1. Let k be a number field and let f be a polynomial in the variables $x_1, \ldots, x_n$ over k. Write $\hat{x}$ for $x_2, \ldots, x_n$. Then, there are $M > 0$ and a nonzero polynomial g in one variable over k, such that for all $K \in \mathcal{A}_{k,M}$, all $a \in k$ with $v_K(g(a)) \leq 0$, and all basic step functions φ on K^{n-1}, one has that

$$I_K(b, s, \varphi) := \int_{\mathcal{O}_K^{n-1}} \varphi(\hat{x}) |f(b, \hat{x})|^s |d\hat{x}|,$$

with $s > 0$ a real variable and $|d\hat{x}|$ the normalized Haar measure, is independent of $b \in a + \mathcal{M}_K$. □

Proof. We give a proof based on motivic integration and constructible functions as developed in [5]. By [5] (or already with some work by Pas [16]), there are $\mathcal{L}_{DP}(k)$-formulas, such that their interpretation in K, with K a field in $\mathcal{A}_{k,M}$ with M big enough yields definable functions

$$\alpha_i(K), \beta_i(K) : K \to \mathbf{Z}$$

and definable subsets

$$A_i(K) \subset K \times k_K^{(n-1)+\ell}$$

of K times a power of the residue field of K, such that the following holds for K in $\mathcal{A}_{k,M}$ with M big enough: with φ_z, the characteristic function of the product of the residue classes mod $\mathcal{M}_K$ of $z = (z_2, \ldots, z_n) \in k_K^{n-1}$, one has that $I_K(b, s, \varphi_z)$ for $b \in K$ is a $\mathbf{Q}$-linear combination of products of factors of the form

$$a_{iz}(K)(b) := \sharp\big(A_i(K) \cap \{b\} \times \{z\} \times k_K^\ell\big),$$
$$q_K^{\alpha_i(K)s+\beta_i(K)},$$
$$\frac{1}{1 - q_K^{cs+d}},$$

and

$$\beta_i(K),$$

with pairs of integers $(c, d) \neq (0, 0)$. We show that there is a nonzero polynomial g over k and $M > 0$, such that the $a_{iz}(K)$, $\alpha_i(K)$, and $\beta_i(K)$ are constant on $a + \mathcal{M}_K$ for all $a \in k$ with $v_K(g(a)) \leq 0$, uniformly in K in $\mathcal{A}_{k,M}$. But this follows at once from Corollary 4.4. For basic step functions, which also involve the characteristic function of Cartesian products with $\mathcal{O}_K$, one works similarly, say, with some less (witnessing) z-variables. The Lemma then follows. ■

Definition 4.2. **(One-dimensional cells)** Let L be any field of characteristic zero containing k as a subfield. Let $X \subset L(t)$ be a $\mathcal{L}_{DP}(k)$-definable set and

$$f : X \to S$$
$$c : S \to L(t)$$

$\mathcal{L}_{DP}(k)$-definable functions, with S a Cartesian product of some power of L and some power of $\mathbf{Z}$. The tuple (X, f, c) is called a (1)-cell with presentation f and center c, when for each $s \in f(X)$, there exists $\xi \in L^\times$ and $a \in \mathbf{Z}$, such that the fiber $f^{-1}(s)$ is an open ball of the form

$$\{x \in L((t)) \mid \overline{\mathrm{ac}}(x - c(s)) = \xi \wedge \mathrm{ord}(x - c(s)) = a\}.$$

The tuple (X, f, c) is called a (0)-cell with presentation f and center c when $c \circ f$ is the identity on X. The set X is called a cell when it is either a (1)-cell or a (0)-cell. □

Lemma 4.3. **([16], [3], One-dimensional cell decomposition)** Let k be a number field and L a field over k. Let g_j be finitely many polynomials in *one* variable over k. Then,

$L(t)$ can be written as the disjoint union of finitely many $\mathcal{L}_{DP}(k)$-definable cells X_i with $\mathcal{L}_{DP}(k)$-definable presentations f_i and $\mathcal{L}_{DP}(k)$-definable centers c_i, such that for each i, j, the restrictions $\mathrm{ord} g_{j|X_i}$ and $\overline{\mathrm{ac}} g_{j|X_i}$ factorize through f_i. Moreover, the formulas describing the X_i, f_i, and c_i can be taken independently of L. □

Proof. This form of Denef–Pas cell decomposition is proven in [3] with $k = \mathbf{Q}$, where it is a relative cell decomposition relative to $\mathbf{Z}^\ell \times L^\ell$ under the map $(\mathrm{ord}(g_j), \overline{\mathrm{ac}}(g_j))_j$. The Lemma for general k follows easily from it by first replacing the parameters from k by variables, then applying the form of [3] to all the variables, and then plugging in the parameters for the variables again. ■

Corollary 4.4. Let the functions $a_{iz}(K)$, $\alpha_i(K)$, and $\beta_i(K)$ be as in the proof of Lemma 4.1. Then, there are $M > 0$ and a nonzero polynomial g in one variable over k, such that for all $K \in \mathcal{A}_{k,M}$, all $a \in k$ with $v_K(g(a)) \le 0$, and all $z \in k_K^{n-1}$, one has that these functions $a_{iz}(K)$, $\alpha_i(K)$, and $\beta_i(K)$ are constant functions in the variable b, when b runs over $a + \mathcal{M}_K$. □

Proof. Let g_ℓ be the (finitely many) polynomials over k in one valued field variable that occur in the valued field quantifier-free formulas that describe the α_i, β_i, and A_i. Apply Lemma 4.3 for the polynomials g_ℓ to find cells X_j with centers c_j and presentations f_j. By quantifier elimination in the language $\mathcal{L}_{DP}(k)$ and by examining valued field quantifier-free formulas in one free-valued field variables, it follows that the images of the centers c_j are contained in the zero set of a (nonzero) polynomial g over k. Now, the corollary follows from the definition of cells and the evident ultraproduct argument to go to K in $\mathcal{A}_{k,M}$ for M big enough. Namely, for all $K \in \mathcal{A}_{k,M}$ for M big, for $a \in k$ with $v_K(g(a)) \le 0$, the set $a + \mathcal{M}_K$ is contained in $X_j(K)$ for some unique j, and $f_j(K)(a + \mathcal{M}_K)$ is a singleton, where $X_j(K)$ and $f_j(K)$ are the K-interpretations of X_j and f_j. ■

5 Estimation of the Motivic Oscillation Index

Let f be a polynomial in n variables over a number field k. Let C_f, S_f, δ_f, and ε_f be as in the section on notation. Let α_f be the motivic oscillation index as before.

Theorem 5.1. Let f be a polynomial in n variables over a number field k. Then,

$$\frac{-n + \delta_f}{2} \le \alpha_f. \tag{5.1.1}$$

□

Proof. Since one can always replace k by a finite field extension and f by $f-a$ for some critical value $a \in k$, it is clear that it is enough to prove that

$$\frac{-n+\varepsilon_f}{2} \leq \alpha_f. \tag{5.1.2}$$

We may suppose that $\varepsilon_f \geq 0$, since otherwise, $\varepsilon_f = -\infty$ and then (5.1.2) is trivial. We then prove the following statement by induction on n when $\varepsilon_f \geq 0$.

For each M, there exist L in $\mathcal{A}_{k,M}$, a basic step function φ_L on L^n (in the sense of Section 4), a real number s_0 with

$$\frac{-n+\varepsilon_f}{2} \leq s_0 \leq 0,$$

and a complex number t_0 with absolute value $q_L^{-s_0}$, such that t_0 is a nontrivial pole (in the sense of Definition 2.1) of the rational function in $T = q_L^{-s}$, defined for $s > 0$ by

$$Z_{f,L,\varphi_L}(s) := \int_{L^n} \varphi_L(x)|f(x)|^s |dx|.$$

This statement implies (5.1.2) (since a nontrivial pole of $Z_{f,L,\varphi_L}(s)$ corresponds to an effectively appearing λ in (5.1.1) by [13], Proposition 2.7).

The case that $n = 1$ is easy by explicit calculation using factorization of f into irreducible polynomials over the algebraic closure of k.

For $n > 1$, first suppose that $\varepsilon_f = 0$. Let M be big enough and choose L in $\mathcal{A}_{k,M}$, such that $S_f(k_L)$ is nonempty. By Theorem 2.2 and Remark 2.3, it follows that $Z_{f,L,\varphi_{\text{triv}}}(s)$, considered as a rational function in $T = q_L^{-s}$, has a nontrivial pole, with φ_{triv} the basic step function which is the characteristic function of $\mathcal{O}_L^n$. Since each complex pole t_0 of $Z_{f,L,\varphi_{\text{triv}}}(s)$ (still considered as a rational function in $T = q_L^{-s}$) with absolute value $q_L^{-s_0}$ satisfies $-n/2 \leq s_0 \leq 0$, when $n > 1$ by results by Segers [18], the case that $\varepsilon_f = 0$ and $n > 1$ follows.

Now, we treat the case $n > 1$ and $\varepsilon_f > 0$. There is a coordinate, say x_1, such that for infinitely many points a in the algebraic closure of k, $\varepsilon_{f_a} \geq \varepsilon_f - 1$, with f_a the polynomial in $\hat{x} := (x_2, \ldots, x_n)$ over $k(a)$ defined by $f_a(\hat{x}) = f(a, \hat{x})$. Indeed, since $\varepsilon_f > 0$, there is a coordinate, say x_1, to which an irreducible component of S_f of dimension ε_f projects dominantly, and then one compares the equations defining S_f and the S_{f_a}. Let g be a polynomial in one variable as in Lemma 4.1 for the polynomial f. Then, by replacing k by a bigger field extension, we can suppose that there is $c \in k$, such that $g(c) \neq 0$ and such that $\varepsilon_{f_c} \geq \varepsilon_f - 1$. Note that f_c is a polynomial in $n-1$ variables. By the induction hypothesis, there exist L in $\mathcal{A}_{k,M}$ with $v_L(g(c)) = 0$, a basic step function ψ_L on L^{n-1}, and a real number s_0 with

$$\frac{-n+\varepsilon_f}{2} = \frac{-(n-1)+(\varepsilon_f-1)}{2} \leq \frac{-(n-1)+\varepsilon_{f_c}}{2} \leq s_0 \leq 0$$

and a complex number with absolute value $q_L^{-s_0}$, which is a nontrivial pole of $Z_{f_c,L,\psi_L}(s)$, considered as a rational function in $T = q_L^{-s}$. Now let φ_L be the basic step function on L^n, which is the product of the characteristic function of $c + \mathcal{M}_L$ with ψ_L. Then t_0 is also a nontrivial pole of $Z_{f,L,\varphi_L}(s)$ (considered as a rational function in $T = q_L^{-s}$), since $Z_{f,L,\varphi_L}(s)$ is a product integral by the above application of Lemma 4.1 to f and since $v_L(g(c)) = 0$. Namely, $Z_{f,L,\varphi_L}(s) = q_L^{-1} Z_{f_c,L,\psi_L}(s)$. This proves the above statement and thus, the theorem. ∎

Remark 5.2. A similar but more refined proof than that of Theorem 5.1 might give the stronger result

$$\frac{-n+\delta_f(K)}{2} \leq \alpha_{f,K} \tag{5.2.1}$$

for all $K \in \mathcal{C}_{k,M}$ with M sufficiently big and $\delta_f(K)$ the Zariski dimension of the Zariski closure of $C_f(K)$ in $\mathbf{A}_k^n$. Note that this implies (5.1.1), since there are enough K with $\delta_f(K) = \delta_f$. □

6 Igusa's Conjecture Modulo p and Modulo p^2

Theorem 6.1. Let f be a homogeneous polynomial in n variables over the number field k. Then there exists $M > 0$ and a constant c, such that for each K in $\mathcal{C}_{k,M}$ and each $y \in K$ with $v_K(y) = -1$ or $v_K(y) = -2$, one has

$$|E_{f,K}(y)| < c|y|_K^{\alpha_f}. \tag{6.1.1}$$

Moreover, the statement for y with $v_K(y) = -2$ holds for all polynomials over k. □

Proof. Let δ_f be as in the section on notation. First note that for homogeneous f, one has $\delta_f = -\infty$, if and only if f is linear, and, in the linear case, one has $\alpha_f = -\infty$ and

$E_{f,K}(y) = 0$ for all $K \in \mathcal{C}_{k,M}$ for M big enough and all $y \in K$ with $v_K(y) < 0$. The case that f is a constant is trivial, cf. Proposition 3.6.

Suppose now, that the degree of f is ≥ 2. Thus, $n > \delta_f > -\infty$. Let H_f be the (not necessarily reduced) projectivization in $\mathbf{P}_k^n$ of the closed subscheme of $\mathbf{A}_k^n$ given by $f = 0$, and let δ be the (projective) dimension of the singular locus of the intersection of H_f with the hyperplane at infinity. Then,

$$\delta + 1 = \delta_f. \tag{6.1.2}$$

Then, by Theorem 4 of Katz [15] and by (6.1.2), there exist constants C and M, such that for all K in $\mathcal{C}_{k,M}$ and all y in K with $v_K(y) = -1$, one has

$$|E_{f,K}(y)| < C q_K^{\frac{-n+\delta_f}{2}}, \tag{6.1.3}$$

with q_K the number of elements of the residue field of K. Together with (5.1.1) of Theorem 5.1, this implies the statement for y with $v_K(y) = -1$.

Now let f be a general polynomial over k in n variables. For $K \in \mathcal{C}_{k,M}$ with M big enough, define

$$A_f(K) := \left\{ x \in \mathcal{O}_K^n \,\middle|\, \operatorname{grad} f(x) \equiv 0 \bmod \mathcal{M}_K \right\}$$

and

$$B_f(K) := \left\{ x \in \mathcal{O}_K^n \,\middle|\, \operatorname{grad} f(x) \not\equiv 0 \bmod \mathcal{M}_K \right\}.$$

By Noether's Normalization Theorem, the number of elements of the set $C_f(k_K)$ is $\leq D q_K^{\delta_f}$ for some D independent of q_K, when the characteristic of k_K is big enough. Hence, there exist some M and D, such that for all K in $\mathcal{A}_{k,M}$, the measure of $A_f(K)$ against the normalized Haar measure $|dx|$ satisfies

$$\int_{A_f(K)} |dx| \leq D q_K^{-n+\delta_f}.$$

On the other hand, for M big enough, K in $\mathcal{A}_{k,M}$, and $y \in K$ with $v_K(y) = -2$,

$$0 = \int_{B_f(K)} \psi_K(yf(x)) |dx|_K,$$

by Hensel's Lemma and the fact that the sum

$$\sum_{z\in\mathbf{F}_q}\psi_q(z)$$

equals zero for any nontrivial additive character ψ_q on $\mathbf{F}_q$. Hence, for such M and K, and for all y in K with $v_K(y)=-2$,

$$\begin{aligned}|E_{f,K}(y)| &= \left|\int_{A_f(K)}\psi_K(yf(x))|dx|_K+\int_{B_f(K)}\psi_K(yf(x))|dx|_K\right|\\ &= \left|\int_{A_f(K)}\psi_K(yf(x))|dx|_K\right|\\ &\le \int_{A_f(K)}|dx|_K\\ &\le Dq_K^{-n+\delta_f}\\ &\le D|y|^{\alpha_f}, \end{aligned} \tag{6.1.4}$$

where D is independent of K and where inequality (6.1.4) follows from $|y|=q_K^2$ and (5.1.1). ■

Corollary 6.2. Igusa's Conjecture 1.1 with argument of order -1 or -2 holds for all homogeneous polynomials f over k. Namely, for f a homogeneous polynomial in n variables over k and $\pi=\pi_f$ a resolution as in the introduction, there exists a constant c, such that for each p-adic completion K of k and each $y\in K$ with $v_K(y)=-1$ or $v_K(y)=-2$, one has

$$|E_{f,K}(y)|<c|y|_K^{\alpha(\pi)}.$$

□

Proof. This follows from Theorem 6.1 and (3.4.1) for big residue characteristics, and from (1.2.1) for small residue characteristics. ■

Remark 6.3. The factor $|v_K(y)|^{n-1}$ featuring in the original conjecture clearly becomes unnecessary when one focuses on y with $v_K(y)=-1$ and $v_K(y)=-2$. When $v_K(y)$ varies over $\mathbf{Z}$, the factor $|v_K(y)|^{n-1}$ can, in general, not be omitted. □

7 Nonhomogeneous Polynomials

Let f be a polynomial in n variables over a number field k. Let H_f be the (not necessarily reduced) projectivization in $\mathbf{P}_k^n$ of the scheme $f = 0$ in $\mathbf{A}_k^n$. Let X_f be the scheme-theoretic intersection of H_f with the hyperplane at infinity. Let δ be the dimension of the singular locus of X_f when this singular locus is nonempty, and put $\delta = -1$ when X_f is smooth. We say that f satisfies Tameness Condition (7.0.1), if

$$\delta_f \geq \delta + 1. \tag{7.0.1}$$

Note that by (6.1.2), a homogeneous polynomial of degree ≥ 2 always satisfies Tameness Condition (7.0.1). Under this Tameness Condition, we get the analogue of Theorem 6.1

Theorem 7.1. Let f be a polynomial in n variables over a number field k. Suppose that f satisfies Tameness Condition (7.0.1). Then there exists a number M and a constant c, such that for each $K \in \mathcal{C}_{k,M}$ and each $y \in K$ with $v_K(y) = -1$, one has

$$|E_{f,K}(y)| < c|y|^{\alpha_f}, \tag{7.1.1}$$

□

Proof. Same proof as for Theorem 6.1, using $\delta + 1$ instead of δ_f in (6.1.3), which holds by the same Theorem 4 of [15]. ■

Example 7.2. For $f(x_1, x_2) = x_1^2 x_2 - x_1$, one has $\alpha_f = -\infty$, but $E_{f,K}(y) \neq 0$ for any $K \in \mathcal{C}_{\mathbf{Q},2}$ when $v_K(y) = -1$. Hence, a literal analogue of Conjecture 1.1 would not make sense for general polynomials. Note that f does not satisfy Tameness Condition (7.0.1). □

8 Igusa's Conjecture Revisited

We come back to the sums $E_f(N)$ from the introduction and study the growth of $|E_f(N)|$ with N. As noted before, it is sufficient to bound $|E_f(p^m)|$ in terms of primes p and positive integers m. We focus on $m > 1$, since the case $m = 1$ is rather well understood by work by Deligne, Laumon, Katz, and others.

In general, one has the following situation which follows from [10] or [4]. Let f be any polynomial over $\mathbf{Q}$ in n variables. Then there exist constants $\beta, \gamma \geq 0$, such that for all big enough primes p and all $m \geq \gamma$, one has

$$|E_f(p^m)| < p^\beta m^{n-1} p^{\alpha_f m}, \tag{8.0.1}$$

with α_f the motivic oscillation index.

A natural sharpening of Igusa's Conjecture 1.1 is the conjecture that one can take $\beta = \gamma = 0$ in (8.0.1), when f is homogeneous. (Conjecture 1.1 follows from this sharpening by (1.2.1) and by Corollary 3.4.) An alternative sharpening of Igusa's conjecture by Denef and Sperber in [12] is based on complex oscillation indices and might be equivalent with our sharpening (that is, the supremum of the complex oscillation indices of f around each of the points of $\mathbf{C}^n$ might equal α_f).

We introduce the following mysterious invariant of f, which might play a role in understanding $E_f(p^m)$ for general f:

Definition 8.1. Let f be a polynomial in n variables over a number field k. Define β_f as the limit over M and γ of the infima of all real $\beta \geq 0$, such that

$$|E_{f,K}(y)| < q_K^\beta |v_K(y)|^{n-1} |y|_K^{\alpha_f}$$

for all $y \in K$ with $v_K(y) < -\gamma$ and all $K \in \mathcal{A}_{k,M}$. Call β_f the *flaw of f w.r.t. α_f*. □

For non-homogeneous f, β_f is not even conjecturally understood.

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References

[1] Birch, B. J., and K. McCann. "A criterion for the p-adic solubililty of Diophantine equations." *The Quarterly Journal of Mathematics* 18, no. 1 (1967): 59–63.

[2] Cluckers, R. "Igusa–Denef–Sperber conjecture on nondegenerate p-adic exponential sums." *Duke Mathematical Journal* (2006): preprint arXiv:math.NT/0606269.

[3] Cluckers, R., and F. Loeser. "b-minimality." *Journal of Mathematical Logic* (2006): preprint arXiv:math.LO/0610183.

[4] Cluckers, R., and F. Loeser. "Constructible exponential functions, motivic Fourier transform and transfer principle." (2005): preprint arXiv:math.AG/0512022.

[5] Cluckers, R., and F. Loeser. "Constructible motivic functions and motivic integration." (2004): preprint arXiv:math.AG/0410203.

[6] Cluckers, R., and F. Loeser. "Fonctions constructibles exponentielles, transformation de Fourier motivique et principe de transfert." *Comptes rendus de l'AcadÈmie des Sciences* 341, no. 12 (2005): 741–46.

[7] Deligne, P. "La conjecture de Weil 1" [Weil's conjecture 1]. *Publications Mathematiques Institut de Hautes Etudes Scientifiques* 43 (1974): 273–307.

[8] Denef, J. "The rationality of the Poincaré series associated to the p-adic points on a variety." *Inventiones Mathematicae* 77 (1984): 1–23.

[9] Denef, J. "On the degree of Igusa's local zeta function." *American Journal of Mathematics* 109 (1987): 991–1008.

[10] Denef, J. "Report on Igusa's local zeta function." *Séminaire Bourbaki* 33, no. 741 (1990–1991): 28.

[11] Denef, J., and F. Loeser. "Definable sets, motives and p-adic integrals." *Journal of the American Mathematical Society* 14, no. 2 (2001): 429–69.

[12] Denef, J., and S. Sperber. "Exponential sums mod p^n and Newton polyhedra." *Bulletin of the Belgian Mathematical Society* (2001): S55–S63.

[13] Denef, J., and W. Veys. "On the holomorphy conjecture for Igusa's local zeta function." *Proceedings of the American Mathematical Society* 123, no. 10 (1995): 2981–8.

[14] Igusa, J. "Lectures on forms of higher degree (notes by S. Raghavan)." *Lectures on Mathematics and Physics*, vol. 59. Tata Institute of Fundamental Research. Berlin: Springer, 1978.

[15] Katz, N. "Estimates for 'singular' exponential sums." *International Mathematics Research Notices* 1999, no. 16 (1999): 875–99.

[16] Pas, J. "Uniform p-adic cell decomposition and local zeta functions." *Journal für die reine und angewandte Mathematik* 399 (1989): 137–72.

[17] Segers, D., and W. Veys. Private communication, Leuven, Belgium, 2006.

[18] Segers, Dirk. "Lower bound for the poles of Igusa's p-adic zeta functions." *Mathematische Annalen* 336, no. 3 (2006): 659–69.