Motivic integration and transfer principles

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joint work with Gordon, Halupczok, Loeser

In this overview I will explain:

- Definable functions
- Motivic (exponential) functions
- Properties
- Transfer principles

prehistory:
Basic p-adic integrals like

$$\int_{\mathbb{Z}_p} |x|^s |dx|, \ s > 0$$

were studied by Tate and Weil in the context of Poisson summation

and generalizations to

$$\int_{\mathbb{Z}_p^n} |f(x)|^s |dx|$$

-) for f a polynomial by Igusa
-) for f a definable function by Denef.

Definable means 'given by a single formula'.

Novelty presented in the series: uniform in all p-adic fields.

The studies by Tate and Weil were precursors for some aspects of the Langlands program.

The abstract Igusa/Denef context has found recent applications in the Langlands program and representation theory of p-adic reductive groups.

Definable functions

A function is called definable when its graph is a definable set.

A set is called definable if it is given by a single, finite-length, sensible formula (a first order formula), in the Denef-Pas language.

In fact, any such a formula gives rise to many sets, similar to schemes by taking rational points.

Definable functions (in the language of Denef-Pas)

Context: integration on $\mathbb{Q}_p^n$ and on $\mathbb{F}_p((t))^n$.

The language to make formulas, where $\mathbb{Q}_p$ could be replaced by any valued field, is:

three sorts of variables, running over $\mathbb{Q}_p$, $\mathbb{F}_p$, and $\mathbb{Z}$,
three sorts of quantifiers,

ring operations on $\mathbb{Q}_p$,
ring operations on $\mathbb{F}_p$,
group operation and ordering on $\mathbb{Z}$

the order map $\mathrm{ord} : \mathbb{Q}_p^\times \to \mathbb{Z}$,
a multiplicative map $\mathrm{ac} : \mathbb{Q}_p \to \mathbb{F}_p$.

logical symbols for Boolean combinations,
equality for each sort of variables and brackets.

By construction, the class of definable sets is stable under Boolean combinations and under taking images under projections.

Suited to study geometry piecewise (no global geometry).

Definable sets can be stratified into analytic manifolds.

Any definable function is C^1 almost everywhere, more precisely, away from a definable set of lower dimension.

ac: depends on choice of uniformizer ϖ.
$\mathrm{ac} : \mathbb{Q}_p \to \mathbb{F}_p$ sends nonzero x to $x\varpi^{-\mathrm{ord}\, x} \bmod (\varpi)$.

In order to work uniformly in all p-adic fields, we must use the more general Denef-Pas language,

Let K be a p-adic field with valuation ring $\mathcal{O}_K$ and maximal ideal $\mathcal{M}_K$, integer $n > 0$,

$$K_n := \mathcal{O}_K/(n\mathcal{M}_K)$$

and

$$\mathrm{ac}_n : K \to K_n$$

multiplicative map, based on choice of uniformizer, sending nonzero x to $x\varpi^{-\mathrm{ord}\, x} \bmod (n\varpi)$.

Recap:
'Definable' means 'given by a single formula'.

a function is definable when its graph is,

a set is definable when it is given by a general Denef-Pas formula.

Such a formula makes sense in any henselian valued field, thus giving rise to many sets.

A collection of sets $(X_K)_K$, where K runs over all p-adic fields, is called definable when these sets are given by a single (one and the same) formula.

There is a natural dimension theory for definable sets.

Motivic (exponential) functions

Definition in two steps:
1) motivic functions
2) motivic exponential functions

A motivic function is, for each p-adic field K with chosen uniformizer ϖ the data of a function

$$F_K : \begin{cases} X_K \to \mathbb{C} \\ x \mapsto \sum_i \prod_j a_{ij}(q_K)(\sum_{y \in Y_{i,j,x}} g_{i,j,K}(x,y) q_K^{f_{i,j,K}(x,y)}) \end{cases}$$

where:

the $f_{i,j,K},\ g_{i,j,K} :: Y_{i,j,K} \to \mathbb{Z}$,

the X_K, and the $Y_{i,j,K} \subset X_K \times K_N^M$ are definable, for some N, M, and

the a_{ij} lie in in $\mathbb{Z}[T, \{\frac{1}{1-T^i}\}_{i<0}]$.

Let ψ run over additive characters on K which are trivial on $\mathcal{M}_K$ and nontrivial on $\mathcal{O}_K$.

A motivic exponential function is, for each p-adic field K with chosen uniformizer ϖ and each ψ the data of a function

$$F_K : X_K \to \mathbb{C} : x \mapsto$$

$$\sum_i \prod_j a_{ij}(q_K)\Big(\sum_{y \in Y_{i,j,x}} g_{i,j,K}(x,y) q_K^{f_{i,j,K}(x,y)} \psi(h_{i,j,K}(x,y) + \frac{e_{i,j,K}(x,y)}{n})\Big)$$

where:

the $f_{i,j,K}$, $g_{i,j,K}$ on $Y_{i,j,K}$ and a_{ij} are as before,

the $h_{i,j,K}$ are K-valued definable, and the $e_{i,j,K}$ are K_n-valued definable, all on $Y_{i,j,K}$.

Properties

Motivic (exponential) functions are stable under integrating some of the variables out.

Theorem (C. Gordon Halupczok)

Let F be a motivic (exponential) function in $n+m$ variables. Then there exists a motivic (exponential) function G in n variables such that the following hold for each p-adic field K and any ψ on K.

$$\int_{y \in K^m} F_{K,\psi}(x,y)|dy| = G_{K,\psi}(x)$$

whenever the integral makes sense.

Moreover, they are natural:
Motivic functions form the smallest class of functions with this property and containing $\pm$ the characteristic functions of definable sets.

Motivic exponential functions are also stable under Fourier transformation on the additive group of K.

? What about Fourier transforms on other groups than the additive groups ?

Representatives

Given a motivic function, to choose formulas behind the definable functions is called to choose representatives.

Once representatives are chosen, the motivic functions also make sense for $\mathbb{F}_p((t))$ for large p.

Also all calculations, and very many properties to calculate integrals in this context make sense on $\mathbb{F}_p((t))$ for large p.

For example:

For each definable bijection, there is a change of variables formula. Indeed, non-vanishing of the jacobian is a definable condition.

Transfer principles

Let F and G be motivic exponential functions in n, resp. in m, variables.

As soon as the residue field characteristic of K is sufficiently large, then the following hold:

(Transfer Principle I, Cluckers - Loeser, Ann. Math. 2010)

Whether

$$\int_{K^n} F_{K,\psi} = \int_{K^m} G_{K,\psi} \text{ for each nontrivial } \psi,$$

only depends on the residue field of K.

Similarly when integrating only some of the variables out.

Let F be a motivic exponential functions in n variables.
As soon as the residue field characteristic of K is sufficiently large, then the following hold:

(Transfer Principle II, Cluckers - Gordon - Halupczok, Duke 2014)

Whether

$$F_{K,\psi} \text{ is integrable over } K^n \text{ for each nontrivial } \psi$$

only depends on the residue field of K.

Similarly in families (namely with parameters).

Similarly for local integrability.

Bounds

Theorem

Let H be a constructible motivic function on $W \times \mathbb{Z}^n$, where W is a definable set. Then there exist integers a, b and M such that for all p-adic fields K the following holds.
If there exists any function $\alpha_K : \mathbb{Z}^n \to \mathbb{R}$ such that

$$|H_K(w,\lambda)|_{\mathbb{C}} \leq \alpha_K(\lambda) \text{ on } W_K \times \mathbb{Z}^n,$$

then

$$|H_K(w,\lambda)|_{\mathbb{C}} \leq q_K^{a+b\|\lambda\|} \text{ on } W_K \times \mathbb{Z}^n.$$

Bounds

Theorem (Transfer)

Let H be an exponential motivic function on $W \times \mathbb{Z}^n$, where W is a definable set, and let a and b be integers. Then, for large enough residue characteristic, the truth of the statement

$$|H_K(w,\lambda)|_{\mathbb{C}} \leq q_K^{a+b\|\lambda\|} \text{ for all } (w,\lambda) \in W_K \times \mathbb{Z}^n \qquad (1)$$

only depends on the isomorphism class of the residue field of K.

Applications

First application: The Fundamental Lemma of the Langlands Program follows in characteristic zero,
as an alternative and more general strategy to work by Waldspurger. (joint work with Hales and Loeser.)

Second application: Let the root datum for a connected, reductive group be given. Then there exists N, such that for each prime p larger than N, and for each irreducible admissible representation π of $G(\mathbb{F}_q((t)))$ (assuming existence of a mock exponential), the Harish-Chandra character of π is an everywhere locally integrable distribution. (joint work with Gordon and Halupczok.)

Third application: see appendix to paper by Shin and Templier.

Tools and techniques

We use certain notions of loci involving analytic conditions, and we relate them to zero loci.

Tools and techniques

Definition

For X a set, for T a space with a complete measure, and for $f : X \times T \to \mathbb{C}$ a function, define the locus of integrability of f in X as the set

$$\mathrm{Int}(f, X) := \{x \in X \mid t \mapsto f(x,t) \text{ is integrable over } T\},$$

Loci of boundedness can be defined similarly, requiring the image of $f(x, \cdot)$ to be bounded.

Similar notion for local integrability, etc.

We prove the following results:

Theorem (C., Gordon, Halupczok)

Let F be a motivic exponential function in $n+m$ variables. Then there exists a motivic exponential function G in n variables such that the following holds
for each local field K of large enough residual characteristic, and any nontrivial additive character ψ_K on K:

$$\mathrm{Int}(F_{K,\psi_K}, K^n) = \text{ the zero locus of } G_{K,\psi_K} \text{ in } K^n \text{ .}$$

This relates the analytic condition of integrability to the geometric condition of lying in a zero locus.

Of course, some forms of quantifier elimination results for definable sets are behind, as well as logical compactness.

Also some cell decomposition techniques and relations like

$$|f(x) - f(y)| = |f'| \cdot |x - y|$$

for definable f are behind to reduce various situations to linear behavior.

Thank you!