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Motivic integration in mixed characteristic with bounded ramification: a summary

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1.1 Introduction

Though one can consider Motivic Integration to have quite satisfactory foundations in residue characteristic zero after [9], [10] and [18], much remains to be done in positive residue characteristic. The aim of the present paper is to explain how one can extend the formalism and results from [9] to mixed characteristic.

Let us start with some motivation. Let K be a fixed finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and let K_d denote its unique unramified extension of degree d, for $d \geq 1$. Let us write by $\mathcal{O}_d$ the ring of integers of K_d and fix a polynomial $h \in \mathcal{O}_1[x_1, \cdots, x_m]$. For each d one can consider the Igusa local zeta function

$$Z_d(s) = \int_{\mathcal{O}_d^m} |h(x)|_d^s |dx|_d,$$

with $|\_|_d$ and $|dx|_d$ the corresponding norm and Haar measure such that the measure of $\mathcal{O}_d$ is 1 and such that $|a|_d$ for any $a \in K_d$ equals the measure of $a\mathcal{O}_d$. Meuser in [20] proved that there exist polynomials G and H in $\mathbb{Z}[T, X_1, \cdots, X_t]$ and complex numbers $\lambda_1, \cdots, \lambda_t$ such that, for all $d \geq 1$,

$$Z_d(s) = \frac{G(q^{-ds}, q^{d\lambda_1}, \cdots, q^{d\lambda_t})}{H(q^{-ds}, q^{d\lambda_1}, \cdots, q^{d\lambda_t})}.$$

Later Pas [22], [23] extended Meuser's result to more general integrals. In view of [15] and [16], it is thus natural to expect that there exists some motivic rational function $Z_{\mathrm{mot}}(T)$ with coefficients in some localization of the Grothendieck ring $G_{\mathbb{F}_q}$ of definable sets over $\mathbb{F}_q$ such that, for every $d \geq 1$, $Z_d(s)$ is obtained from $Z_{\mathrm{mot}}(T)$ by using the morphism $G_{\mathbb{F}_q} \to \mathbb{Z}$ counting rational points over $\mathbb{F}_{q^d}$ and letting T go to q^{-ds}. The theory presented here allows to prove such a result (more generally for h replaced by a definable function).

Another motivation for the present work lies in a joint project with J. Nicaise [12], where we prove some cases of a conjecture of Chai on the additivity of his base change conductor for semi-abelian varieties [2] and [3] by using Fubini's theorem for Motivic Integration from this paper for the mixed and from [9] for the equal characteristic zero case.

One of the achievements of motivic integration is the definition of measure and integrals on more general Henselian valued fields than just locally compact ones, for example on Laurent series fields over a characteristic zero field [17], [14], on complete discrete valuation rings with perfect residue field [19], [24], [21], and on algebraically closed valued fields [18]. A second achievement, since [16], and continued in [5] [9] [10], is that motivic integration can be used to interpolate p-adic integrals for all finite field extension of $\mathbb{Q}_p$ and integrals over $\mathbb{F}_q((t))$, uniformly in big primes p and its powers q. A third main achievement which goes together with the introduction of motivic additive characters and their motivic integrals, is the Transfer Principle of [10] which allows one to transfer equalities between integrals defined over $\mathbb{Q}_p$ to equalities of integrals defined over $\mathbb{F}_p((t))$ and vice versa. This is useful to change the characteristic in statements like the Fundamental Lemma in the Langlands program as is done in [4] [25]. In this paper we will focus on the first two mentioned achievements of motivic integration, in a mixed characteristic context. Firstly, for fixed prime p and integer $e > 0$, we will define the motivic measure and integrals on all Henselian discretely valued fields of mixed characteristic $(0, p)$ and ramification degree e, which will coincide with the standard measure in the case of p-adic fields. Secondly, our approach will be uniform in all unramified, Henselian field extensions, and hence, it will give an interpolation of p-adic integrals for all p-adic fields with ramification degree e.

Let us explain why the third mentioned achievement of motivic integration is not implemented.

Of course it would be possible to implement a motivic additive character and motivic 'oscillating' integrals involving the character in our context, but this is left out since it is natural to do this as a next step. However, a Transfer Principle would not make sense in the present context. We will explain this in the most basic instance of the Transfer Principle, namely in a concrete setting falling under the Ax-Kochen Ershov principle. Suppose that one has a definable subassignment φ as in [9], namely, a definable subset in the equicharacteristic zero valued field context. Then it is explained in [10] that, for p big enough, φ gives a subset of some Cartesian power of $\mathbb{Q}_p$, and similarly of some Cartesian power of $\mathbb{F}_p((t))$, which is independent of the choice of the formulas which give φ. This independence can be easily obtained from Gödel's Completeness Theorem (together with the knowledge of the appropriate theory of Henselian valued fields as used in loc. cit.). Indeed, the fact that two formulas ψ_1 and ψ_2 both give the same definable subassignment φ is a first order sentence, and hence, by Gödel's result, can be deduced from finitely many axioms from the theory. A finite collection of axioms can at most specify that the residue field characteristic is different from some concrete primes, but it can never specify that the characteristic is zero. Hence, the fields $\mathbb{Q}_p$ for p big enough are also models of this finite collection of axioms and hence ψ_1 and ψ_2 both give the same definable set over $\mathbb{Q}_p$ (and similarly over $\mathbb{F}_p((t))$) for big p. Note that this is a basic instance of having some result in equicharacteristic 0, and deriving the analogous result in big enough residue field characteristic. More generally the Transfer Principle transfers results about integrals from a $\mathbb{Q}_p$ context to an $\mathbb{F}_p((t))$ context and vice versa, even when the results not necessarily hold motivically in equicharacteristic zero. It is clear that such techniques are not applicable to our context since the fact that a valued field has mixed characteristic $(0, p)$ and ramification degree e is completely expressible by a finite set of axioms. Hence, we cannot change the characteristic for any of the properties obtained for integrals on the mixed characteristic valued fields. On the other hand, what we gain (as opposed to the equicharacteristic zero context of [9] and [10]), is that any motivic relation, calculation, equality, and so on, will hold for all the p-adic fields of the correct ramification degree and correct residue characteristic (as opposed to for p big enough as in [9] and [10]).

A basic tool in our approach is to use higher order angular components maps $\overline{\mathrm{ac}}_n$ for integers $n \geq 1$, already used by Pas in [22], where $\overline{\mathrm{ac}}_n$ is a certain multiplicative map from the valued field K to the residue ring $\mathcal{O}_K/\mathcal{M}_K^n$ with $\mathcal{M}_K$ the maximal ideal of the valuation ring $\mathcal{O}_K$. We use several structure results about definable sets and definable functions in first order languages involving the $\overline{\mathrm{ac}}_n$, one of which is called cell decomposition and goes back to [22] and [5]. Note that the approach of this paper with the $\overline{\mathrm{ac}}_n$ would also work in equicharacteristic zero discretely valued Henselian fields, and it has the advantage of providing much more definable sets than with $\overline{\mathrm{ac}} = \overline{\mathrm{ac}}_1$ only, for instance all cylinders over definable sets are definable with the $\overline{\mathrm{ac}}_n$, which is not the case if one uses only the usual angular component $\overline{\mathrm{ac}}$. In mixed characteristic there is a basic interplay between the residue characteristic p, the ramification degree e, and the angular component maps $\overline{\mathrm{ac}}_n$. Indeed, suppose for example that $p = 2$, and that we want to lift a root x_0 of a polynomial f by Hensel's Lemma. If it happens that $f'(x_0) = 2$ for some x_0, then we typically have to know that $f(x_0)$ is zero modulo 2^2 in order to uniquely lift x_0 to a zero of f. Hence, one should be able to speak about approximate roots modulo $\mathcal{M}_K^n$ with $n = 2e$. Such basic phenomena indicate the need of considering higher order residue rings in the setup, instead of only considering the residue field as in [9] [10].

Similarly as in [9] we consequently study families of motivic integrals, and we obtain many similar results as in [9]. However, we give a more direct approach to definitions and properties of the motivic measure and functions than in [9]: instead of the existence-uniqueness theorem of section 10 of [9], we explicitly define the motivic integrals and the integrability conditions and we do this step by step, as an iteration of more simple integrals. These explicit definitions give the same motivic measure and integrals as the ones that come from a direct image framework. Of course one has to be careful when translating conditions about integrability of an integral over a product space to conditions on the iterated integral. Since the value group is the group of integers, the origin of all integrability issues in our approach is summation on the integers, where one has the following Tonelli variant of Fubini's Theorem:

> Suppose that $f : \mathbb{Z}^{k+\ell} \to \mathbb{R}$ is a function taking non-negative values in $\mathbb{R}$. Then one has in $\mathbb{R} \cup \{+\infty\}$ that (without any convergence conditions)
>
> $$\sum_{(x,y)\in\mathbb{Z}^{k+\ell}} f(x,y) = \sum_{x\in\mathbb{Z}^k} g(x)$$
>
> with $g : \mathbb{Z}^k \to \mathbb{R} \cup \{+\infty\} : x \mapsto \sum_{y\in\mathbb{Z}^\ell} f(x,y)$. Hence, f is summable over $\mathbb{Z}^{k+\ell}$ if and only if g is real valued and summable over $\mathbb{Z}^k$.

Hence, the Tonelli variant of Fubini's Theorem for non-negatively valued functions translates integrability issues to iterated, more simple integrals. The notion of non-negativity for motivic functions is defined as in [9], namely, by using semi-rings as value rings of the motivic functions (as opposed to rings). Note that in a semi-ring, every element is considered as nonnegative, like one does in the semi-ring $\mathbb{N}$.

One new feature that does not appear in [9], and which provides more flexibility in view of future applications, is the usage of the abstract notion of $\mathcal{T}$-fields, where $\mathcal{T}$ stands for a first order theory. The reader has the choice to work with some of the listed more concrete examples of $\mathcal{T}$-fields (which are close to the concrete semi-algebraic setup of [9] or the subanalytic setup of [5]) or with axiomatic, abstract $\mathcal{T}$-fields. Thus, $\mathcal{T}$-fields allow one to work with more general theories $\mathcal{T}$ than the theories in the original work by Pas. Note that also this feature of integration on $\mathcal{T}$-fields would work similarly in the equicharacteristic zero context, as a generalization of [9].

Some of the highlights of our formalism coincide with the highlights of [9], consisting of a general change of variables formula, a general Fubini Theorem, the ability to specialize to previously known theories of motivic integration (e.g. as in [19]), the ability to interpolate many p-adic integrals, avoidance of any completion process on the Grothendieck ring level, and, very importantly, the ability to work in parameter set-ups where the parameters can come from the valued field, the residue field, and the value group (this last property has been very useful in [4] and [25]).

To make our work more directly comparable and linked with [9], we write down in Section 11 how our more direct definitions of integrable constructible motivic functions lead naturally to a direct image formalism, analogous to the one in [9]. Let us indicate how [9] and this paper complement each other, by an example. Having an equality between two motivic integrals as in [9] implies that the analogous equality will hold over all p-adic fields for p big enough and all fields $\mathbb{F}_q((t))$ of big enough characteristic (the lower bound can be computed but is usually very bad). This leaves one with finitely 'small' primes p, say, primes which are less than N. For the fields $\mathbb{F}_q((t))$ of small characteristic, very little is known in general and one must embark on a case by case study. On the other hand, in mixed characteristic, one could use the framework of this paper finitely many times to obtain the equality for all p-adic fields with residue characteristic less than N and bounded ramification degree. Note that knowing an equality for a small prime p and all possible ramification degrees is more or less equivalent to knowing it in $\mathbb{F}_p((t))$, which as we mentioned can be very hard.

We end Section 11 with a comparison with work by J. Sebag and the second author on motivic integration in a smooth, rigid, mixed characteristic context. This comparison plays a role in [12].

This is a summary and contains almost no proofs. We refer to our paper in preparation [11] for complete proofs. Also in [11], we will implement both the mixed characteristic approach of this paper and the equicharacteristic zero approach, generalizing and complementing [9].

1.2 A concrete setting

1.2.1

A discretely valued field L is a field with a surjective group homomorphism $\mathrm{ord} : L^\times \to \mathbb{Z}$, satisfying the usual axioms of a non-archimedean valuation. A ball in L is by definition a set of the form $\{x \in L \mid \gamma \leq \mathrm{ord}(x-a)\}$, where $a \in L$ and $\gamma \in \mathbb{Z}$. The collection of balls in L forms a base for the so-called valuation topology on L. The valued field L is called Henselian if its valuation ring $\mathcal{O}_L$ is a Henselian ring. Write $\mathcal{M}_L$ for the maximal ideal of $\mathcal{O}_L$.

In the whole paper we will work with the notion of $\mathcal{T}$-fields, which is more specific than the notion of discretely valued field, but which can come with additional structure if one wants. The reader who wants to avoid the formalism of $\mathcal{T}$-fields may skip Section 1.3 and directly go to Section 1.4 and use the following concrete notion of $(0,p,e)$-fields instead of $\mathcal{T}$-fields.

1.2.2 Definition Fix an integer $e > 0$ and a prime number p. A $(0,p,e)$-field is a Henselian, discretely valued field K of characteristic 0, residue field characteristic p, and ramification degree e, together with a chosen uniformizer π_K of the valuation ring $\mathcal{O}_K$ of K. That the ramification degree of K is e means that $\mathrm{ord}\pi_K^e = \mathrm{ord} p = e$.

Note that $\mathbb{Q}_p$ together with, for example, p as a uniformizer is a $(0,p,1)$-field, as well as the algebraic closure of $\mathbb{Q}$ inside $\mathbb{Q}_p$, or any unramified, Henselian field extension of $\mathbb{Q}_p$. A $(0,p,e)$-field K comes with natural so-called higher order angular component maps for $n \geq 1$,

$$\overline{\mathrm{ac}}_n : K^\times \to (\mathcal{O}_K \bmod \mathcal{M}_K^n) : x \mapsto \pi_K^{-\mathrm{ord}x} x \bmod \mathcal{M}_K^n$$

extended by $\overline{\mathrm{ac}}_n(0) = 0$. Sometimes one writes $\overline{\mathrm{ac}}$ for $\overline{\mathrm{ac}}_1$. Each map $\overline{\mathrm{ac}}_n$ is multiplicative on K and coincides on $\mathcal{O}_K^\times$ with the natural projection $\mathcal{O}_K \to \mathcal{O}_K/\mathcal{M}_K^n$.

1.2.3

To describe sets in a field independent way, we will use first order languages, where the following algebraic one is inspired by languages of Denef and Pas. Its name comes from the usage of higher order angular component maps, namely modulo positive powers of the maximal ideal. Consider the following basic language $\mathcal{L}_{\mathrm{high}}$ which has a sort for the valued field, a sort for the value group, and a sort for each residue ring of the valuation ring modulo π^n for integers $n > 0$. On the collection of these sorts, $\mathcal{L}_{\mathrm{high}}$ consists of the language of rings for the valued field together with a symbol π for the uniformizer, the language of rings for each of the residue rings, the Presburger language $(+,-,0,1,\leq,\{\cdot \equiv \cdot \bmod n\}_{n>1})$ for the value group, a symbol ord for the valuation map, symbols $\overline{\mathrm{ac}}_n$ for integers $n > 0$ for the angular component maps modulo the n-th power of the maximal ideal, and projection maps $p_{n,m}$ between the residue rings for $n \geq m$. On each $(0,p,e)$-field K, the language $\mathcal{L}_{\mathrm{high}}$ has its natural meaning, where π stands for π_K, ord for the valuation $K^\times \to \mathbb{Z}$, $\overline{\mathrm{ac}}_n$ for the angular component map $K \to \mathcal{O}_K/\mathcal{M}_K^n$, and $p_{n,m}$ for the natural projection map from $\mathcal{O}_K/\mathcal{M}_K^n$ to $\mathcal{O}_K/\mathcal{M}_K^m$.

Let $\mathcal{T}_{(0,p,e)}$ be the theory in the language $\mathcal{L}_{\mathrm{high}}$ of sentences that are true in all $(0,p,e)$-fields. Thus, in particular, each $(0,p,e)$-field is a model of $\mathcal{T}_{(0,p,e)}$. In this concrete setting, we let $\mathcal{T}$ be

$\mathcal{T}_{(0,p,e)}$ in the language $\mathcal{L}_{\text{high}}$, and $\mathcal{T}$-field means $(0,p,e)$-field. One can give a concrete list of axioms that imply the whole theory $\mathcal{T}_{(0,p,e)}$ (see [22]), but this is not relevant to this paper.

1.3 Theories on $(0,p,e)$-fields

In total we give three approaches to $\mathcal{T}$-fields in this paper, so that the reader can choose which one fits him best. The first one is the concrete setting of Section 1.2; the second one consists of a list of more general and more adaptable settings in Section 1.3.1, and the third approach is the axiomatic approach for theories and languages on $(0,p,e)$-fields in Section 1.3.2. Recall that for the first approach one takes $\mathcal{T} = \mathcal{T}_{(0,p,e)}$ in the language $\mathcal{L}_{\text{high}}$, and $\mathcal{T}$-field just means $(0,p,e)$-field.

1.3.1 A list of theories

In our second approach, we give a list of theories and corresponding languages which can be used throughout the whole paper.

1. Most closely related to the notion of $(0,p,e)$-fields is that of $(0,p,e)$-fields over a given ring R_0, for example a ring of integers, using the language $\mathcal{L}_{\text{high}}(R_0)$. Namely, for R_0 a subring of a $(0,p,e)$-field, let $\mathcal{L}_{\text{high}}(R_0)$ be the language $\mathcal{L}_{\text{high}}$ with coefficients (also called parameters) from R_0, and let $\mathcal{T}_{(0,p,e)}(R_0)$ be the theory of $(0,p,e)$-fields over R_0 in the language $\mathcal{L}_{\text{high}}(R_0)$. In this case one takes $\mathcal{T} = \mathcal{T}_{(0,p,e)}(R_0)$ with language $\mathcal{L}_{\text{high}}(R_0)$. By a $(0,p,e)$-field K over R_0 we mean in particular that the order and angular component maps on K extend the order and angular component maps on R_0.
2. In order to include analytic functions, let K be a $(0,p,e)$-field, and for each integer $n \geq 1$ let $K\{x_1,\ldots,x_n\}$ be the ring of those formal power series $\sum_{i\in\mathbb{N}^n} a_i x^i$ over K such that $\mathrm{ord}(a_i)$ goes to $+\infty$ whenever $i_1+\ldots+i_n$ goes to $+\infty$. Let $\mathcal{L}_K$ be the language $\mathcal{L}_{\text{high}}$ together with function symbols for all the elements of the rings $K\{x_1,\ldots,x_n\}$, for all $n > 0$. Each complete $(0,p,e)$-field L over K allows a natural interpretation of the language $\mathcal{L}_K$, where f in $K\{x_1,\ldots,x_n\}$ is interpreted naturally as a function from $\mathcal{O}_L^n$ to L. Let $\mathcal{T}_K$ be the theory in the language $\mathcal{L}_K$ of the collection of complete $(0,p,e)$-fields L over K. In this case one takes $\mathcal{T} = \mathcal{T}_K$ with language $\mathcal{L}_K$. For an explicit list of axioms that implies $\mathcal{T}_K$, see [5].
3. More generally than in the previous example, any of the analytic structures of [7] can be used for the language with corresponding theory $\mathcal{T}$, provided that $\mathcal{T}$ has at least one $(0,p,e)$-field as model.
4. For $\mathcal{T}_0$ and $\mathcal{L}_0$ as in any of the previous three items let $\mathcal{T}$ and $\mathcal{L}$ be any expansion of $\mathcal{T}_0$ and $\mathcal{L}_0$, which enriches $\mathcal{T}_0$ and $\mathcal{L}_0$ exclusively on the residue rings sorts. Suppose that $\mathcal{T}$ has at least one model which is a $(0,p,e)$-field.

For any of the listed theories in the corresponding languages, a $\mathcal{T}$-field is by definition a $(0,p,e)$-field that is a model for $\mathcal{T}$.

1.3.2 The axiomatic set-up

Our third approach to $\mathcal{T}$-fields consists of a list of axioms which should be fulfilled by an otherwise unspecified theory $\mathcal{T}$ in some language $\mathcal{L}$. The pairs of theories and languages for $(0,p,e)$-fields in the prior two approaches are examples of this axiomatic set-up by Proposition 1.3.10 (see Proposition 1.3.11 for more examples).

In our third approach, we start with a language $\mathcal{L}$ which contains $\mathcal{L}_{\rm high}$ and has the same sorts as $\mathcal{L}_{\rm high}$, and a theory $\mathcal{T}$ which contains $\mathcal{T}_{(0,p,e)}$ and which is formulated in the language $\mathcal{L}$. The sort for the valued field is called the main sort, and each of the other sorts (namely the residue ring sorts and the value group sort) are called auxiliary. It is important that no extra sorts are created along the way.

The list of axioms will be about all models of $\mathcal{T}$, and not only about $(0,p,e)$-fields. Note that any model $\mathcal{K}$ of the theory $\mathcal{T}_{(0,p,e)}$ with valued field K carries an interpretation of all the symbols of $\mathcal{L}_{\rm high}$ with the usual first order properties, even when K is not a $(0,p,e)$-field[1]. We will use the notation π_K, $\overline{\rm ac}_n$ and so on for the meaning of the symbols π and $\overline{\rm ac}_n$ of $\mathcal{L}_{\rm high}$, as well as the notion of balls, and so on, for all models of $\mathcal{T}_{(0,p,e)}$. The axioms below will involve parameters, which together with typical model theoretic compactness arguments will yield all the family-versions of the results we will need for motivic integration. To see in detail how such axioms are exploited, we refer to [8]. By definable, resp. A-definable, we will mean $\mathcal{L}$-definable without parameters, resp. $\mathcal{L}$-definable allowing parameters from A, unless otherwise stated.

The following two Jacobian properties treat close-to-linear (local) behavior of definable functions in one variable.

1.3.3 Definition (Jacobian property for a function) Let K be the valued field of a model of $\mathcal{T}_{(0,p,e)}$. Let $F : B \to B'$ be a function with $B, B' \subset K$. We say that F has the Jacobian property if the following conditions hold all together:

(i) F is a bijection and B, B' are balls in K,

(ii) F is C^1 on B,

(iii)

$$\frac{\partial F}{\partial x} \text{ is nonvanishing and } {\rm ord}\big(\frac{\partial F}{\partial x}\big) \text{ is constant on } B,$$

(iv) for all $x, y \in B$ with $x \neq y$, one has

$${\rm ord}\big(\frac{\partial F}{\partial x}\big) + {\rm ord}(x-y) = {\rm ord}(F(x)-F(y)).$$

If moreover $n > 0$ is an integer, we say that F has the n-Jacobian property if also the following hold

(v) $\overline{\rm ac}_n\big(\frac{\partial F}{\partial x}\big)$ is constant on B,

(vi) for all $x, y \in B$ one has

$$\overline{\rm ac}_n\big(\frac{\partial F}{\partial x}\big) \cdot \overline{\rm ac}_n(x-y) = \overline{\rm ac}_n(F(x)-F(y)).$$

Very often, the Jacobian property is used in families (with a model theoretic compactness argument), which explains our choice for the partial derivative notation in the above definition.

1.3.4 Definition (Jacobian property for $\mathcal{T}$) Say that the Jacobian property holds for the $\mathcal{L}$-theory $\mathcal{T}$ if for any model $\mathcal{K}$ with Henselian valued field K the following holds.

For any finite set A in $\mathcal{K}$ (serving as parameters in whichever sorts), any integer $n > 0$, and any A-definable function $F : K \to K$ there exists an A-definable function

$$f : K \to S$$

with S a Cartesian product of (the $\mathcal{K}$-universes of) sorts not involving K (these are also called

[1] This can happen, for example, when K is not discretely valued.

auxiliary sorts), such that each infinite fiber $f^{-1}(s)$ is a ball on which F is either constant or has the n-Jacobian property (as in Definition 1.3.3).

The following notion of $\mathcal{T}$ being split is related to the model-theoretic notions of orthogonality and stable embeddedness.

1.3.5 Definition (Split) Call $\mathcal{T}$ split if the following conditions hold for any model $\mathcal{K}$ with Henselian valued field K, value group Γ and residue rings $\mathcal{O}_K/\mathcal{M}_K^n$

(i). Any $\mathcal{K}$-definable subset of Γ^r is Γ-definable in the language $(+,-,0,<)$.
(ii). For any finite set A in $\mathcal{K}$, any A-definable subset $X \subset (\prod_{i=1}^s \mathcal{O}_K/\mathcal{M}_K^{m_i}) \times \Gamma^r$ is equal to a finite disjoint union of $Y_i \times Z_i$ where the Y_i are A-definable subsets of $\prod_{i=1}^s \mathcal{O}_K/\mathcal{M}_K^{m_i}$, and the Z_i are A-definable subsets of Γ^r.

The general notion of b-minimality is introduced in [8]. Here we work with a version which is more concretely adapted to the valued field setting.

1.3.6 Definition (Finite b-minimality) Call $\mathcal{T}$ finitely b-minimal if the following hold for any model $\mathcal{K}$ with Henselian valued field K. Each locally constant $\mathcal{K}$-definable function $g: K^\times \to K$ has finite image, and, for any finite set A in $\mathcal{K}$ (serving as parameters in whichever sorts) and any A-definable set $X \subset K$, there exist an integer n, an A-definable function

$$f: X \to S$$

with S a Cartesian product of (the $\mathcal{K}$-universes of) sorts not involving K (also called auxiliary sorts), and an A-definable function

$$c: S \to K$$

such that each nonempty fiber $f^{-1}(s)$ for $s \in S$ is either

1. equal to the singleton $\{c(s)\}$, or,
2. equal to the ball $\{x \in K \mid \overline{\mathrm{ac}}_n(x - c(s)) = \xi(s),\ \mathrm{ord}(x - c(s)) = z(s)\}$ for some $\xi(s)$ in $\mathcal{O}_K/\mathcal{M}_K^n$ and some $z(s) \in \Gamma$.

Note that in the above definition, the values $z(s)$ and $\xi(s)$ depend uniquely on s in the case that $f^{-1}(s)$ is a ball and can trivially be extended when $f^{-1}(s)$ is not a ball so that $s \mapsto z(s)$ and $s \mapsto \xi(s)$ can both be seen as A-definable functions on S.

1.3.7 Lemma *For any model $\mathcal{K}$ with valued field K of a finitely b-minimal theory, any definable function from a Cartesian product of (the $\mathcal{K}$-universes of) auxiliary sorts to K has finite image, and so does any definable, locally constant function from any definable set $X \subset K^n$ to K.*

1.3.8 Corollary *A finitely b-minimal theory is in particular b-minimal (as defined in [8]).*

Finally we come to the most general notion of $\mathcal{T}$-fields, namely the axiomatic one of our third approach.

1.3.9 Definition Let $\mathcal{T}$ be a theory containing $\mathcal{T}_{(0,p,e)}$ in a language $\mathcal{L}$ with the same sorts as $\mathcal{L}_{\mathrm{high}}$, which is split, finitely b-minimal, has the Jacobian property, and has at least one $(0,p,e)$-field as model. Then by a $\mathcal{T}$-field we mean a $(0,p,e)$-field which is a model of $\mathcal{T}$.

We have the following variant of the cell decomposition statement and related structure results on definable sets and functions of [7] for our more concrete theories.

1.3.10 Theorem ([7]) *The theory $\mathcal{T}_{(0,p,e)}$ as well as the listed theories in 1.3.1 satisfy the conditions of Definition 1.3.9.*

Finally we indicate how one can create new theories with properties as in Definition 1.3.9.

1.3.11 Proposition *Let $\mathcal{T}$ be a theory that satisfies the conditions of Definition 1.3.9. Then so does the theory $\mathcal{T}(R)$ in the language $\mathcal{L}(R)$ for any ring R which is a subring of a $\mathcal{T}$-field, where $\mathcal{T}(R)$ is the theory of all $\mathcal{T}$-fields which are algebras over R (and which extend* ord *and the* $\overline{\mathrm{ac}}_n$ *on R).*

From now on we fix one of the notions of $\mathcal{T}$, $\mathcal{L}$, and $\mathcal{T}$-fields as in Definition 1.3.9 for the rest of the paper, which includes the possibility of $\mathcal{T}$ and $\mathcal{L}$ being as in Sections 1.2, 1.3.1, or as in Proposition 1.3.11. We will often write K for a $\mathcal{T}$-field instead of writing the pair K, π_K where π_K is a uniformizer of $\mathcal{O}_K$.

1.4 Definable subassignments and definable morphisms

1.4.1

We recall that definable means $\mathcal{L}$-definable without parameters[2]. For any integers $n, r, s \geq 0$ and for any tuple $m = (m_1, \ldots, m_s)$ of nonnegative integers, denote by $h[n, m, r]$ the functor sending a $\mathcal{T}$-field K to

$$h[n, m, r](K) := K^n \times (\mathcal{O}_K/\mathcal{M}_K^{m_1}) \times \cdots \times (\mathcal{O}_K/\mathcal{M}_K^{m_s}) \times \mathbb{Z}^r.$$

The data of a subset X_K of $h[n, m, r](K)$ for each $\mathcal{T}$-field K is called a definable subasssignment (in model theory sometimes loosely called a definable set), if there exists an $\mathcal{L}$-formula φ in tuples of free variables of the corresponding lengths and in the corresponding sorts such that X_K equals $\varphi(K)$, the set of the points in $h[n, m, r](K)$ satisfying φ.

An example of a definable subassignment of $h[1, 0, 0]$ is the data of the subset $P_2(K) \subset K$ consisting of the nonzero squares in K for each $\mathcal{T}$-field K, which can be described by the formula $\exists y(y^2 = x \wedge x \neq 0)$ in one free variable x and one bounded variable y, both running over the valued field[3].

A definable morphism $f : X \to Y$ between definable subassignments X and Y is given by a definable subassignment G such that $G(K)$ is the graph of a function $X(K) \to Y(K)$ for any $\mathcal{T}$-field K. We usually write f for the definable morphism, $\mathrm{Graph}(f)$ for G, and f_K for the function $X(K) \to Y(K)$ with graph $G(K)$. A definable isomorphism is by definition a definable morphism which has an inverse.

Denote by Def (or $\mathrm{Def}(\mathcal{T})$ in full) the category of definable subassignments with the definable morphisms as morphisms. More generally, for Z a definable subassignment, denote by Def_Z the category of definable subassignments X with a specified definable morphism $X \to Z$ to Z, with as morphisms between X and Y the definable morphisms which make commutative diagrams with the specified $X \to Z$ and $Y \to Z$. We will often use the notation $X_{/Z}$ for X in Def_Z. In the prior publications [9] and [10], we consequently used the notation $X \to Z$ instead of the shorter $X_{/Z}$.

For every morphism $f : Z \to Z'$ in Def, composition with f defines a functor $f_! : \mathrm{Def}_Z \to \mathrm{Def}_{Z'}$, sending $X_{/Z}$ to $X_{/Z'}$. Also, fiber product defines a functor $f^* : \mathrm{Def}_{Z'} \to \mathrm{Def}_Z$, namely, by sending

[2] Note that parameters from, for example, a base ring can be used, see Section 1.3.1 and Proposition 1.3.11.

[3] Note that, as is standard, to determine $\varphi(K)$, each variable occurring in φ (thus also the variables which are bound by a quantifier and hence not free), runs over exactly one set out of K, $\mathbb{Z}$, or a residue ring $\mathcal{O}_K/\mathcal{M}_K^{\ell}$.

$Y_{/Z'}$ to $(Y \otimes_{Z'} Z)_{/Z}$, where for each $\mathcal{T}$-field K the set $(Y \otimes_{Z'} Z)(K)$ is the set-theoretical fiber product of $Y(K)$ with $Z(K)$ over $Z'(K)$ with the projection as specified function to $Z(K)$.

Let Y and Y' be in Def. We write $Y \times Y'$ for the subassignment corresponding to the Cartesian product and we write $Y[n,m,r]$ for $Y \times h[n,m,r]$. (We fix in the whole paper h to be the definable subassignment of the singleton $\{0\}$, that is, $h(K) = \{0\} = K^0$ for all K, so that $h[n,m,r]$, as previously defined, is compatible with the notation of $Y[n,m,r]$ for general Y.)

By a point on a definable subassignment X we mean a tuple $x = (x_0, K)$ where K is a $\mathcal{T}$-field and x_0 lies in $X(K)$. We denote $|X|$ for the collection of all points that lie on X.

1.4.2 Dimension

Since $\mathcal{T}$ is in particular b-minimal in the sense of [8] by Corollary 1.3.8, for each $\mathcal{T}$-field K and each definable subassignment φ we can take the dimension of $\varphi(K)$ to be as defined in [8], and use the dimension theory from [8]. In the context of finite b-minimality, for nonempty and definable $X \subset h[n,m,r](K)$, this dimension is defined by induction on n, where for $n=0$ the dimension of X is defined to be zero, and, for $n=1$, $\dim X = 1$ if and only if $p(X)$ contains a ball where $p : h[1,m,r](K) \to K$ is the coordinate projection, and one has $\dim X = 0$ otherwise. For general $n \geq 1$, the dimension of such X is the maximal number $r > 0$ such that for some coordinate projection $p : h[n,m,r](K) \to K^r$, $p(X)$ contains a Cartesian product of r balls if such r exists and the dimension is 0 otherwise. Note that a nonempty definable $X \subset h[n,m,r](K)$ has dimension zero if and only if it is a finite set.

The dimension of a definable subassignment φ itself is defined as the maximum of all $\varphi(K)$ when K runs over all $\mathcal{T}$-fields.

For $f : X \to Y$ a definable morphism and K a $\mathcal{T}$-field, the relative dimension of the set $X(K)$ over $Y(K)$ (of course along f_K) is the maximum of the dimensions of the fibers of f_K, and the relative dimension of the definable assignment X over Y (along f) is the maximum of these over all K.

One has all the properties of [8] for the dimensions of the sets $\varphi(K)$ and the related properties for the definable subassignments themselves, analogous to the properties of the so-called K-dimension of [9].

1.5 Summation over the value group

We consider a formal symbol $\mathbb{L}$ and the ring

$$\mathbb{A} := \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1-\mathbb{L}^{-i}}\Big)_{i>0}\Big],$$

as subring of the ring of rational functions in $\mathbb{L}$ over $\mathbb{Q}$. Furthermore, for each real number $q > 1$, we consider the ring morphism

$$\theta_q : \mathbb{A} \to \mathbb{R} : r(\mathbb{L}) \mapsto r(q),$$

that is, one evaluates the rational function $r(\mathbb{L})$ in $\mathbb{L}$ at q.

Recall that $h[0,0,1]$ can be identified with $\mathbb{Z}$, since $h[0,0,1](K) = \mathbb{Z}$ for all $\mathcal{T}$-fields K. Let S be in Def, that is, let S be a definable subassignment. A definable morphism $\alpha : S \to h[0,0,1]$ gives rise to a function (also denoted by α) from $|S|$ to $\mathbb{Z}$ which sends a point (s, K) on S to $\alpha_K(s)$. Likewise, such α gives rise to the function $\mathbb{L}^\alpha$ from $|S|$ to $\mathbb{A}$ which sends a point (s,K) on S to $\mathbb{L}^{\alpha_K(s)}$.

We define the ring $\mathcal{P}(S)$ of constructible Presburger functions on S as the subring of the ring of functions $|S| \to \mathbb{A}$ generated by

1. all constant functions into $\mathbb{A}$,
2. all functions $\alpha : |S| \to \mathbb{Z}$ with $\alpha : S \to h[0,0,1]$ a definable morphism,
3. all functions of the form $\mathbb{L}^{\beta}$ with $\beta : S \to h[0,0,1]$ a definable morphism.

Note that a general element of $\mathcal{P}(S)$ is thus a finite sum of terms of the form $a\mathbb{L}^{\beta}\prod_{i=1}^{\ell}\alpha_i$ with $a \in \mathbb{A}$, and the β and α_i definable morphisms from S to $h[0,0,1] = \mathbb{Z}$.

For any $\mathcal{T}$-field K, any $q > 1$ in $\mathbb{R}$, and f in $\mathcal{P}(S)$ we write $\theta_{q,K}(f) : S(K) \to \mathbb{R}$ for the function sending $s \in S(K)$ to $\theta_q(f(s,K))$.

Define a partial ordering on $\mathcal{P}(S)$ by setting $f \geq 0$ if for every $q > 1$ in $\mathbb{R}$ and every s in $|S|$, $\theta_q(f(s)) \geq 0$. We denote by $\mathcal{P}(S)_+$ the set $\{f \in \mathcal{P}(S) \mid f \geq 0\}$. Write $f \geq g$ if $f - g$ is in $\mathcal{P}_+(S)$. Similarly, write $\mathbb{A}_+$ for the sub-semi-ring of $\mathbb{A}$ consisting of the non-negative elements of $\mathbb{A}$, namely those elements a with $\theta_q(a) \geq 0$ for all real $q > 1$.

Recall the notion of summable families in $\mathbb{R}$ or $\mathbb{C}$, cf. [1] VII.16. In particular, a family $(z_i)_{i\in I}$ of complex numbers is summable if and only if the family $(|z_i|)_{i\in I}$ is summable in $\mathbb{R}$.

We shall say a function φ in $\mathcal{P}(h[0,0,r])$ is integrable if for each $\mathcal{T}$-field K and for each real $q > 1$, the family $(\theta_{q,K}(\varphi)(i))_{i\in\mathbb{Z}^r}$ is summable.

More generally we shall say a function φ in $\mathcal{P}(S[0,0,r])$ is S-integrable if for each $\mathcal{T}$-field K, for each real $q > 1$, and for each $s \in S(K)$, the family $(\theta_{q,K}(\varphi)(s,i))_{i\in\mathbb{Z}^r}$ is summable. The latter notion of S-integrability is key to all integrability notions in this paper.

We denote by $\mathrm{I}_S\mathcal{P}(S[0,0,r])$ the collection of S-integrable functions in $\mathcal{P}(S[0,0,r])$. Likewise, we denote by $\mathrm{I}_S\mathcal{P}_+(S[0,0,r])$ the collection of S-integrable functions in $\mathcal{P}_+(S[0,0,r])$. Note that $\mathrm{I}_S\mathcal{P}(S[0,0,r])$, resp. $\mathrm{I}_S\mathcal{P}_+(S[0,0,r])$, is a $\mathcal{P}(S)$-module, resp. a $\mathcal{P}_+(S)$-semi-module.

The following is inspired by results in [13] and appears in this form in [9].

1.5.1 Theorem-Definition For each φ in $\mathrm{I}_S\mathcal{P}(S[0,0,r])$ there exists a unique function $\psi = \mu_{/S}(\varphi)$ in $\mathcal{P}(S)$ such that for all $q > 1$, all $\mathcal{T}$-fields K, and all s in $S(K)$

$$\theta_{q,K}(\psi)(s) = \sum_{i\in\mathbb{Z}^r}\theta_{q,K}(\varphi)(s,i). \tag{1.1}$$

Moreover, the mapping $\varphi \mapsto \mu_{/S}(\varphi)$ yields a morphism of $\mathcal{P}(S)$-modules

$$\mu_{/S} : \mathrm{I}_S\mathcal{P}(S\times\mathbb{Z}^r) \longrightarrow \mathcal{P}(S).$$

The proof of 1.5.1 is based on finite b-minimality, the fact that $\mathcal{T}$ is split, and explicit calculations, mainly of geometric series and their derivatives. Clearly, the above map $\mu_{/S}$ sends $\mathrm{I}_S\mathcal{P}_+(S\times\mathbb{Z}^r)$ to $\mathcal{P}_+(S)$. For Y a definable subassignment of S, we denote by $\mathbf{1}_Y$ the function in $\mathcal{P}(S)$ with value 1 on Y and zero on $S \setminus Y$. We shall denote by $\mathcal{P}^0(S)$ (resp. $\mathcal{P}^0_+(S)$) the subring (resp. sub-semi-ring) of $\mathcal{P}(S)$ (resp. $\mathcal{P}_+(S)$) generated by the functions $\mathbf{1}_Y$ for all definable subassignments Y of S and by the constant function $\mathbb{L} - 1$.

If $f : Z \to Y$ is a morphism in Def, composition with f yields natural pullback morphisms $f^* : \mathcal{P}(Y) \to \mathcal{P}(Z)$ and $f^* : \mathcal{P}_+(Y) \to \mathcal{P}_+(Z)$. These pullback morphisms and the subrings $\mathcal{P}^0(S)$ will play a role for the richer class of motivic constructible functions. First we turn our attention to another ingredient for motivic constructible functions, coming from the residue rings. Afterwards we will glue these two ingredients along the common subrings $\mathcal{P}^0_+(S)$ to define motivic constructible functions.

1.6 Integration over the residue rings

On the integers side we have defined rings of (nonnegative) constructible Presburger functions $\mathcal{P}_+(\cdot)$ and a summation procedure over subsets of $\mathbb{Z}^r$. On the residue rings side we will proceed differently.

Let Z be a definable subassignment in Def. Define the semi-group $\mathcal{Q}_+(Z)$ as the quotient of the free abelian semi-group over symbols $[Y]$ with $Y_{/Z}$ a subassignment of $Z[0,m,0]$ for some $m=(m_1,\dots,m_s)$ with $m_i \geq 0$ and $s \geq 0$, with as distinguished map from Y to Z the natural projection, by the following relations.

$$[\emptyset] = 0, \text{ where } \emptyset \text{ is the empty subassignment.} \tag{1.2}$$

$$[Y] = [Y'] \tag{1.3}$$

if there exists a definable isomorphism $Y \to Y'$ which commutes with the projections $Y \to Z$ and $Y' \to Z$.

$$[Y_1 \cup Y_2] + [Y_1 \cap Y_2] = [Y_1] + [Y_2] \tag{1.4}$$

for Y_1 and Y_2 definable subassignments of a common $Z[0,m,0]$ for some m.

$$[Y[0,m',0]] = [Y'] \tag{1.5}$$

if for the projection $p : Z[0,m+m',0] \to Z[0,m,0]$ one has $Y' = p^{-1}(Y)$ for some definable $Y \subset Z[0,m,0]$ and some $m_i, m_i' \geq 0$.

We will still write $[Y]$ for the class of $[Y]$ in $\mathcal{Q}_+(Z)$ for $Y \subset Z[0,m,0]$. In [9], the longer notation $SK_0(\mathrm{RDef}_Z)$ is used instead of $\mathcal{Q}_+(Z)$. Note that in [9] relation (1.5) is left out since it is redundant if one only uses $\overline{\mathrm{ac}}_1$ instead of all the $\overline{\mathrm{ac}}_n$. The semi-group $\mathcal{Q}_+(Z)$ carries a semi-ring structure with multiplication for $Y \subset Z[0,m,0]$ and $Y' \subset Z[0,m',0]$ given by

$$[Y]\cdot[Y'] := [Y \otimes_Z Y'],$$

where the fibre product is taken along the coordinate projections to Z. Similarly, for $f : Z_1 \to Z_2$ any morphism in Def, there is a natural pullback homomorphism of semi-rings $f^* : \mathcal{Q}_+(Z_2) \to \mathcal{Q}_+(Z_1)$ which sends $[Y]$ for some $Y \subset Z_2[0,m,0]$ to $[Y \otimes_{Z_2} Z_1]$. Write $\mathbb{L}$ for the class of $Z[0,1,0]$ in $\mathcal{Q}_+(Z)$. Then, by relation (1.5), one has that the class of $Z[0,m,0]$ in $\mathcal{Q}_+(Z)$ equals $\mathbb{L}^{|m|}$ with $m=(m_i)_i$ and $|m| = \sum_i m_i$. Clearly, for each $a \in \mathcal{Q}_+(Z)$, there exists a tuple m and a $Y \subset Z[0,m,0]$ such that $a=[Y]$.

To preserve a maximum of information at the level of the residue rings, we will integrate functions in $\mathcal{Q}_+(\cdot)$ over residue ring variables in a formal way. Suppose that $Z = X[0,k,0]$ for some tuple k, let a be in $\mathcal{Q}_+(Z)$ and write a as $[Y]$ for some $Y \subset Z[0,n,0]$. We write $\mu_{/X}$ for the corresponding formal integral in the fibers of the coordinate projection $Z \to X$

$$\mu_{/X} : \mathcal{Q}_+(Z) \to \mathcal{Q}_+(X) : [Y] \to [Y],$$

where the class of Y is first taken in $\mathcal{Q}_+(Z)$ and then in $\mathcal{Q}_+(X)$. Note that this allows one to integrate functions from $\mathcal{Q}_+$ over residue ring variables, but of course not over valued field neither over value group variables. To integrate over any kind of variables, we will need to combine the value group part $\mathcal{P}_+$ and the residue rings part $\mathcal{Q}_+$.

1.7 Putting $\mathcal{P}_+$ and $\mathcal{Q}_+$ together to form $\mathscr{C}_+$

Many interesting functions on Henselian valued fields have a component that comes essentially from the value group and one that comes from residue rings. For Z in Def, we will glue the pieces $\mathcal{P}_+(Z)$ and $\mathcal{Q}_+(Z)$ together by means of the common sub-semi-ring $\mathcal{P}^0_+(Z)$. Recall that $\mathcal{P}^0_+(Z)$ is the sub-semi-ring of $\mathcal{P}_+(Z)$ generated by the characteristic functions $\mathbf{1}_Y$ for all definable subassignments $Y \subset Z$ and by the constant function $\mathbb{L} - 1$.

Using the canonical semi-ring morphism $\mathcal{P}^0_+(Z) \to \mathcal{Q}_+(Z)$, sending $\mathbf{1}_Y$ to $[Y]$ and $\mathbb{L}-1$ to $\mathbb{L}-1$, we define the semi-ring $\mathscr{C}_+(Z)$ as

$$\mathcal{P}_+(Z) \otimes_{\mathcal{P}^0_+(Z)} \mathcal{Q}_+(Z).$$

We call elements of $\mathscr{C}_+(Z)$ (nonnegative) constructible motivic functions on Z.

If $f : Z \to Y$ is a morphism in Def, we find natural pullback morphisms $f^* : \mathscr{C}_+(Y) \to \mathscr{C}_+(Z)$, by the tensor product definition of $\mathscr{C}_+(\cdot)$. Namely, f^* maps $\sum_{i=1}^r a_i \otimes b_i$ to $\sum_i f^*(a_i) \otimes f^*(b_i)$, where $a_i \in \mathcal{P}_+(Y)$ and $b_i \in \mathcal{Q}_+(Y)$.

1.7.1 Interpretation in non-archimedean local fields.

An important feature of our setting (as well as in the settings of [9] and [16]) is that the motivic constructible functions and their integrals interpolate actual functions and their integrals on non-archimedean local fields, and even more generally on $\mathcal{T}$-fields with finite residue field.

Let $X \subset h[n, m, r]$ be in Def, let φ be in $\mathscr{C}_+(X)$, and let K be a $\mathcal{T}$-field with finite residue field. In this case φ gives rise to an actual set-theoretic function φ_K from $X(K)$ to $\mathbb{Q}_{\geq 0}$, defined as follows:

For a in $\mathcal{P}_+(X)$, one gets $a_K : X(K) \to \mathbb{Q}_{\geq 0}$ by replacing $\mathbb{L}$ by q_K, the number of elements in the residue field of K.

For $b = [Y]$ with Y a subassignment of $X[0, m, 0]$ in $\mathcal{Q}_+(X)$, if one writes $p : Y(K) \to X(K)$ for the projection, one defines $b_K : X(K) \to \mathbb{Q}_{\geq 0}$ by sending $x \in X(K)$ to $\#(p^{-1}(x))$, that is, the number of points in $Y(K)$ that lie above $x \in X(K)$.

For our general φ in $\mathscr{C}_+(X)$, write φ as a finite sum $\sum_i a_i \otimes b_i$ with $a_i \in \mathcal{P}_+(X)$ and $b_i \in \mathcal{Q}_+(X)$. Our general definitions are such that the function

$$\varphi_K : X(K) \to \mathbb{Q}_{\geq 0} : x \mapsto \sum_i a_{iK}(x) \cdot b_{iK}(x)$$

does not depend on the choices made for a_i and b_i.

1.7.2 Integration over residue rings and value group

We have the following form of independence (or orthogonality) between the integer part and the residue rings part of $\mathscr{C}_+(\cdot)$.

1.7.3 Proposition *Let S be in* Def*. The canonical morphism*

$$\mathcal{P}_+(S[0, 0, r]) \otimes_{\mathcal{P}^0_+(S)} \mathcal{Q}_+(S[0, m, 0]) \longrightarrow \mathscr{C}_+(S[0, m, r])$$

is an isomorphism of semi-rings, where the homomorphisms $p^* : \mathcal{P}^0_+(S) \to \mathcal{P}_+(S[0, 0, r])$ and $q^* : \mathcal{P}^0_+(S) \to \mathcal{Q}_+(S[0, m, 0])$ are induced by the pullback homomorphisms of the projections $p : S[0, 0, r] \to S$ and $q : S[0, m, 0] \to S$.

The mentioned canonical morphism of Proposition 1.7.3 sends $a \otimes b$ to $p_1^*(a) \otimes p_2^*(b)$, where $p_1 : S[0,m,r] \to S[0,0,r]$ and $p_1 : S[0,m,r] \to S[0,m,0]$ are the projections. Of course the proof of Proposition 1.7.3 uses that $\mathcal{T}$ is split. Recall that for a in $\mathcal{Q}_+(X)$, one can write $a = [Y]$ for some Y in Def_X, say, with specified morphism $f : Y \to X$. We shall write $\mathbf{1}_a := \mathbf{1}_{f(Y)}$ for the characteristic function of $f(Y)$, the "support" of a.

1.7.4 Lemma-Definition Let φ be in $\mathscr{C}_+(Z)$ and suppose that $Z = X[0,m,r]$ for some X in Def. Say that φ is X-integrable if one can write $\varphi = \sum_{i=1}^{\ell} a_i \otimes b_i$ with $a_i \in \mathcal{P}_+(X[0,0,r])$ and $b_i \in \mathcal{Q}_+(X[0,m,0])$ as in Proposition 1.7.3 such that moreover the a_i lie in $\mathrm{I}_X\mathcal{P}_+(X[0,0,r])$ in the sense of Section 1.5. If this is the case, then

$$\mu_{/X}(\varphi) := \sum_i \mu_{/X}(a_i) \otimes \mu_{/X}(b_i) \ \in \mathscr{C}_+(X)$$

does not depend on the choice of the a_i and b_i and is called the integral of φ in the fibers of the coordinate projection $Z \to X$.

The following lemma is a basic form of a projection formula which concerns pulling a factor out of the integral if the factor depends on other variables than the ones that one integrates over.

1.7.5 Lemma *Let φ be in $\mathscr{C}_+(Z)$ such that φ is X-integrable, where $Z = X[0,m,r]$ for some X in* Def. *Let ψ be in $\mathscr{C}_+(X)$ and let $p : Z \to X$ be the projection. Then $p^*(\psi)\varphi$ is X-integrable and*

$$\mu_{/X}(p^*(\psi)\varphi) = \psi\mu_{/X}(\varphi)$$

holds in $\mathscr{C}_+(X)$.

Note that Lemma 1.7.5 is immediate when $m = 0$. Using the natural morphisms $\mathcal{P}_+(Z) \to \mathscr{C}_+(Z)$ which sends ψ to $\psi \otimes [Z]$, and $\mathcal{Q}_+(Z) \to \mathscr{C}_+(Z)$ which sends ν to $\mathbf{1}_Z \otimes \nu$, we can formulate the following. (Note that $\mathcal{P}_+(Z) \to \mathscr{C}_+(Z) : b \mapsto \mathbf{1}_Z \otimes b$ is not necessarily injective neither necessarily surjective.)

1.7.6 Lemma *For any $\varphi \in \mathscr{C}_+(Z)$ there exist ψ in $\mathcal{P}_+(Z[0,m,0])$ and ν in $\mathcal{Q}_+(Z[0,0,r])$ for some m and r such that ν is Z-integrable and $\varphi = \mu_{/Z}(\psi) = \mu_{/Z}(\nu)$.*

Here is a first instance of the feature that relates integration of motivic functions with actual integration (or summation) on $\mathcal{T}$-fields with finite residue field.

1.7.7 Lemma *Let φ be in $\mathscr{C}_+(Z)$ and suppose that $Z = X[0,m,r]$ for some X in* Def. *Let K be a $\mathcal{T}$-field with finite residue field and consider φ_K as in Section 1.7.1. If φ is X-integrable then, for each $x \in X(K)$, $\varphi_K(x,\cdot) : y \mapsto \varphi_K(x,y)$ is integrable against the counting measure, and if one writes ψ for $\mu_{/X}(\varphi)$, then*

$$\psi_K(x) = \sum_y \varphi_K(x,y)$$

where the summation is over those y such that $(x,y) \in Z(K)$.

1.8 Integration over one valued field variable

For the moment let K be any discretely valued field. For a ball $B \subset K$ and for any real number $q > 1$, define $\theta_q(B)$ as the real number $q^{-\mathrm{ord} b}$, where $b \in K^\times$ is such that $B = a + b\mathcal{O}_K$ for some $a \in K$. We call $\theta_q(B)$ the q-volume of B.

Next we will define a naive and simple notion of step-function. Finite b-minimality will allow us to reduce part of the integration procedure to step-functions. A finite or countable collection of balls in K, each with different q-volume, is called a step-domain. We will identify a step-domain S with the union of the balls in S. This is harmless since one can recover the individual balls from their union since they all have different q-volume. Call a nonnegative real valued function $\varphi : K \to \mathbb{R}_{\geq 0}$ a step-function if there exists a unique step-domain S such that φ is constant and nonzero on each ball of S and zero outside $S \cup \{a\}$ for some $a \in K$. Note that uniqueness of the step-domain S for φ is automatic, except possibly when the residue field has two elements.

Let $q > 1$ be a real number. Say that a step-function $\varphi : K \to \mathbb{R}_{\geq 0}$ with step-domain S is q-integrable over K if and only if

$$\sum_{B \in S} \theta_q(B) \cdot \varphi(B) < \infty, \tag{1.1}$$

where one sums over the balls B in S, and then the expression (1.1) is called the q-integral of φ over K. Using Theorem 1.5.1 one proves the following.

1.8.1 Lemma-Definition Suppose that $Z = X[1,0,0]$ for some X in Def. Let φ be in $\mathcal{P}_+(Z)$. Call φ an X-integrable family of step-functions if for each $\mathcal{T}$-field K, for each $x \in X(K)$, and for each $q > 1$, the function

$$\theta_{q,K}(\varphi)(x,\cdot) : K \to \mathbb{R}_{\geq 0} : t \mapsto \theta_{q,K}(\varphi)(x,t) \tag{1.1}$$

is a step-function which is q-integrable over K. If φ is such a family, then there exists a unique function ψ in $\mathcal{P}_+(X)$ such that $\theta_{q,K}(\psi)(x)$ equals the q-integral over K of (1.1) for each $\mathcal{T}$-field K, each $x \in X(K)$, and each $q > 1$. We then call φ X-integrable, we write

$$\mu_{/X}(\varphi) := \psi$$

and call $\mu_{/X}(\varphi)$ the integral of φ in the fibers of $Z \to X$.

Finally we define how to integrate a general motivic constructible function over one valued field variable, in families.

1.8.2 Lemma-Definition Let φ be in $\mathscr{C}_+(Z)$ and suppose that $Z = X[1,0,0]$. Say that φ is X-integrable if there exists ψ in $\mathcal{P}_+(Z[0,m,0])$ with $\mu_{/Z}(\psi) = \varphi$ as in Lemma 1.7.6 such that ψ is $X[0,m,0]$-integrable in the sense of Lemma-Definition 1.8.1 and then

$$\mu_{/X}(\varphi) := \mu_{/X}(\mu_{/X[0,m,0]}(\psi)) \ \in \mathscr{C}_+(X)$$

is independent of the choices and is called the integral of φ in the fibers of $Z \to X$.

The proof of 1.8.2 is similar to the proofs in section 9 of [9].

1.9 General integration

In this section we define the motivic measure and the motivic integral of motivic constructible functions in general. For uniformity results and for applications it is important that we do this in families, namely, in the fibers of projections $X[n,m,r] \to X$ for X in Def. We define the integrals in the fibers of a general coordinate projection $X[n,m,r] \to X$ by induction on $n \geq 0$.

1.9.1 Lemma-Definition Let φ be in $\mathscr{C}_+(Z)$ and suppose that $Z = X[n,m,r]$ for some X in Def. Say that φ is X-integrable if there exist a definable subassignment $Z' \subset Z$ whose complement in Z has relative dimension $< n$ over X, and an ordering of the coordinates on $X[n,m,r]$ such that $\varphi' := \mathbf{1}_{Z'}\varphi$ is $X[n-1,m,r]$-integrable and $\mu_{/X[n-1,m,r]}(\varphi')$ is X-integrable. If this holds then

$$\mu_{/X}(\varphi) := \mu_{/X}(\mu_{/X[n-1,m,r]}(\varphi')) \ \in \mathscr{C}_+(X)$$

does not depend on the choices and is called the integral of φ in the fibers of $Z \to X$, and is compatible with the definitions made in 1.8.2.

More generally, let φ be in $\mathscr{C}_+(Z)$ and suppose that $Z \subset X[n,m,r]$. Say that φ is X-integrable if the extension by zero of φ to a function $\widetilde{\varphi}$ in $\mathscr{C}_+(X[n,m,r])$ is X-integrable, and define $\mu_{/X}(\varphi)$ as $\mu_{/X}(\widetilde{\varphi})$. If X is $h[0,0,0]$ (which is a final object in Def), then we write μ instead of $\mu_{/X}$ and we call $\mu(\varphi)$ the integral of φ over Z.

One can prove 1.9.1 in two ways (both relying on all the properties of $\mathcal{T}$-fields of Definition 1.3.9): using more recent insights from [6] to reverse the order of the coordinates, or, using the slightly longer approach from [9] with a calculation on bi-cells.

One of the main features is a natural relation between motivic integrability and motivic integration on the one hand, and classical measure theoretic integrability and integration on local fields on the other hand:

1.9.2 Proposition *Let φ be in $\mathscr{C}_+(X[n,m,r])$ for some X in* Def. *If φ is X-integrable, then, for each local field K which is a $\mathcal{T}$-field and for each $x \in X(K)$ one has that $\varphi_K(x,\cdot)$ is integrable (in the standard measure-theoretic sense). If one further writes ψ for $\mu_{/X}(\varphi)$, then, for each $x \in X(K)$,*

$$\psi_K(x) = \int_y \varphi_K(x,y),$$

where the integral is against the product measure of the Haar measure on K with the counting measure on $\mathbb{Z}$ and on the residue rings for y running over $h[n,m,r](K)$, and where the Haar measure gives $\mathcal{O}_K$ measure one.

1.10 Further properties

As mentioned before, the projection formula allows one to pull a factor out of the integral if that factor depends on other variables than the ones that one integrates over.

1.10.1 Proposition (Projection formula) *Let φ be in $\mathscr{C}_+(Z)$ for some $Z \subset X[n,m,r]$ and some X in* Def. *Suppose that φ is X-integrable, let ψ be in $\mathscr{C}_+(X)$ and let $p : Z \to X$ be the projection. Then $p^*(\psi)\varphi$ is X-integrable and*

$$\mu_{/X}(p^*(\psi)\varphi) = \psi\mu_{/X}(\varphi)$$

holds in $\mathscr{C}_+(X)$.

In other words, if one would write $\mathrm{I}_X\mathscr{C}_+(Z)$ for the X-integrable functions in $\mathscr{C}_+(Z)$, then

$$\mu_{/X} : \mathrm{I}_X\mathscr{C}_+(Z) \to \mathscr{C}_+(Z) : \varphi \mapsto \mu_{/X}(\varphi)$$

is a morphism of $\mathscr{C}_+(X)$-semi-modules, where the semi-module structure on $\mathrm{I}_X\mathscr{C}_+(Z)$ comes from the homomorphism $p^* : \mathscr{C}_+(X) \to \mathscr{C}_+(Z)$ of semi-rings, with $p : Z \to X$ the projection.

We will now explain Jacobians and relative Jacobians, first in a general, set-theoretic setting, and then for definable morphisms. This is done in several steps.

For any function $h : A \subset K^n \to K^n$ (in the set-theoretic sense of function) for some $\mathcal{T}$-field K and integer $n > 0$, let $\mathrm{Jac}h : A \to K$ be the determinant of the Jacobian matrix of h where this matrix is well-defined (on the interior of A) and let $\mathrm{Jac}h$ take the value 0 elsewhere in A.

In the relative case, consider a function $f : A \subset C \times K^n \to C \times K^n$ which makes a commutative diagram with the projections to C, with K a $\mathcal{T}$-field and with some set C. Write $\mathrm{Jac}_{/C} f : A \to K$ for the function satisfying for each $c \in C$ that $(\mathrm{Jac}_{/C} f)(c, z) = \mathrm{Jac}(f_c)(z)$ for each $c \in C$ and each $z \in K^n$ with $(c, z) \in A$, and where $f_c : A_c \to K^n$ is the function sending z to t with $f(c, z) = (c, t)$ and $(c, z) \in A$.

The existence of the relative Jacobian $\mathrm{Jac}g_{/X}$ in the following definable context is clear by the definability of the partial derivatives and continuity properties.

1.10.2 Lemma-Definition Consider a definable morphism $g : A \subset X[n, 0, 0] \to X[n, 0, 0]$ over X for some definable subassignment X. By $\mathrm{Jac}_{/X} g$ denote the unique definable morphism $A \to h[1, 0, 0]$ satisfying for each $\mathcal{T}$-field K that $(\mathrm{Jac}_{/X} g)_K = \mathrm{Jac}_{/X_K}(g_K)$ and call it the relative Jacobian of g over X.

We can now formulate the change of variables formula, in a relative setting.

1.10.3 Theorem (Change of variables) *Let $F : Z \subset X[n, 0, 0] \to Z' \subset X[n, 0, 0]$ be a definable isomorphism over X for some X in* Def. *Then there exists a definable subassignment $Y \subset Z$ whose complement in Z has dimension $< n$ over X, and such that the relative Jacobian $\mathrm{Jac}_{/X} F$ of F over X is nonvanishing on Y. Moreover, if we take the unique φ' in $\mathscr{C}_+(Z')$ with $F^*(\varphi') = \varphi$, then $\varphi \mathbb{L}^{-\mathrm{ord}\mathrm{Jac}_{/X} F}$ is X-integrable if and only if φ' is X-integrable, and then*

$$\mu_{/X}(\varphi \mathbb{L}^{-\mathrm{ord}\mathrm{Jac}_{/X} F}) = \mu_{/X}(\varphi')$$

in $\mathscr{C}_+(X)$, with the convention that $\mathbb{L}^{-\mathrm{ord}(0)} = 0$.

The proof of Theorem 1.10.3 relies on the Jacobian property for $\mathcal{T}$. Finally we formulate a general Fubini Theorem, in the Tonelli variant for non-negatively valued functions.

1.10.4 Theorem (Fubini-Tonelli) *Let φ be in $\mathscr{C}_+(Z)$ for some $Z \subset X[n, m, r]$ and some X in* Def. *Let $X[n, m, r] \to X[n - n', m - m', r - r']$ be a coordinate projection. Then φ is X-integrable if and only if there exists a definable subassignment Y of Z whose complement in Z has dimension $< n$ over X such that, if we put $\varphi' = \mathbf{1}_Y \varphi$, then φ' is $X[n - n', m - m', r - r']$-integrable and $\mu_{/X[n-n',m-m',r-r']}(\varphi')$ is X-integrable. If this holds, then*

$$\mu_{/X}(\mu_{/X[n-n',m-m',r-r']}(\varphi')) = \mu_{/X}(\varphi)$$

in $\mathscr{C}_+(X)$.

The proof of Theorem 1.10.4 relies essentially on Lemma-Definition 1.9.1.

1.11 Direct image formalism

Let Λ be in Def. From now on, all objects will be over Λ, where we continue to use the notation $\star_{/\Lambda}$ instead of $\star \to \Lambda$ to denote that some object $\star$ is considered over Λ.

Consider X in Def_Λ. For each integer $d \geq 0$, let $\mathscr{C}_+^{\leq d}(X_{/\Lambda})$ be the ideal of $\mathscr{C}_+(X)$ generated by characteristic functions $\mathbf{1}_Z$ of $Z \subset X$ which have relative dimension $\leq d$ over Λ. Furthermore, we put $\mathscr{C}_+^{\leq -1}(X_{/\Lambda}) = \{0\}$.

For $d \geq 0$, define $C_+^d(X_{/\Lambda})$ as the quotient of semi-groups $\mathscr{C}_+^{\leq d}(X_{/\Lambda})/\mathscr{C}_+^{\leq d-1}(X_{/\Lambda})$; its nonzero elements can be seen as functions having support of dimension d and which are defined almost everywhere, that is, up to definable subassignments of dimension $< d$.

Finally, put

$$C_+(X_{/\Lambda}) := \bigoplus_{d \geq 0} C_+^d(X_{/\Lambda}),$$

which is actually a finite direct sum since $C_+^d(X_{/\Lambda}) = \{0\}$ for d larger than the relative dimension of X over Λ.

We introduce a notion of isometries for definable subassignments. This is some work since also residue ring and integer variables play a role.

1.11.1 Definition (Isometries) Consider $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, +\infty\}$. Extend the natural order on $\mathbb{Z}$ to $\overline{\mathbb{Z}}$ so that $+\infty$ is the biggest element, and $-\infty$ the smallest.

Define $\overline{\mathrm{ord}}$ on $h[1,0,0]$ as the extension of ord by $\overline{\mathrm{ord}}(0) = +\infty$. Define $\overline{\mathrm{ord}}$ on $h[0,m,r]$ by sending 0 to $+\infty$ and everything else to $-\infty$. Define $\overline{\mathrm{ord}}$ on $h[n,m,r]$ by sending $x = (x_i)_i$ to $\inf_i \overline{\mathrm{ord}}(x_i)$.

Call a definable isomorphism $f : Y \to Z$ between definable subassignments Y and Z an isometry if and only if

$$\overline{\mathrm{ord}}(y - y') = \overline{\mathrm{ord}}(f_K(y) - f_K(y'))$$

for all $\mathcal{T}$-fields K and all y and y' in $Y(K)$, where $y - y' = (y_i - y_i')_i$. In the relative setting, let $f : Y \to Z$ be a definable isomorphism over Λ. Call f an isometry over Λ if for all $\mathcal{T}$-fields K and for all $\lambda \in \Lambda(K)$, one has that $f_\lambda : Y_\lambda \to Z_\lambda$ is an isometry, where Y_λ is the set of elements in $Y(K)$ that map to λ, and f_λ is the restriction of f_K to Y_λ.

1.11.2 Definition (Adding parameters) Let $f : Y \to Z$ and $f' : Y' \to Z'$ be morphisms in Def with $Y' \subset Y[0,m,r]$ and $Z' \subset Z[0,s,t]$ for some m, r, s, and t. Say that f' is obtained from f by adding parameters, if the natural projections $p : Y' \to Y$ and $r : Z' \to Z$ are definable isomorphisms and if moreover the composition $r \circ f'$ equals $f \circ p$.

We will now define the integrable functions (over Λ) inside $C_+(X_{/\Lambda})$, denoted by $\mathrm{I}C_+(X_{/\Lambda})$, for any definable subassignment $X_{/\Lambda}$. The main idea here is that integrability conditions should not change under pull-backs along isometries and under maps obtained from the identity function by adding parameters. Consider φ in $\mathscr{C}_+^{\leq d}(X_{/\Lambda})$ and its image $\overline{\varphi}$ in $C_+^d(X_{/\Lambda})$ for some definable subassignment $X_{/\Lambda}$ over Λ. In general, one can write X as a disjoint union of definable subassignments X_1, X_2 such that there exists a definable morphism $f = f_2 \circ f_1 : Z \subset \Lambda[d,m,r] \to X_2$ for some m, r, $\mathbf{1}_{X_1}\varphi = 0$, f_2 is an isometry over Λ, and f_1 is obtained from the identity function $X_2 \to X_2$ by adding parameters. Call $\overline{\varphi}$ integrable if and only if $f^*(\varphi)$ is Λ-integrable as in Lemma-Definition 1.9.1. Note that this condition is independent of the choice of the X_i and f. This defines the grade d part $\mathrm{I}C_+^d(X_{/\Lambda})$ of $\mathrm{I}C_+(X_{/\Lambda})$, and one sets

$$\mathrm{I}C_+(X_{/\Lambda}) := \sum_{d \geq 0} \mathrm{I}C_+^d(X_{/\Lambda}).$$

The following theorem gives the existence and uniqueness of integration in the fibers relative over Λ (in all relative dimensions over Λ), in the form of a direct image formalism, by associating to any morphim $f : Y \to Z$ in Def_Λ a morphism of semi-groups $f_!$ from $\mathrm{I}C_+(Y_{/\Lambda})$ to $\mathrm{I}C_+(Z_{/\Lambda})$. This association happens to be a functor and the map $f_!$ sends a function to its integral in the fibers relative over Λ (in the correct relative dimensions over Λ). The underlying idea is that isometries, inclusions, and definable morphisms obtained by adding parameters from an identity map should yield a trivial $f_!$ coming from the inverse of the pullback f^*, and further there is a change of variables situation and a Fubini-Tonelli situation that should behave as in Section 1.10.

1.11.3 Theorem *There exists a unique functor from* Def_Λ *to the category of semi-groups, which sends an object* Z *in* Def_Λ *to the semi-group* $\mathrm{IC}_+(Z_{/\Lambda})$*, and a definable morphism* $f : Y \to Z$ *to a semi-group homomorphism* $f_! : \mathrm{IC}_+(Y_{/\Lambda}) \to \mathrm{IC}_+(Z_{/\Lambda})$*, such that, for* φ *in* $\mathrm{IC}^d_+(Y_{/\Lambda})$ *and a representative* φ^0 *in* $\mathscr{C}^{\leq d}_+(Y_{/\Lambda})$ *of* φ *one has:*

M1 (Basic maps):

If f *is either an isometry or is obtained from an identity map* $C \to C$ *for some* C *in* Def *by adding parameters, then* $f_!(\varphi)$ *is the class in* $\mathrm{IC}^d_+(Z_{/\Lambda})$ *of* $(f^{-1})^*(\varphi^0)$.

M2 (Inclusions):

If $Y \subset Z$ *and* f *is the inclusion function, then* $f_!(\varphi)$ *is the class in* $\mathrm{IC}^d_+(Z_{/\Lambda})$ *of the unique* ψ *in* $\mathscr{C}^{\leq d}_+(Z_{/\Lambda})$ *with* $f^*(\psi) = \varphi^0$ *and* $\psi \mathbf{1}_Y = \psi$.

M3 (Fubini-Tonelli):

If $f : Y = \Lambda[d, m, r] \to Z = \Lambda[d - d', m - m', r - r']$ *is a coordinate projection, then* φ^0 *can be taken by Theorem 1.10.4 such that it is* $\Lambda[d - d', m - m', r - r']$*-integrable and then* $f_!(\varphi)$ *is the class in* $IC^{d-d'}_+(\Lambda[d - d', m - m', r - r']_{/\Lambda})$ *of*

$$\mu_{/\Lambda[d-d',m-m',r-r']}(\varphi^0).$$

M4 (Change of variables):

If f *is a definable isomorphism over* $\Lambda[0, m, r]$ *with* $Y \subset \Lambda[d, m, r]$ *and* $Z \subset \Lambda[d, m, r]$ *then* φ^0 *and a* $\psi \in \mathscr{C}^{\leq d}_+(Y_{/\Lambda})$ *can be taken by Theorem 1.10.3 such that* $\varphi^0 = \psi \mathbb{L}^{-\mathrm{ordJac}_{/\Lambda[0,m,r]} f}$*, and then* $f_!(\varphi)$ *is the class in* $\mathrm{IC}^d_+(Z_{/\Lambda})$ *of* $(f^{-1})^*(\psi)$.

Theorem 1.11.3 thus yields a functor from the category Def_Λ to the category with objects $\mathrm{IC}_+(Z/\Lambda)$ and with homomorphisms of semi-groups (or even of semi-modules over $\mathscr{C}_+(\Lambda)$) as morphisms. This functor is an embedding (that is, injective on objects and on morphisms). The functoriality property $(g \circ f)_! = g_! \circ f_!$ is a flexible form of a Fubini Theorem.

Note that $C_+(X_{/\Lambda})$ is a graded $\mathscr{C}_+(X)$-semi-module (but not so for $\mathrm{IC}_+(X_{/\Lambda})$ which is just a graded $\mathscr{C}_+(\Lambda)$-semi-module). Using this module structure, we can formulate the following form of the projection formula.

1.11.4 Proposition *For every morphism* $f : Y \to Z$ *in* Def_Λ*, and every* α *in* $\mathscr{C}_+(Z)$ *and* β *in* $\mathrm{IC}_+(Y_{/\Lambda})$*,* $\alpha f_!(\beta)$ *belongs to* $\mathrm{IC}_+(Z_{/\Lambda})$ *if and only if* $f^*(\alpha)\beta$ *is in* $\mathrm{IC}_+(Y_{/\Lambda})$*. If these conditions are verified, then* $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

The analogy with the direct image formalism of Theorem 14.1.1 of [9] with $S = \Lambda$ is now complete. An important ingredient of the proof of Theorem 1.11.3 is that general definable morphisms can be factored, at least piecewise, into definable morphisms of the specified simple types falling under M1 up to M4.

1.11.5 Comparison with [19]

We end the paper by comparing our motivic integrals with the ones in [19] by means of specialization. Let R be a complete discrete valuation ring with fraction field K and perfect residue field k.

In [19], motivic integration on smooth rigid varieties over K was developed. We shall now compare it with the approach in the present paper in unequal characteristic (although it works similarly in equal characteristic zero).

Let X be a smooth quasi-compact and separated rigid variety over K endowed with a gauge form ω. Assume that K has unequal characteristic. Note that, if we write p for the residue field characteristic of K and e for the ramification degree of K, one can naturally consider $X(L)$ for any complete $(0,p,e)$-field L. Consider the analytic theory $\mathcal{T} = \mathcal{T}_K$ with language $\mathcal{L}_K$, as in 2 of Section 1.3.1. Then we can look at X as a definable subassignment (by using affine charts). One may define $\int_X |\omega|$ using definable morphisms as charts and as transition functions, say for X of dimension n. For each chart in a finite disjoint covering of X by (definable) charts, one takes the pullback of ω on that chart to get a volume form on an open definable subassignment O of $h[n,0,0]$. One then expresses the latter as a multiple f of $dx_1 \wedge \ldots \wedge dx_n$, where clearly f is a definable morphism to $h[1,0,0]$. Finally one integrates $\mathbb{L}^{-\mathrm{ord} f}$ on O as is defined in section 1.9, with the convention that $\mathbb{L}^{-\mathrm{ord}(0)} = 0$, and one takes the sum for all the charts, where integrability follows from the quasi-compactness assumption. This is well defined thanks to the Change of Variables Theorem 1.10.3.

We will now link this integral $\int_X |\omega|$ to the integral as defined in [19]. Let $K_0(\mathrm{Var}_k)$ be the Grothendieck ring of varieties over k, moded out by the extra relations $[X] = [f(X)]$, for $f : X \to Y$ radicial. Note that radicial means that for every algebraically closed field ℓ over k the induced map $X(\ell) \to Y(\ell)$ is injective, and that $[f(X)]$ is an abbreviation for the class of the constructible set given as the image of f by Chevalley's Theorem. (In the case that the characteristic of k is zero, the extra relations would be redundant.) In any case, this ring is isomorphic to the Grothendieck ring of definable sets with coefficients from k for the theory of algebraically closed fields in the language of rings.

Similarly as the morphism γ as in Section 16.3 of [9], there is a canonical morphism δ : $\mathscr{C}_+(\mathrm{point}) \to K_0(\mathrm{Var}_k)\otimes\mathbb{A}$. Indeed, we may note that $\mathscr{C}_+(\mathrm{point})$ is isomorphic to $\mathbb{A}_+ \otimes_{\mathbb{Z}} \mathcal{Q}_+(\mathrm{point})$, and for $x = \sum_{i=1}^r a_i \otimes b_i$ with $a_i \in \mathbb{A}_+$ and $b_i \in \mathcal{Q}_+(\mathrm{point})$, we may set $\delta(x) = \sum_i [b_i] \otimes a_i$, where $[b_i]$ is the class in $K_0(\mathrm{Var}_k)$ of the constructible set obtained from b_i by elimination of quantifiers for the theory of algebraically closed fields in the language of rings (Chevalley's Theorem).

In [19] an integral $\int_X^{LS} |\omega|$ in the localization of $K_0(\mathrm{Var}_k)$ with respect to the class of the affine line is defined. Hence we may consider the image of $\int_X^{LS} |\omega|$ in the further localization $K_0(\mathrm{Var}_k)\otimes\mathbb{A}$.

1.11.6 Proposition *Let X be a smooth separated rigid variety over K endowed with a gauge form ω. Assume that K has unequal characteristic and that X is quasi-compact. Then, with the above notation, $\delta(\int_X |\omega|)$ is equal to the image of $\int_X^{LS} |\omega|$ in $K_0(\mathrm{Var}_k) \otimes \mathbb{A}$.*

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