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ORBITAL INTEGRALS FOR LINEAR GROUPS

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Abstract For a linear group G acting on an absolutely irreducible variety X over $\mathbb{Q}$, we describe the orbits of $X(\mathbb{Q}_p)$ under $G(\mathbb{Q}_p)$ and of $X(\mathbb{F}_p((t)))$ under $G(\mathbb{F}_p((t)))$ for p big enough. This allows us to show that the degree of a wide class of orbital integrals over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ is less than or equal to 0 for p big enough, and similarly for all finite field extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.

Keywords: orbital integrals; Igusa's local zeta functions; motivic integrals; Denef–Pas language; Galois cohomology; pseudo-finite fields

AMS 2000 *Mathematics subject classification:* Primary 11S40; 03C98
Secondary 22E35; 11S20; 11U09; 11M41

1. Introduction

Let F be a number field with ring of integers $\mathcal{O}_F$. Let $\mathcal{A}_F$ be the collection of all finite field extensions of non-archimedean completions of F. Let $\mathcal{B}_F$ be the collection of all fields of the form $\mathbb{F}_q((t))$ which are rings over $\mathcal{O}_F$. For K in $\mathcal{A}_F \cup \mathcal{B}_F$, let $\mathcal{O}_K$ be its valuation ring, M_K its maximal ideal, k_K its residue field, and $q_K := \sharp k_K$. For $N > 0$, let $\mathcal{C}_N$ be

$$\mathcal{C}_N := \{K \in \mathcal{A}_F \cup \mathcal{B}_F \mid \mathrm{char}(k_K) > N\}.$$

For any $K \in \mathcal{C}_1$, let $\overline{\mathrm{ac}} : K^\times \to k_K^\times$ be a multiplicative map extending the projection $\mathcal{O}_K^\times \to k_K^\times$, put $\overline{\mathrm{ac}}(0) = 0$, and let $\mathrm{ord} : K^\times \to \mathbb{Z}$ be the order.

Let G be a linear algebraic group over F, rationally acting on an absolutely irreducible† algebraic variety X over F. Suppose that X is a homogeneous G-space, that is, the action of $G(\mathbb{C})$ on $X(\mathbb{C})$ is transitive. For K a field over F and $x \in X(K)$, let $G(K)(x)$ be the orbit of x under the action of $G(K)$.

Theorem 1.1. *Let $U \subset X$ be an affine open. Then there are finitely many regular functions $f_i : U \to \mathbb{A}^1_F$ and integers $N > 0$ and $d > 0$ such that, for any $K \in \mathcal{C}_N$ and any $x \in X(K)$, the set $G(K)(x) \cap U(K)$ depends only on $\overline{\mathrm{ac}}(f_i(x))$ and $\mathrm{ord}(f_i(x)) \bmod d$.*

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† See Remark 5.4 to loosen the condition of absolute irreducibility of X.

Definition 1.2. Let $U \subset X$ be an affine open and let $f, g_i : U \to \mathbb{A}^1_F$ be regular functions. Let ω be a volume form on U, that is, a degree n rational differential form on U when X is of dimension n. For each $K \in \mathcal{C}_1$ let $W(K)$ be $\bigcap_i g_i^{-1}(U_i)$ with U_i either $\mathcal{O}_K$ or M_K. For each $K \in \mathcal{C}_1$ and for each x in $X(K)$, under the condition of integrability for all $s > 0$, consider the orbital integral

$$I_{K,x}(s) := \int_{G(K)(x)\cap W(K)} |f|^s |\omega|_K, \tag{1.2.1}$$

with $|\omega|_K$ the measure on $U(K)$ associated with ω. If for some K and x this is not integrable, put $I_{K,x}(s) := 0$ for this K and x.

Theorem 1.3. *Let $I_{K,x}(s)$ be as in Definition 1.2. Then, there exists $N > 0$ such that $I_{K,x}(s)$ is rational and of degree less than or equal to 0 in q_K^{-s} for each $K \in \mathcal{C}_N$ and for each $x \in X(K)$.*

Addendum to Theorem 1.3. *Let $I_{K,x}(s)$ be as in Definition 1.2. Then there exist $N > 0$ and $F(\mathbb{L}, T)$ in $\mathbb{Q}(\mathbb{L}, T)$ of the form*

$$F(\mathbb{L}, T) = T^a \prod_{i=1}^{m} (1 - \mathbb{L}^{a_i} T^{b_i}),$$

with $a, a_i, b_i \in \mathbb{Z}$, $b_i > 0$, and $m \geqslant 0$, such that

$$F(q_K, q_K^{-s}) I_{K,x}(s)$$

is a polynomial in q_K^{-s} for each K in $\mathcal{C}_N$ and each $x \in X(K)$. In particular, by Theorem 1.3, the degrees of the numerator and the denominator of $I_{K,x}(s)$ are uniformly bounded and only finitely many real poles with bounded multiplicities in s can occur in $I_{K,x}(s)$ when K varies over $\mathcal{C}_N$ for suitable N and when x varies in $X(K)$.

1.1.

Orbital integrals as in Theorem 1.3 occur in representation theory and in the study by Igusa [**15, 16**] and others, of prehomogeneous vector spaces, where $X = \mathbb{A}^n_F$, G acts linearly on X, f is a relative invariant of this action, and $W(K)$ is a Cartesian product of sets of the form $\mathcal{O}_K$ and $a + M_K$ with $a \in \mathcal{O}_K$.* For such X, f and $W(K)$, Igusa [**15**] determines, under some extra conditions, the poles of the integral (1.2.1) in terms of explicit group theoretical invariants. Since the multiplicity of the poles is *a priori* bounded by the dimension of X, this gives an explicit bound on the degree of the denominator of (1.2.1). Combining this with Theorem 1.3 then also gives a bound on the degree of the numerator in terms of these group theoretical data.

For general f, Theorem 1.3 and its addendum give a single finite set of candidate poles in $t = q_K^{-s}$ of (1.2.1) and uniform bounds on the degree of the denominator and the numerator of (1.2.1) when K varies in $\mathcal{C}_N$, generalizing greatly the situation of [**15**].

* Note that the characteristic function of such $W(K)$ is an important kind of Schwartz–Bruhat function, since it is the essential building block for Schwartz–Bruhat functions on adeles and ideles.

2. The logical setting and pseudo-finite fields

2.1. The languages $\mathcal{L}_{\mathrm{DP}}$ and $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$

Let $\mathbb{L}_{\mathrm{Ord}} = (+, -, \leqslant, 0)$ be the language of ordered groups.

For K a valued field, M the maximal ideal of its valuation ring R, and k the residue field, an *angular component modulo* M (or angular component for short) is a map $\overline{\mathrm{ac}} : K \to k$ such that the restriction to $K^\times$ is a multiplicative homomorphism to $k^\times$, the restriction to $R^\times$ coincides with the restriction to $R^\times$ of the natural projection $R \to k$, and such that $\overline{\mathrm{ac}}(0) = 0$.

The *language* $\mathcal{L}_{\mathrm{DP}}$ *of Denef–Pas* is defined as the three sorted language

$$(\mathbb{L}_{\mathrm{Rings}}, \mathbb{L}_{\mathrm{Rings}}, \mathbb{L}_{\mathrm{Ord}}, \overline{\mathrm{ac}}, \mathrm{ord}).$$

The sorts of $\mathcal{L}_{\mathrm{DP}}$ are a valued field K with residue field k and value group G. The function symbol ord is the additively written valuation $\mathrm{ord} : K^\times \to G$, and $\overline{\mathrm{ac}} : K \to k$ is an angular component.* The first ring language is used for the valued field, the second for the residue field, and $\mathbb{L}_{\mathrm{Ord}}$ is used for the value group.

The language $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$ is obtained from $\mathcal{L}_{\mathrm{DP}}$ by removing the value group sort and replacing it by infinitely many sorts for the quotients G/nG, $n = 2, 3, \ldots$, where G is the value group. Here, G/nG is considered as a group and the language for each of these sorts is the group language $(+, -, 0)$ together with projection maps $\pi_{nm} : G/nG \to G/mG$ for $m \geqslant 2$ a divisor of n and order maps $\mathrm{ord}_n : K^\times \to G/nG$ making commuting diagrams, extended by $\mathrm{ord}_n(0) = 0$.

2.2. Theories $\mathcal{T}_0$, $\mathcal{T}_\infty$ and $\mathcal{T}_\infty^{(d)}$

In all that follows, $\mathcal{T}$ is any theory in $\mathcal{L}_{\mathrm{DP}}$ that contains the $\mathcal{L}_{\mathrm{DP}}$-theory $\mathcal{T}_0$ of Henselian valued fields with angular component modulo the maximal ideal and with residual characteristic zero. This theory has elimination of valued field quantifiers in $\mathcal{L}_{\mathrm{DP}}$ by [**18**]. Recall that a perfect field k is called pseudo-finite if it has a unique field extension of any given finite degree and if any absolutely irreducible variety over k has a k-rational point. The theory of pseudo-finite fields is a first-order theory which can be expressed by an infinite axiom scheme in the language of rings.

Let $\mathcal{T}_\infty$ be the theory containing $\mathcal{T}_0$ which expresses that the value group is elementary equivalent to the additive group $\mathbb{Z}$ and that the residue field is a pseudo-finite field. Each model of $\mathcal{T}_\infty$ is elementary equivalent with $k((t))$ for some pseudo-finite field k, because of either Ax and Kochen [**1**, **2**], Eršov [**9**], Cohen [**6**], Pas [**18**] or others.

For any $d \in \mathbb{N}_0$, let $\mathbb{Z}^{(d)}$ be the additive group $\mathbb{Z}[r^{-1}]_{r \in I_d}$ with $I_d = \{r \in \mathbb{N}_0 \mid \gcd(r, d) = 1\}$ and $\mathbb{N}_0 = \{z \in \mathbb{Z} \mid z > 0\}$. Let $\mathcal{T}_\infty^{(d)}$ be the theory containing $\mathcal{T}_0$ expressing

* The problem that the function ord is not defined globally on K is easily settled and the reader may choose a way to do so. For example, the reader may choose a value of $\mathrm{ord}(0)$ in the value group and treat the cases that the argument of ord equals zero always separately, or, the reader may add a symbol $+\infty$ to the language $\mathbb{L}_{\mathrm{Ord}}$ that is bigger than any element of the value group, and make the natural changes. We will always make clear what we mean by expressions as $\mathrm{ord}(x) \leqslant \mathrm{ord}(y)$ and so on when x or y can be zero.

that the value group is elementary equivalent with $\mathbb{Z}^{(d)}$ and that the residue field is a pseudo-finite field.

If k is a pseudo-finite field of characteristic zero, we denote by $k((t))^{(d)}$ the field

$$\bigcup_{r\in I_d} k((t^{1/r})),$$

where we take the union in a compatible way, that is, such that $(t^{1/r})^m = (t^{1/r'})^{m'}$ for any integers $m, m' \geqslant 0$, $r, r' > 0$, whenever $m'/r' = m/r$. Note that $k((t))^{(d)}$ is a model of $\mathcal{T}_\infty^{(d)}$ and that each model of $\mathcal{T}_\infty^{(d)}$ is elementary equivalent with $k((t))^{(d)}$ for some pseudo-finite field k, because of similar Ax–Kochen–Eršov principles as cited above.

Let $\mathcal{T}$ be as before.

Definition 2.1. A collection P of relations P_K on K^n for each model K of $\mathcal{T}$ which is uniformly definable by a formula in $\mathcal{L}_{\mathrm{DP}}$ is called a *definable relation over* $\mathcal{T}$. Here, $n \in \mathbb{N}_0$.

Definition 2.2. A collection $\sim$ of equivalence relations $\sim_K$ on K^n for each model K of $\mathcal{T}$ which is uniformly definable by a formula $\psi(x, y)$ in $\mathcal{L}_{\mathrm{DP}}$, where x and y run over K^n, is called a *definable equivalence relation over* $\mathcal{T}$. Here, $n \in \mathbb{N}_0$.

Definition 2.3. Let $\sim$ be a definable equivalence relation over $\mathcal{T}$. If there exists $N \in \mathbb{N}_0$ such that, for any model K of $\mathcal{T}$, the number of equivalence classes of $\sim_K$ is less than or equal to N, then $\sim$ is called *finite over* $\mathcal{T}$.

If K is a model of $\mathcal{T}$, we say that $\sim$ is *finite over* K if the number of equivalence classes of $\sim_K$ is finite.

Definition 2.4. A formula in $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$ with no quantifiers running over the valued field sort is called a *tame formula*.

Definition 2.5. A definable relation P over $\mathcal{T}$ is called *tame over* $\mathcal{T}$ (respectively *tame over* K, for any given model K of $\mathcal{T}$), if there exists a tame formula $\varphi(x)$ such that

$$\mathcal{T} \vdash \varphi(x) \leftrightarrow P(x)$$

(respectively $K \models \varphi(x) \leftrightarrow P(x)$).

Definition 2.6. Let $\sim$ be a definable equivalence relation over $\mathcal{T}$. Say that *the imaginaries of* $\sim$ *are tame over* $\mathcal{T}$, if there exists a tame formula $\psi(x, \xi, m)$, with ξ a tuple of residue field variables and m a tuple of variables running over $(G/nG)_{n=2,3,\ldots}$, such that

$$\mathcal{T} \vdash (\forall x)(\exists \xi)(\exists m)(\forall y)(x \sim y \leftrightarrow \psi(y, \xi, m)). \tag{2.6.1}$$

For K a model of $\mathcal{T}$, say that *the imaginaries of* $\sim$ *are tame over* K if there exists a ψ as above (that may depend on K) that satisfies the same condition but with $\mathcal{T} \vdash$ replaced by $K \models$.

Lemma 2.7. *Let* $\sim$ *be a definable equivalence relation over* $\mathcal{T}_\infty$. *Suppose that* $\sim$ *is finite over* $k((t))$ *for each pseudo-finite field* k *of characteristic zero. Then* $\sim$ *is finite over* $\mathcal{T}_\infty$.

Proof. This follows by compactness in a standard way, cf. the proof of Lemma 2.9 for a less standard compactness argument. □

Proposition 2.8. *The theory $\mathcal{T}_\infty^{(d)}$ has elimination of valued field quantifiers in $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$. Moreover, any $\mathcal{L}_{\mathrm{DP}}$-formula without free value group variables is equivalent over $\mathcal{T}_\infty^{(d)}$ to a $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$-formula without valued field quantifiers.*

Proof. We first prove that the collection of groups $G = \mathbb{Z}^{(d)}$, $G/2G$, $G/3G$, ... has elimination of G-quantifiers in the multisorted language $\mathbb{L}_0$ consisting of $\mathbb{L}_{\mathrm{ord}}$ for G, together with the language of groups for each of the groups G/nG, and the natural projection maps $\pi_{mn} : G/mG \to G/nG$ and $\pi_n : G \to G/nG$ for n dividing $m > 0$.

It is enough to eliminate the quantifier $(\exists x)$ from a formula $\varphi(y)$ of the form:

$$(\exists x)\Big(\bigwedge_i f_i(y) = K_i x \bigwedge_j g_j(y) \neq L_j x \bigwedge_\ell \Big(h_\ell(y) \mathop{\square}_\ell M_\ell x\Big) \wedge H_1(y) \wedge H_2(\pi_m(x,y))\Big) \tag{2.8.2}$$

with $y = (y_1, \ldots, y_n)$ running over G^n, f_i, g_j, h_ℓ linear homogeneous forms in y over $\mathbb{Z}$, $m > 0$ and the K_i, L_j, M_ℓ integers, the $\square_\ell$ either $<$ or no condition, the H_i formulae without G-quantifiers, and $\pi_m(x,y) = (\pi_m(x), \pi_m(y_1), \ldots, \pi_m(y_n))$. By changing H_1 if necessary and because the order on each coset of mG in G is dense, we may suppose that all the $\square_\ell$ are no condition. Suppose first that at least one of the K_i is non-zero. Then, again changing H_1 if necessary, we may suppose that there are no polynomials g_j. But then (2.8.2) is equivalent with a certain G-quantifier free formula $H_3(y)$. Suppose finally that there are no polynomials f_i. Then we may suppose that there are no polynomials g_j since the fibres of the maps π_m are infinite. Then clearly (2.8.2) is equivalent with a certain G-quantifier free formula $H_4(y)$. This proves the G-quantifier elimination in $\mathbb{L}_0$.

Now we prove the statement for the theory $\mathcal{T}_\infty^{(d)}$. The theory $\mathcal{T}_\infty^{(d)}$ has elimination of valued field quantifiers and of G-quantifiers in the language $\mathcal{L}_{\mathrm{DP}} \cup \mathbb{L}_0$ by [**18**] and by the above quantifier elimination result for G-quantifiers in $\mathbb{L}_0$. We have to show that we can remove the sort for G. Let $\varphi(x, \xi, \alpha)$ be an $\mathcal{L}_{\mathrm{DP}} \cup \mathbb{L}_0$-formula without valued field quantifiers and G-quantifiers, where x are valued field variables, ξ are residue field variables, and α runs over $(G/nG)_{n=2,3,\ldots}$, but possibly containing the symbol ord. For f_i polynomials over $\mathbb{Z}$, the condition

$$\operatorname{ord} f_i(x) < \operatorname{ord} f_j(x)$$

with possibly $f_j(x) = 0$ and $f_i(x) \neq 0$, is equivalent to the condition

$$\overline{\mathrm{ac}}(f_i(x) + f_j(x)) = \overline{\mathrm{ac}}(f_i(x) + 2f_j(x)) = \overline{\mathrm{ac}}(f_i(x) + 3f_j(x))$$

and one can rewrite conditions $\operatorname{ord} f_i(x) \leqslant \operatorname{ord} f_j(x)$ and $\operatorname{ord} f_i(x) = \operatorname{ord} f_j(x)$ similarly. This easily shows that $\varphi(x, \xi, \alpha)$ is equivalent to a $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$-formula without valued field quantifiers. □

Lemma 2.9. *Let P be a definable relation over $\mathcal{T}_\infty$. Suppose that P is tame over $k((t))$ for each pseudo-finite field k of characteristic zero. Then P is tame over $\mathcal{T}_\infty$.*

Proof. Let x be the tuple consisting of the variables that occur freely in P. For any tame formula $\psi(x)$, let C_ψ be the sentence

$$(\forall x)(P(x) \leftrightarrow \psi(x)).$$

By the supposition of the lemma and by compactness,

$$\mathcal{T}_\infty \vdash C_{\psi_1} \vee \cdots \vee C_{\psi_n}$$

for some formulae $\psi_1, \ldots, \psi_n$ and some n. For each $j = 1, \ldots, n$, let D_j be the sentence

$$C_{\psi_j} \wedge (\neg C_{\psi_1} \wedge \cdots \wedge \neg C_{\psi_{j-1}}),$$

where $\neg$ is the negation. Because D_j has no free variables, we see by elimination of valued field quantifiers [**18**] that D_j is equivalent over $\mathcal{T}_\infty$ with a sentence in the residue field language, and hence equivalent over $\mathcal{T}_\infty$ with a tame formula.

Now let $\psi(x)$ be the formula

$$(D_1 \to \psi_1(x)) \wedge \cdots \wedge (D_n \to \psi_n(x)).$$

Then, ψ is equivalent over $\mathcal{T}_\infty$ with a tame formula and

$$\mathcal{T}_\infty \vdash \psi(x) \leftrightarrow P(x)$$

by the construction of the proof, and hence, P is tame over $\mathcal{T}_\infty$. □

Lemma 2.10. *Let $\sim$ be a definable equivalence relation over $\mathcal{T}_\infty^{(d)}$. Suppose that the imaginaries of $\sim$ are tame over $k((t))^{(d)}$ for each pseudo-finite field k of characteristic zero. Then the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty^{(d)}$.*

Proof. Although the proof is similar to the proof of Lemma 2.9, we give the details.

Suppose that $\sim$ is an equivalence relation in n variables and let $x = (x_1, \ldots, x_n)$ run over the valued field. For any tame formula $\psi(x, \xi, m)$, with ξ a tuple of residue field variables and m a tuple of variables running over $(G/nG)_{n=2,3,\ldots}$, let C_ψ be the sentence

$$(\forall x)(\exists \xi)(\exists m)(\forall y)(x \sim y \leftrightarrow \psi(y, \xi, m)).$$

By the supposition of the lemma and by compactness,

$$\mathcal{T}_\infty^{(d)} \vdash C_{\psi_1} \vee \cdots \vee C_{\psi_n}$$

for some formulae $\psi_1, \ldots, \psi_n$ and some n. By taking the tuples ξ and m big enough, we may suppose that the ψ_j have free variables included in x, ξ, m. For each $j = 1, \ldots, n$, let D_j be the sentence

$$C_{\psi_j} \wedge (\neg C_{\psi_1} \wedge \cdots \wedge \neg C_{\psi_{j-1}}),$$

where $\neg$ is the negation. Now let $\psi(x, \xi, m)$ be the formula

$$(D_1 \to \psi_1(x, \xi, m)) \wedge \cdots \wedge (D_n \to \psi_n(x, \xi, m)).$$

Then, ψ is $\mathcal{T}_\infty^{(d)}$-equivalent with a tame formula, because of the same reason as in the proof of Lemma 2.9, and

$$\mathcal{T}_\infty^{(d)} \vdash (\forall x)(\exists \xi)(\exists m)(\forall y)(x \sim y \leftrightarrow \psi(y, \xi, m))$$

by the construction of the proof. Hence, the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty^{(d)}$. □

Proposition 2.11 (criterion). *Let P be a definable relation over $\mathcal{T}_\infty$ that can be defined by $\theta(x)$ in $\mathcal{L}_{\mathrm{DP}}$ that is existential with respect to the valued field variables and the value group variables. Suppose that for each pseudo-finite field k of characteristic zero there exists $d \in \mathbb{N}_0$ such that for all $r \in \mathbb{N}_0$ with $\gcd(r, d) = 1$ we have for all tuples x over $k((t))$*

$$k((t)) \models \theta(x) \quad \leftrightarrow \quad k((t^{1/r})) \models \theta(x).$$

Then P is tame over $\mathcal{T}_\infty$.

Proof. By Lemma 2.9 it suffices to prove that P is tame over $k((t))$ for k an arbitrary pseudo-finite field k of characteristic zero. Note that P induces a relation on $k((t))^{(d)}$ which is defined by $\theta(x)$. By the hypothesis of the proposition,

$$k((t)) \models \theta(x) \quad \leftrightarrow \quad k((t))^{(d)} \models \theta(x).$$

Now apply Proposition 2.8 to θ to obtain a $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$-formula ψ without valued field and $(G/nG)_{n=2,3,\dots}$-quantifiers such that

$$k((t))^{(d)} \models \theta(x) \quad \leftrightarrow \quad k((t))^{(d)} \models \psi(x).$$

We may suppose that only maps ord_n with n dividing a power of d occur in ψ. For n dividing a power of d, the natural map from G/nG for $k((t))$ to G/nG for $k((t))^{(d)}$ is an isomorphism. Hence,

$$k((t)) \models \psi(x) \quad \leftrightarrow \quad k((t))^{(d)} \models \psi(x),$$

and thus, P is tame over $k((t))$. □

Conjecture 2.12. *Let $\sim$ be a definable equivalence relation over $\mathcal{T}_\infty$. If $\sim$ is finite and tame over $\mathcal{T}_\infty$, then the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty$.*

Conjecture 2.13. *Let $\sim$ be a definable equivalence relation over $\mathcal{T}_\infty^{(d)}$. If $\sim$ is finite over $\mathcal{T}_\infty^{(d)}$, then the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty^{(d)}$.*

Proposition 2.14. *Conjecture 2.13 implies Conjecture 2.12.*

Proof. This follows from Lemma 2.16, the 'transfer lemma'. □

Remark 2.15. If Conjecture 2.13 is true, then Theorem 1.1 follows from it, from Lemma A, from our 'criterion for tameness' (Proposition 2.11), and from Proposition 2.14, by a standard ultraproduct argument (this way, one thus avoids the use of Lemmas B and C of § 3). Conjectures 2.12 and 2.13 are related to elimination of imaginaries as studied in model theory. In fact, Hrushovski [**13**] recently proved Conjecture 2.13 by using techniques for obtaining elimination of imaginaries (cf. [**12**]).

Lemma 2.16 (transfer lemma). *Let $\sim$ be a definable equivalence relation over $\mathcal{T}_\infty$, which is tame over $\mathcal{T}_\infty$, say, defined by a tame formula $\theta(x,y)$. Then there exists a $d_0 \in \mathbb{N}_0$ such that for all multiples d of d_0 the formula $\theta(x,y)$ defines an equivalence relation $\sim_d$ over $\mathcal{T}_\infty^{(d)}$ such that the following assertions hold.*

(i) *For each model of the form $k((t))^{(d)}$ of $\mathcal{T}_\infty^{(d)}$, the relation $\sim_d$ is the union of the relations $\sim$ on the subfields $k((t^{1/r}))$.*

(ii) *If $\sim$ is finite over $\mathcal{T}_\infty$, then $\sim_d$ is finite over $\mathcal{T}_\infty^{(d)}$.*

(iii) *If the imaginaries of $\sim_{d_0}$ are tame over $\mathcal{T}_\infty^{(d_0)}$, then the imaginaries of $\sim_d$ are tame over $\mathcal{T}_\infty^{(d)}$ and the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty$.*

Proof. Let d_0 be the product of all the n such that the map ord_n occurs in θ. Then (i) follows immediately. Suppose now that $\sim$ is finite over $\mathcal{T}_\infty$. Then, by definition, there exists $N > 0$ such that in any model of $\mathcal{T}_\infty$ there are at most N equivalence classes. Let $\theta_N(a_1,\ldots,a_{N+1})$ be the tame formula $\bigvee_{i\neq j}\theta(a_i,a_j)$; if it holds for all tuples a_i then there are at most N equivalence classes. Since θ_N is a tame formula only involving ord_n with n dividing d_0, $\theta_N(a_1,\ldots,a_{N+1})$ holds for any tuples a_i in the models $k((t))^{(d)}$ and thus (ii) follows. Now suppose that the imaginaries of $\sim_{d_0}$ are tame over $\mathcal{T}_\infty^{(d_0)}$, say, the tame formula $\psi(x,\xi,m)$ satisfies

$$\mathcal{T}_\infty^{(d_0)} \vdash (\forall x)(\exists\xi)(\exists m)(\forall y)(x \sim_{d_0} y \leftrightarrow \psi(y,\xi,m)). \tag{2.16.3}$$

We may suppose that the only maps ord_n that occur in ψ are such that n divides a power of d_0. Take a tuple x over some model $k((t))^{(d)}$ of $\mathcal{T}_\infty^{(d)}$. Since $k((t))^{(d)} \subset k((t))^{(d_0)}$ and by (2.16.3), we can take ξ, m such that

$$k((t))^{(d_0)} \models (\forall y)(x \sim_{d_0} y \leftrightarrow \psi(y,\xi,m)).$$

For n dividing a power of d_0, the natural maps from G/nG for $k((t))$ to G/nG for $k((t))^{(d)}$ and from both these to G/nG for $k((t))^{(d_0)}$ are isomorphisms. Hence,

$$k((t))^{(d)} \models (\forall y)(x \sim_d y \leftrightarrow \psi(y,\xi,m))$$

and if moreover $x \in k((t))$ then also

$$k((t)) \models (\forall y)(x \sim y \leftrightarrow \psi(y,\xi,m)).$$

In other words,

$$k((t))^{(d)} \models (\forall x)(\exists\xi)(\exists m)(\forall y)(x \sim_d y \leftrightarrow \psi(y,\xi,m))$$

and

$$k((t)) \models (\forall x)(\exists\xi)(\exists m)(\forall y)(x \sim y \leftrightarrow \psi(y,\xi,m)),$$

which finishes the proof of (iii) by Proposition 2.10. □

3. Cohomological lemmas

When K is a field, we denote by K^{a} an algebraic closure of K. For a field K and a discrete topological group G with a continuous action of the Galois group $\mathrm{Gal}(K^{\mathrm{a}}/K)$, we denote the first cohomology set of G by $H^1(K,G)$.

Moreover, if L is a Galois extension of K, we will also consider $H^1(L,G)$ and $H^1(L/K,G(L))$, where the last one is the cohomology set of the group $G(L)$, of all elements in G which are fixed under $\mathrm{Gal}(K^{\mathrm{a}}/L)$, with respect to the obvious action of $\mathrm{Gal}(L/K)$ on $G(L)$.

For all these notions we refer to [**20**, I.5, III]. Note that these cohomology sets are pointed sets: they are equipped with a distinguished element, so that we can speak about exact sequences.

Proposition 3.1. *Let k be a pseudo-finite field of characteristic zero, and G a linear algebraic group over $K := k((t))$. Then we have*

(a) *there are only a finite number of fields between K and K^{a} of any given finite degree over K;*

(b) *$H^1(K,G)$ is finite;*

(c) *if G is semi-simple and simply connected, then $H^1(K,G) = 0$.*

Proof. (a) Let L be such a field of degree n over K and with ramification index e. Then $L = k'((t))(\sqrt[e]{at})$ with $[k' : k] = n/e$ and $a \in k'$. Thus $L \subset k''((t))(\sqrt[e]{t})$, where k'' contains all extensions of k' inside k^{a} with degree less than or equal to e.

(b) This follows from (a) and a theorem of Borel and Serre (see [**20**, III.4.3, Théorème 4]).

(c) A pseudo-finite field has cohomological dimension less than or equal to 1 (see, for example, [**10**, Corollary 10.19]). But a theorem of Bruhat–Tits [**4**] asserts that $H^1(K,G) = 0$, when G is semi-simple and simply connected over K, whenever K is a complete field with respect to a discrete valuation, whose residue field has cohomological dimension less than or equal to 1. (In the p-adic case this is known as Kneser's Theorem.) □

Lemma A. *Let k be a pseudo-finite field of characteristic zero, and G a linear algebraic group over $K = k((t))$. Then there exists an integer $d \geqslant 1$ having the following property: if L is any field extension of K with $n := [L : K] < \infty$, and $\gcd(d,n) = 1$, then the restriction map*

$$H^1(K,G) \xrightarrow{\text{res}} H^1(L,G)$$

is injective.

Proof. By twisting (see [**20**, I.5.3, III.1.3]) with cocycles in a set of representatives of $H^1(K,G)$, which is finite by Proposition 3.1, it suffices to prove that the kernel of res is zero. There are four cases, as detailed below.

(1) *G* is connected and reductive

Let G^{ss} be the (semi-simple) derived group of G, and G^{sc} the universal covering group of G^{ss}. Let T be the connected component of the centre of G. It is a torus. The kernel Δ of the epimorphism $T \times G^{\mathrm{sc}} \to G$, induced by the multiplication map $T \times G^{\mathrm{ss}} \to G$, is finite and contained in the centre of $T \times G^{\mathrm{sc}}$ (see [**3**, p. 325] and [**21**, 2.2.2.(3), p. 37]). This yields an exact sequence

$$0 \to \Delta \to T \times G^{\mathrm{sc}} \to G \to 0.$$

By Proposition 3.1 (c), this induces a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
H^1(K,\Delta) & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(K,G) & \longrightarrow & H^2(K,\Delta) \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
H^1(L,\Delta) & \longrightarrow & H^1(L,T) & \longrightarrow & H^1(L,G) & \longrightarrow & H^2(L,\Delta)
\end{array}$$

The cohomology sets of T and Δ are abelian groups and for these one can consider the corestriction maps. Let d be the least common multiple of the orders of the groups Δ and $H^1(K,T)$. Let L be a finite extension of K whose degree n is relatively prime to d. Then the first, second and fourth vertical arrows in the above diagram are injective, because composing them with the corestriction morphisms yields multiplication by n, which is bijective on elements that are annihilated by d. Each element of $H^1(K,T)$ which is mapped to zero in $H^1(L,G)$ belongs to the image of $H^1(K,\Delta)$, because corestriction commutes with the most left horizontal arrows in the above diagram. Straightforward diagram chasing now shows that the third vertical arrow has indeed a trivial kernel.

(2) *G* is connected

Let G_{u} be the unipotent radical of G. Because $H^1(K,\cdot)$ is zero on any twist of G_{u}, by [**20**, III.2.1, Proposition 6], we see that $H^1(K,G)$ injects into $H^1(K,G/G_{\mathrm{u}})$. Apply now the previous case to the connected reductive group G/G_{u} to obtain the desired result.

(3) *G* is finite

Since $H^1(K,G)$ is finite (by Proposition 3.1), there exists a finite Galois extension K' of K such that the inflation map (which is always injective)

$$H^1(K'/K, G(K')) \to H^1(K,G)$$

is bijective, and such that $G(K') = G$. Put $d = [K' : K]$. Let L be any finite extension of K whose degree n is relatively prime with d. Consider the commutative diagram

$$\begin{array}{ccc}
H^1(K'/K, G(K')) & \xrightarrow{\ \cong\ } & H^1(K,G) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{res}} \\
H^1(K'\cdot L/L, G(K'\cdot L)) & \longrightarrow & H^1(L,G)
\end{array}$$

where α is the obvious natural map. Note that α is a bijection because the natural map from $\mathrm{Gal}(K' \cdot L/L)$ to $\mathrm{Gal}(K'/K)$ is an isomorphism (since $K' \cap L = K$) and $G(K' \cdot L) = G(K') = G$. Moreover, the bottom horizontal map in the above diagram is injective, because it is an inflation map. Thus res is indeed injective.

(4) G is any linear algebraic group

Let G_0 be the identity component of G, and let $E = G/G_0$ be the finite quotient. Let K' be a finite Galois extension of K such that $E(K') = E$. Let d be a positive integer such that Lemma A is true for G replaced by G_0 (Case (2)), and for G replaced by E (Case (3)), and such that d is divisible by $[K' : K]$. For L any finite extension of K whose degree n is relatively prime with d, one has $E(K) = E(L)$. Indeed, the natural map from $\mathrm{Gal}(K' \cdot L/L)$ to $\mathrm{Gal}(K'/K)$ is an isomorphism since $K' \cap L = K$. The desired result is now obtained by straightforward diagram chasing in the following diagram with exact rows:

$$\begin{array}{ccccccc}
E(K) & \longrightarrow & H^1(K, G_0) & \longrightarrow & H^1(K, G) & \longrightarrow & H^1(K, E) \\
\Vert & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
E(L) & \longrightarrow & H^1(L, G_0) & \longrightarrow & H^1(L, G) & \longrightarrow & H^1(L, E)
\end{array}$$

This finishes the proof of Lemma A. □

Remark 3.2. From the above proof, and the material in the proof of Proposition 3.1, it is clear that Lemma A remains true when K is any complete characteristic zero field with respect to a discrete valuation, whose residue field has cohomological dimension less than or equal to 1, such that there are only a finite number of fields between K and K^{a} of any given finite degree over K. When K is a p-adic field and G connected, this was proved by a different method in a paper of Sansuc [**19**, Remarque 4.8.1].

Lemma B. *Let K be a field of characteristic zero, and let k be a subfield of K. Let G be a linear algebraic group over K. Suppose that G is obtained from a linear algebraic group G_k over k by base change and denote by $G(k^{\mathrm{a}})$ the group of rational points on G_k over the algebraic closure k^{a} of k in K^{a}. Then the natural map*

$$H^1(K, G(k^{\mathrm{a}})) \to H^1(K, G)$$

is surjective.

Proof. We consider the following three cases.

Case 1: G has a normal algebraic subgroup T defined over k which is a torus, such that $E := G/T$ is finite.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^1(K, T(k^{\mathrm{a}})) & \longrightarrow & H^1(K, G(k^{\mathrm{a}})) & \longrightarrow & H^1(K, E) \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \Vert \\
H^1(K, T) & \longrightarrow & H^1(K, G) & \longrightarrow & H^1(K, E)
\end{array}$$

Let $b \in H^1(K, G)$. We have to prove that b is in the image of α. Let c be the image of b in $H^1(K, E)$, and let $\bar{c}$ be a cocycle representing c. We denote by $\Delta(\bar{c})$ the image of $\bar{c}$ in $H^2(K, {}_{\bar{c}}T(k^{\mathrm{a}}))$ (cf. [**20**, I.5.6]), where ${}_{\bar{c}}T(k^{\mathrm{a}})$ is obtained from $T(k^{\mathrm{a}})$ by twisting with the cocycle $\bar{c}$.

Claim 1. The natural map from $H^2(K, {}_{\bar{c}}T(k^{\mathrm{a}}))$ to $H^2(K, {}_{\bar{c}}T)$ is injective. (Here ${}_{\bar{c}}T$ is obtained from T by twisting with $\bar{c}$.)

We prove this claim later, and first proceed with the proof of Case 1. The natural image of $\bar{c}$ in $H^2(K, {}_{\bar{c}}T)$ is zero, because c is the image of $b \in H^1(K, G)$. Hence, Claim 1 implies that $\Delta(\bar{c}) = 0$ in $H^2(K, {}_{\bar{c}}T(k^{\mathrm{a}}))$. Thus c belongs to the image of $H^1(K, G(k^{\mathrm{a}}))$ in $H^1(K, E)$, and there exists an $a \in H^1(K, G(k^{\mathrm{a}}))$ such that $\alpha(a)$ and b have the same image c in $H^1(K, E)$.

Let $\bar{a}$ be a cocycle representing a. We can twist the above diagram by the cocycle $\bar{a}$. Let ${}_{\bar{a}}G(k^{\mathrm{a}})$, ${}_{\bar{a}}G$, ${}_{\bar{a}}T(k^{\mathrm{a}})$, ${}_{\bar{a}}T$ be obtained from $G(k^{\mathrm{a}}), G, T(k^{\mathrm{a}}), T$, by twisting with $\bar{a}$. By [**20**, I.5.5, Corollary 2] we see that b is in the image of $H^1(K, {}_{\bar{a}}T)$, under the natural map from $H^1(K, {}_{\bar{a}}T)$ to $H^1(K, {}_{\bar{a}}G)$ composed with the canonical bijection between $H^1(K, {}_{\bar{a}}G)$ and $H^1(K, G)$. Hence, to prove that b is in the image of α, it suffices to prove the following claim.

Claim 2. The natural map $H^1(K, {}_{\bar{a}}T(k^{\mathrm{a}})) \to H^1(K, {}_{\bar{a}}T)$ is surjective.

We now prove Claim 2. Let w be any element of the abelian torsion group $H^1(K, {}_{\bar{a}}T)$ and let n be the order of w. Let T_n be the kernel of the nth power map on ${}_{\bar{a}}T(k^{\mathrm{a}})$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H^1(K, T_n) & \longrightarrow & H^1(K, {}_{\bar{a}}T(k^{\mathrm{a}})) & \xrightarrow{(\cdot)^n} & H^1(K, {}_{\bar{a}}T(k^{\mathrm{a}})) \\ \| & & \downarrow{\scriptstyle\alpha} & & \downarrow \\ H^1(K, T_n) & \longrightarrow & H^1(K, {}_{\bar{a}}T) & \xrightarrow{(\cdot)^n} & H^1(K, {}_{\bar{a}}T) \end{array}$$

Notice that w belongs to the image of the first arrow in the second row, because w belongs to the kernel of the second arrow in that row. This implies Claim 2. Next we turn to the proof of Claim 1.

Put $W = {}_{\bar{c}}T / {}_{\bar{c}}T(k^{\mathrm{a}})$. We have an exact sequence

$$H^1(K, W) \to H^2(K, {}_{\bar{c}}T(k^{\mathrm{a}})) \to H^2(K, {}_{\bar{c}}T).$$

Note that W is a uniquely divisible abelian group. Hence $H^1(K, W) = 0$. This finishes the proof of Claim 1, and thus also of Case 1.

Case 2: G is obtained by base change to K from a linear algebraic group over k whose identity component is reductive.

Let T be a maximal torus of G defined over k. Thus T is a Cartan subgroup of G and T has finite index in its normalizer N. A result of Springer (cf. [**20**, III.4.3, Lemma 6]) asserts that the natural map

$$H^1(K, N) \to H^1(K, G)$$

is surjective. Applying Case 1 to N finishes the proof of Case 2.

Case 3: G is obtained by base change to K from any linear algebraic group over k.

Let G_{u} be the unipotent radical of the identity component of G. Note that the natural map from $H^1(K, G)$ to $H^1(K, G/G_{\mathrm{u}})$ is injective (cf. the argument in the proof of Case 2 of Lemma A). Moreover, the natural map

$$H^1(K, G(k^{\mathrm{a}})/G_{\mathrm{u}}(k^{\mathrm{a}})) \to H^1(K, G/G_{\mathrm{u}})$$

is surjective by Case 2. Hence to prove Case 3, it suffices to prove the following claim.

Claim 3. The natural map

$$H^1(K, G(k^{\mathrm{a}})) \to H^1(K, G(k^{\mathrm{a}})/G_{\mathrm{u}}(k^{\mathrm{a}}))$$

is surjective.

We now turn to the proof of Claim 3. This is well known in the special case that $k = K$, cf. [**19**, Lemma 13]; for the general case we need a different argument. We may suppose that $G_{\mathrm{u}} \neq \{1\}$. Let C be the centre of G_{u}, then C is unipotent and $\dim C \geqslant 1$ (see, for example, [**14**, § 17]). Moreover, C is connected, because any unipotent linear algebraic group over a field of characteristic zero is connected (see [**17**, Chapter 3, § 2, Corollary 2 of Theorem 1]).

By induction on $\dim G$, it suffices to prove that the natural map

$$H^1(K, G(k^{\mathrm{a}})) \to H^1(K, G(k^{\mathrm{a}})/C(k^{\mathrm{a}}))$$

is surjective.

Using Proposition 41 in Chapter I.5.6 of [**20**], we see that it suffices to prove that $H^2(K, {}_{\bar{e}}C(k^{\mathrm{a}})) = 0$ for each cocycle $\bar{e}$ with values in $G(k^{\mathrm{a}})/C(k^{\mathrm{a}})$, where ${}_{\bar{e}}C$ is obtained from C by twisting with $\bar{e}$. But this is clear because the abelian group $C(k^{\mathrm{a}})$ is uniquely divisible, since it is unipotent (thus admitting a composition series with successive quotients isomorphic to $k^{\mathrm{a}}, +$). This terminates the proof of Lemma B. □

Lemma C. *Let k be a field of characteristic zero, and let G be a linear algebraic group over $K = k((t))$. Suppose that G is obtained from a linear algebraic group G_k over k by base change. Let k^{a} be the algebraic closure of k in K^{a} and let $\mathcal{O}$ be the valuation ring of K^{a}, i.e. the integral closure of $k[[t]]$ in K^{a}. Denote $G_k(k^{\mathrm{a}})$ by $G(k^{\mathrm{a}})$ and $G_k(\mathcal{O})$ by $G(\mathcal{O})$. Then the map*

$$H^1(K, G(\mathcal{O})) \to H^1(K, G(k^{\mathrm{a}})),$$

induced by reduction modulo the maximal ideal of $\mathcal{O}$, is injective.

Proof. Let I be the kernel of the reduction map $G(\mathcal{O}) \to G(k^{\mathrm{a}})$. We have an exact sequence

$$H^1(K, I) \to H^1(K, G(\mathcal{O})) \to H^1(K, G(k^{\mathrm{a}})).$$

Hence, it suffices to prove that $H^1(K, {}_cI) = 0$ for each 1-cocycle c with values in $G(\mathcal{O})$. Here ${}_cI$ denotes the twist of I by c.

Let L be any finite Galois extension of K, over which c is defined. We are going to prove that $H^1(L/K, {}_cI(L)) = 0$, where $I(L) = I \cap G(L)$. This implies that $H^1(K, {}_cI) = 0$.

We look at the filtration

$$I(L) = I_1 \triangleright I_2 \triangleright I_3 \triangleright \cdots \triangleright I_j \triangleright \cdots ,$$

with

$$I_j := \{g \in G(\mathcal{O}) \cap G(L) \mid g \equiv 1 \bmod \pi^j\},$$

for $j = 1, 2, \ldots$, where π is a generator for the maximal ideal of $\mathcal{O} \cap L$.

Note that I_j/I_{j+1} is an abelian group, for each $j \geqslant 1$. This is easily verified by identifying G with a subgroup of GL_n for a suitable n. Moreover, the abelian group I_j/I_{j+1} is uniquely divisible. Indeed this follows from Hensel's Lemma, because the map $G \to G : g \mapsto g^m$ induces multiplication by m on the tangent space of G at 1. Thus $H^1(L/K, {}_cI_j/{}_cI_{j+1}) = 0$, for all $j \geqslant 1$.

The exact sequence

$$0 \to \frac{{}_cI_j}{{}_cI_{j+1}} \to \frac{{}_cI_1}{{}_cI_{j+1}} \to \frac{{}_cI_1}{{}_cI_j} \to 0$$

induces an exact sequence

$$H^1\left(\frac{L}{K}, \frac{{}_cI_j}{{}_cI_{j+1}}\right) \to H^1\left(\frac{L}{K}, \frac{{}_cI_1}{{}_cI_{j+1}}\right) \to H^1\left(\frac{L}{K}, \frac{{}_cI_1}{{}_cI_j}\right).$$

Using induction on j, we conclude that

$$H^1(L/K, {}_cI_1/{}_cI_j) = 0 \quad \text{for all } j \geqslant 1. \tag{3.2.1}$$

Next we turn to the proof that $H^1(L/K, {}_cI(L)) = 0$. Let $a = (a_\sigma)_{\sigma \in \mathrm{Gal}(L/K)}$ be any 1-cocycle of $\mathrm{Gal}(L/K)$ with values in ${}_cI(L) = {}_cI_1$. From (3.2.1) it follows that for each $j \geqslant 1$ there exists $b_j \in {}_cI_1$ such that for all $\sigma \in \mathrm{Gal}(L/K)$ we have

$$a_\sigma \equiv b_j^{-1}\sigma(b_j) \mod \pi^j. \tag{3.2.2}$$

Let κ be the ramification index of L over K, and assume $\kappa \mid j$. We can choose a basis for $\mathcal{O} \cap \mathcal{L}$ over $k[\![t]\!]$, and write the components of the tuple b_j in terms of that basis (considering the coefficients as unknowns), and obtain in this way a system of polynomial equations whose solvability in $k[\![t]\!]/t^{j/\kappa}$ is equivalent with the existence of b_j satisfying (3.2.2). Applying Greenberg's theorem [**11**] to this system of equations, we see that there exists $b \in {}_cI_1$ such that

$$a_\sigma = b^{-1}\sigma(b)$$

for all $\sigma \in \mathrm{Gal}(L/K)$. This finishes the proof that $H^1(L/K, {}_cI(L)) = 0$. Lemma C is now proven. □

4. Definability of cohomology

For $\mathcal{L}$ a language and $a = (a_i)_i$ a tuple (or a set) of elements of an $\mathcal{L}$-structure, $\mathcal{L}(a)$ denotes the language $\mathcal{L}$ with extra constant symbols for the a_i.

Lemma 4.1. *Let k be a pseudo-finite field of characteristic zero. Let d be an integer. Then the field $K := k((t))^{(d)}$ has only finitely many field extensions of any given finite degree inside an algebraic closure K^{a} of K.*

Proof. The lemma follows from the fact that a field extension L of degree n of K is contained in $K_{m,n} := (k_m((t))^{(d)})[t^{1/n}]$, with k_m the unique field extension of k of degree m and with m big enough so that k_m contains k_n and all nth roots of 1. We leave the details to the reader. □

Let K be $k((t))^{(d)}$. Let e be a positive integer. Choose n such that all d^eth roots of 1 are in k_n, with k_n the unique degree n extension of k. Let M be the finite Galois extension of K:

$$M := k_n((t))^{(d)}.$$

Let L be the finite Galois extension of K:

$$L := M[t^{1/d^e}]. \tag{4.1.1}$$

Note that any finite field extension of K is contained in a field L as in (4.1.1).

Lemma 4.2. *Let $a_i \in k$, $i = 0, \dots, n-1$, be the coefficients of an irreducible polynomial $f_a(x) := x^n + \sum_{i=0}^{n-1} a_i x^i$ over k. Then, the field k_n, the group $\mathrm{Gal}(L/K)$, and its action on k_n are $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a)$-definable, that is, they are isomorphic to an $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a)$-definable field and group which acts definably on the field.*

For $i = 0, \dots, n-1$, let $a'_i \in \mathcal{O}_K$ be such that a'_i lies above a_i and let $a'_n \in \mathcal{O}_K$ be such that $\overline{\mathrm{ac}}(a'_n) = 1$ and such that the image of $\mathrm{ord}(a'_n)$ in G/d^eG is a generator of G/d^eG. Write $a = (a_1, \dots, a_{n-1})$ and $a' = (a'_1, \dots, a'_{n-1}, a'_n)$. Then, the field L with the action of $\mathrm{Gal}(L/K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a', a)$-definable, uniformly in a', that is, it is given by formulae in which the tuple a' may occur but which is independent of the choice of a'.

Proof. If ξ is a zero of f_a, k_n is the vector space k^n with multiplicative structure induced by the isomorphism $k^n \to \bigoplus_{i=0}^{n-1} \xi^i k$, which is independent of the choice of ξ. Hence, k_n is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a)$-definable.

The Galois group of k_n over k is cyclic of order n, say, with generator σ, and each of its powers σ^ℓ corresponds to a matrix $B_\ell = (b_{ij})$ in $GL_n(k)$ by

$$\sigma^\ell(\xi^j) = \sum_{i=0}^{n-1} b_{ij} \xi^i. \tag{4.2.2}$$

Since $\mathrm{Gal}(k_n/k)$ is commutative, each matrix B_ℓ is independent of the choice of the zero ξ. Moreover, the matrix subgroup $B := \{B_\ell\}_\ell$ of $GL_n(k)$ is independent of the choice of generator σ.

Clearly, the field M with the action of $\mathrm{Gal}(M/K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a_1', \dots, a_{n-1}')$-definable, uniformly in $a_1', \dots, a_{n-1}'$.

Now we use $a' = (a_1', \dots, a_n')$ to define L. For a_n' with $\overline{\mathrm{ac}}(a_n') = 1$ and such that the image of $\mathrm{ord}(a_n')$ in G/d^eG is a generator of G/d^eG, there exists by Hensel's lemma $\xi \in K$ and an integer b with $\gcd(b, d) = 1$ such that

$$t^b = \xi^{d^e} a_n'.$$

Hence, $M[t^{1/d^e}]$ is the same field as $M[(a_n')^{1/d^e}]$ within a fixed algebraic closure of M. Thus the field L is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a')$-definable, uniformly in a'.

Let $\mu_{d^e} \in k_n$ be a primitive d^eth root of unity. The Galois group of L over M is cyclic of order d^e and isomorphic to the multiplicative group (μ_{d^e}) generated by μ_{d^e}. Since $\mathrm{Gal}(L/K)$ is a semidirect product $(\mu_n) \rtimes B$, it is clearly $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a)$-definable. The action on k_n is clearly also $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a)$-definable. Thus the field L with the action of $\mathrm{Gal}(L/K)$ is clearly $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a')$-definable, uniformly in a'. □

For x_0 in $X(K)$ (or in $X(\mathcal{O}_K)$), let H_{x_0} be the stabilizer of x_0, considered as a linear group over K, respectively as a group scheme over $\mathcal{O}_K$. Following Serre [**20**, III.4.4], there is an injection Cor_{x_0} from the orbits in $X(K)$ under the action of $G(K)$ into $H^1(K, H_{x_0}(K^{\mathrm{a}}))$. Namely, $\mathrm{Cor}_{x_0}(\text{orbit of } x)$ with $x \in G(K)$ is defined to be the class of the cocycle

$$\sigma \mapsto \tau^{-1}\sigma(\tau), \tag{4.2.3}$$

where $\tau \in G(K^{\mathrm{a}})$ is such that $x = \tau(x_0)$.

Fix $x_0 \in X(\mathcal{O}_K)$ and let $\bar{H}_{x_0}$ be the reduction of H_{x_0} modulo the maximal ideal of $\mathcal{O}_K$. Since $H^1(K, H_{x_0}(K^{\mathrm{a}}))$ is finite by [**20**, III.4.3, Théorème 4] and Lemma 4.1, we may assume that L is big enough (and still of the form (4.1.1)) so that $H^1(K, H_{x_0}(K^{\mathrm{a}}))$ is naturally isomorphic to $H^1(L/K, H_{x_0}(L))$. Hence we can consider the diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{\quad p \quad} & X(K)/G(K) \\ & & \big\downarrow{\scriptstyle \mathrm{Cor}_{x_0}} \\ H^1(L/K, H_{x_0}(\mathcal{O}_L)) & \xrightarrow{\pi_{x_0}} & H^1(L/K, H_{x_0}(L)) \\ \big\downarrow{\scriptstyle i_{x_0}} & & \\ H^1(L/K, \bar{H}_{x_0}(k_n)) & & \end{array} \tag{4.2.4}$$

with $\mathcal{O}_L$ the valuation ring of L, and p, π_{x_0} and i_{x_0} the natural maps.

From now until the end of the proof of Theorem 5.1 we suppose that $F = \mathbb{Q}$. The treatment for general number fields F is completely similar but requires coefficients from F in all the valued field and residue field languages and respective definitions considered in § 2; we leave it to the reader to carry this out (cf. the comments preceding Theorem 5.2 or [**5**, **8**] for adding coefficients to a language).

To say that a certain subset of $X(K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-definable we work with a finite cover with affine charts of X, defined over $\mathbb{Q}$ (cf. [**8**] or [**5**]).

Proposition 4.3. *Let x be in $X(K)$. Suppose that there exists $x_0 \in X(\mathcal{O}_K)$ such that π_{x_0} is surjective and such that i_{x_0} is injective. Then there exists $c \in k^m$ for some m such that the orbit of x under the action of $G(K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-definable. (The point is that no direct reference to the point x can be made in $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$.)*

Proof. Let $E(x_0)$ be the condition on x_0 that $x_0 \in X(\mathcal{O}_K)$, that π_{x_0} is surjective, and that i_{x_0} is injective. Let x_0 satisfy E. By Serre [**20**, III.4.4] we know that Cor_{x_0} is injective. It is enough to prove that the condition

$$(\exists x_0)(E(x_0) \wedge i_{x_0}\pi_{x_0}^{-1} \mathrm{Cor}_{x_0} p(x) = i_{x_0}\pi_{x_0}^{-1} \mathrm{Cor}_{x_0} p(y)) \tag{4.3.5}$$

on $y \in G(K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-definable for some $c \in k^m$, since it cuts out the orbit of x. Take $a \in k^n$ as in Lemma 4.2. Let $D(a')$ be the condition on $a' = (a'_i)_{i=0}^n$ that $a'_i \in \mathcal{O}_K$ lies above a_i for $i < n$, and that $a'_n \in \mathcal{O}_K$ is such that $\overline{\mathrm{ac}}(a'_n) = 1$ and such that the image of $\mathrm{ord}(a'_n)$ in G/d^eG is a generator of G/d^eG. By Lemma 4.2, the field L with the action of $\mathrm{Gal}(L/K)$ on it is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a')$-definable, uniformly in a'. Also k_n, $\mathcal{O}_L$, the projection $\mathcal{O}_L \to k_n$, and the action of $\mathrm{Gal}(L/K)$ on k_n and on $\mathcal{O}_L$ are $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a')$-definable, uniformly in a'. Hence, $H_{x_0}(L)$, $\bar{H}_{x_0}(k_n)$, $H_{x_0}(\mathcal{O}_L)$, the natural maps $H_{x_0}(\mathcal{O}_L) \to \bar{H}_{x_0}(k_n)$ and $H_{x_0}(\mathcal{O}_L) \to H_{x_0}(L)$, and the action of $\mathrm{Gal}(L/K)$ on $H_{x_0}(L)$, $\bar{H}_{x_0}(k_n)$, and on $H_{x_0}(\mathcal{O}_L)$ are $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a', x_0)$-definable, uniformly in a' and x_0. Since $\mathrm{Gal}(L/K)$ is finite, that two given cocycles, a_σ, b_σ representing elements of one of the occurring $H^1(L/K, \cdot)$ are cohomologous is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a', x_0)$-definable over the graphs of a_σ and b_σ (more precisely, over all the entries of all the tuples in these graphs), uniformly in a', x_0, and in the graphs of a_σ and b_σ. One then readily checks that the condition $E(x_0)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, a')$-definable, uniformly in a', that is, it is given in the charts by formulae in which a' may occur but which is independent of the choice of a'.

Let a_σ be a cocycle representing an element of $H^1(L/K, \bar{H}_{x_0}(k_n))$ which lies in $i_{x_0}\pi_{x_0}^{-1} \mathrm{Cor}_{x_0} p(x)$, for some x_0 satisfying E. By identifying cocycles with their graph we may use them in formulae. In particular, (the graph of) a_σ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-definable for some $c \in k^m$. Let $F(a'_\sigma, x_0)$ be the property on a'_σ that a'_σ is a cocycle representing an element of $H^1(L/K, H_{x_0}(\mathcal{O}_L))$ that is mapped to the class of a_σ under i_{x_0}. Likewise, let $G(a'_\sigma, x_0, y)$ be the property on a'_σ, y that the class of a'_σ is mapped to $\mathrm{Cor}_{x_0} py$ under π_{x_0}. As before, the conditions F and G are $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, c, a', x_0)$-definable, uniformly in a' and x_0.

It is clear that

$$(\exists a')(\exists x_0)(\exists a'_\sigma)(D(a') \wedge E(x_0) \wedge F(a'_\sigma, x_0) \wedge G(a'_\sigma, x_0, y)) \tag{4.3.6}$$

is equivalent to (4.3.5). Moreover, (4.3.6) is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(a, c)$-definable, where a, c are the tuples in k as constructed above. This proves the proposition. □

5. Proof of the main results

We prove the following slight generalization of Theorem 1.1.

Theorem 5.1. *Let $U \subset X$ be an affine open, defined over F. Then there are finitely many regular functions $f_i : U \to \mathbb{A}_F^1$ and integers $N > 0$ and $d > 0$ such that, for any*

$K \in \mathcal{C}_N$ and any $x \in X(K)$, the set $G(K)(x) \cap U(K)$ depends only on $\overline{\mathrm{ac}}(f_i(x))$ and a statement in the language of groups on the $\mathrm{ord}(f_i(x)) \bmod d$, *that is, interpreted in the value group modulo d, and where* $\mathrm{ord}(0) \equiv 0 \bmod d$ *by convention.*

Proof. Recall that we suppose till the end of the proof that $F = \mathbb{Q}$. The proof for general F is completely similar but requires coefficients from F in all the valued field and residue field languages and respective definitions considered in § 2; we leave it to the reader to carry this out (cf. the comments preceding Theorem 5.2 or [**5**, **8**] for adding coefficients to a language).

For any field K over $\mathbb{Q}$ let $\sim_K$ be the equivalence relation on $X(K)$ such that two points are equivalent for $\sim_K$ if and only if they lie in the same orbit under the action of $G(K)$. Let $\sim$ be the equivalence relation over $\mathcal{T}_0$ whose interpretation is $\sim_K$ for any K which is a model of $\mathcal{T}_0$.

Let k be a pseudo-finite field of characteristic zero. Since X is absolutely irreducible, there exists $x_0 \in X(k)$.* Lemma A implies the conditions of Criterion 2.11 for $\sim_{k((t))}$ and for some d. This can be seen by applying Lemma A to the group H_{x_0} with $x_0 \in X(k)$ and by using the injectivity and naturality of Cor_{x_0}, as defined in (4.2.3) following [**20**, III.4.4]. Hence, the conclusion of Criterion 2.11 holds, that is, $\sim$ is tame over $\mathcal{T}_\infty$.

Now let K be $k((t))^{(d)}$ with $d > 0$. A straightforward adaptation of the proof of [**20**, I.2.2, Proposition 8] yields that

$$H^1(K, A) = \lim_{\rightarrow} H^1(k((t))[t^{1/r}], A),$$

when A is a discrete topological group with a continuous action of the Galois group $\mathrm{Gal}(K^{\mathrm{a}}/k((t)))$, and where r runs over all integers with $\gcd(r, d) = 1$, directed by divisibility, since the absolute Galois group of K is isomorphic to the projective limit of the absolute Galois groups of the $k((t))[t^{1/r}]$. (Note that formation of equivalence classes commutes with taking direct limits.)

Since direct limits preserve surjectivity and injectivity (in the category of sets), Lemmas B and C imply that, for $x_0 \in X(k) \subset X(\mathcal{O}_K)$ and for well chosen L, the map π_{x_0} is surjective and that the map i_{x_0} is injective in diagram (4.2.4). This holds for all $d > 0$.

Take $x \in U(K)$. Now apply Proposition 4.3. One finds some $c \in k^m$ for some m such that the orbit of x in $X(K)$ under the action of $G(K)$ is $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-definable. By the quantifier elimination result Proposition 2.8, this orbit is given by an $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(c)$-formula $\psi(c, y)$ where the only quantifiers that may occur in ψ run over the residue field. Since $\sim_K$ is finite by [**20**, III.4.3, Théorème 4] and Lemma 4.1, one obtains that the imaginaries of $\sim_K$ are tame over K by combining the formulae $\psi(c, y)$ for all orbits. Namely, let A_i, with $i = 1, \ldots, \ell$ for some ℓ, be the equivalence classes of $\sim_K$, associate with each A_i a formula $\psi_i(c_i, y)$, for some c_i running over k^m for some m as above, take a tuple of variables $\xi = (\xi_0, \xi_1, \ldots, \xi_m)$ running over k^{1+m}, and let ψ be the formula

$$\bigvee_{i=1}^{\ell} (\psi_i(\xi_1, \ldots, \xi_m, y) \wedge \xi_0 = i).$$

* This is the only place where the absolute irreducibility of X is used; see Remark 5.4.

Then

$$K \models (\forall x)(\exists \xi)(\exists m)(\forall y)(x \sim_K y \leftrightarrow \psi(y, \xi, m)). \tag{5.1.1}$$

In other words, the imaginaries of $\sim_K$ are tame over K.

By Lemma 2.10, the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty^{(d)}$ and this holds for any $d > 0$.

By taking an appropriate d and since $\sim$ is tame over $\mathcal{T}_\infty$, the transfer lemma (Lemma 2.16) asserts that the imaginaries of $\sim$ are tame over $\mathcal{T}_\infty$. Now Theorem 5.1 and thus Theorem 1.1 follow for some d, by an ultraproduct argument. □

Write $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$ for the language $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}$ together with coefficients from F in the valued field and residue field sort. We will prove Theorem 5.2 which is a slight generalization of Theorem 1.3. For the notion of definable subassignments and definable morphisms we refer to [**8**] and [**5**]. By $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable subassignments or $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable morphisms we mean definable subassignments, respectively definable morphisms, which are also definable in the language $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$; this means that they are given by finitely many $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-formulae in affine charts defined over F. In particular, one can take K-rational points on any $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable subassignment for any field K over F which carries an $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-structure and similarly for $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable morphisms.

Theorem 5.2. *Write X' for $X \otimes_F \operatorname{Spec}(F((t)))$. Let $f : X' \to \mathbb{A}^1_{F((t))}$ and $g : X' \to \mathbb{A}^1_{F((t))}$ be $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable morphisms. Let ω be a volume form on an affine open in X. Let W be a $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable subassignment of the functor which sends a field E over F to $X'(E((t)))$. For each $K \in \mathcal{C}_1$ and for each x in $X(K)$, under the condition of integrability for all $s > 0$, consider the orbital integral*

$$I_{K,x}(s) := \int_{G(K)(x)\cap W(K)} |g_K|\,|f_K|^s |\omega|_K, \tag{5.2.2}$$

with $|\omega|_K$ the measure on $X(K)$ associated with ω, and g_K, f_K and W_K the K-interpretations of f and W. If for some x and K this is not integrable, put $I_{K,x}(s) := 0$ for this K and x.

Then, the conclusion of Theorem 1.3 and its addendum holds.

Proof. We give two proofs. (In fact, the proof holds for tame integrals, that is, integrals over a tame domain of a tame integrand; cf. [**7**] and [**18**] for the notion of tame integrals and Remark 5.5 for an extension of this notion. That the domain of the integral (5.2.2) is tame follows from Theorem 5.1.)

Firstly, one obtains Theorem 5.2 from Theorem 5.1 by taking a suitable embedded resolution of singularities with normal crossings and calculate the integral on the resolution space (cf. [**7**]). Secondly, one obtains Theorem 5.2 from Theorem 5.1 by using the elementary method of Pas [**18**] to calculate $I_{K,x}(s)$ uniformly in $K \in \mathcal{C}_N$ for N big enough (cf. [**18**]). □

Remark 5.3. Note that the rationality of $I_{K,x}(s)$ as in Theorem 5.2 and the fact that only finitely many poles with bounded multiplicity can occur already follows from the rationality results for motivic integrals in [**5**], by enriching the Denef–Pas languages with constant symbols for a point in each of the orbits for the theory $\mathcal{T}_\infty$; this approach is based on Denef–Pas cell decomposition [**18**].

Remark 5.4. The fact that X is absolutely irreducible is not really needed for Theorem 1.1, Theorem 1.3, its addendum, or the results of this section. Indeed, instead of asking that X is absolutely irreducible, it is enough to ask that at least one irreducible component over F of X is absolutely irreducible. This implies that there exists a point in $X(k)$ for any pseudo-finite field k over F and this is all that the absolute irreducibility of X was used for.

Remark 5.5. In fact, in Theorem 5.2, one can consider still more general integrals for which the same conclusions hold. Indeed, the integrand in (5.2.2) may be a finite sum of terms of the form

$$\biggl(\prod_{j=1}^{\ell} \operatorname{ord}(h_{j,K})\biggr)|g_K|\,|f_K|^s|\omega|_K, \tag{5.5.3}$$

where the $|g_K|$, $|f_K|^s$ and $|\omega|_K$ are as in the theorem, the $h_j : X' \to \mathbb{G}_m$ are $\mathcal{L}_{\mathrm{DP}}^{\mathrm{tame}}(F)$-definable morphisms with K-interpretation $h_{j,K}$, and ord is the order on $K^\times$ extended by zero. This is so because an integral with integrand (5.5.3) over a tame domain can be rewritten as a tame integral over more variables with an integrand as in (5.2.2).

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