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A VERSION OF p-ADIC MINIMALITY

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Abstract. We introduce a very weak language $\mathscr{L}_M$ on p-adic fields K, which is just rich enough to have exactly the same definable subsets of the line K that one has using the ring language. (In our context, definable always means definable with parameters.) We prove that the only definable functions in the language $\mathscr{L}_M$ are trivial functions. We also give a definitional expansion $\mathscr{L}'_M$ of $\mathscr{L}_M$ in which K has quantifier elimination, and we obtain a cell decomposition result for $\mathscr{L}_M$-definable sets.
Our language $\mathscr{L}_M$ can serve as a p-adic analogue of the very weak language $(<)$ on the real numbers, to define a notion of minimality on the field of p-adic numbers and on related valued fields. These fields are not necessarily Henselian and may have positive characteristic.

§1. **Definitions and main results.** When studying minimality, the following concept, which was introduced by Macpherson and Steinhorn [12] is a natural framework.

1.1. Definition. Let $\mathscr{L}$ be an expansion of a language $\mathscr{L}_0$. A structure $(K, \mathscr{L})$ is called $\mathscr{L}_0$-minimal if and only if every $\mathscr{L}$-definable subset of the line K is already $\mathscr{L}_0$-definable (with parameters from the model K). When the property holds for all the models K of an $\mathscr{L}$-theory $\mathscr{T}$, then $\mathscr{T}$ is called $\mathscr{L}_0$-minimal.

Using this notion, o-minimality is in fact $(<)$-minimality. Other examples of this kind are minimality, C-minimality, P-minimality and so on. Note that among the existing notions of minimality that make sense in the p-adic context, namely P-minimality [9] and b-minimality [5], only P-minimality fits into this framework.

Haskell and Macpherson developed the concept of P-minimality as a p-adic version of o-minimality. In P-minimality, only structures $(K, \mathscr{L})$ where $\mathscr{L} \supseteq \mathscr{L}_{\text{ring}}$ are considered. However, also structures which can be defined using weaker languages than $\mathscr{L}_{\text{ring}}$, like ordered groups $(R, +, <)$, may satisfy the definition of o-minimality. We think that this restriction in the scope of P-minimality is somewhat artificial, since for example the semi-affine structure $(\mathbb{Q}_p, +, -, \{x \mapsto cx\}_{c \in \mathbb{Q}_p}, \{P_n\}_n)$, which was studied by Liu in [10], has the same definable subsets of $\mathbb{Q}_p$ as one has with the ring language. Here the symbols P_n are the Macintyre predicates for the nonzero n-th powers in the field.

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In this paper we present a language $\mathscr{L}_M$, as a p-adic alternative to the minimal language $(<)$ of o-minimality (hence the subscript 'M' in $\mathscr{L}_M$, which stands for 'Minimal language'). Giving an explicit theory of the models K in which we are interested, purely in the language $\mathscr{L}_M$, is not yet possible at this point. Instead, we have to assume that the models K also have a field structure (more precisely a structure of valued field, with a finite residue field and a $\mathbb{Z}$-group as value group). Using this field structure, we can axiomatize what the symbols of $\mathscr{L}_M$ mean.

The language $\mathscr{L}_M$ itself is a very weak language, indeed much weaker than the field language. In particular, the only definable functions in the language $\mathscr{L}_M$ are trivial ones: we show that any $\mathscr{L}_M$-definable function $X \subset K^n \to K$ is piecewise a coordinate projection or a constant function, where the pieces can be taken $\mathscr{L}_M$-definable. We introduce a definitional expansion $\mathscr{L}'_M$ of $\mathscr{L}_M$ in which K has quantifier elimination and we prove a cell decomposition theorem. We also discuss non-definability of Skolem functions and mention some difficulties in classifying the definable sets up to definable bijections.

The models K are allowed to be quite general since our axioms do not require K to be Henselian, neither do they require the characteristic to be zero, so that our results are also valid for fields like $\mathbb{F}_q(t)$. For most of the paper, we will be working in the following class of fields:

1.2. Definition. Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{Z}$ the ordered abelian group of integers. We define a $(\mathbb{F}_q, \mathbb{Z})$-field to be a valued field with residue field isomorphic to $\mathbb{F}_q$ and value group elementarily equivalent to $\mathbb{Z}$.

The definition of $(\mathbb{F}_q, \mathbb{Z})$-fields encompasses a very diverse range of fields, like $\mathbb{Q}_p$ and $\mathbb{Q}$ with the p-adic valuation (both with $p = q$), and also $\mathbb{F}_q(t)$ and $\mathbb{F}_q((t))$. Formally p-adic fields as introduced by Ax and Kochen [1] are $(\mathbb{F}_q, \mathbb{Z})$-fields, as are all p-adically closed fields. We should note that the use of the term 'p-adically closed' may differ somewhat among authors. As is common in P-minimality, we use the notion introduced by Prestel and Roquette in [14]. Hence, a p-adically closed field is a henselian $(\mathbb{F}_q, \mathbb{Z})$-field of characteristic zero, where $q = p^r$, and the dimension of $\mathscr{O}_K / p\mathscr{O}_K$ regarded as a vector space over $\mathbb{F}_p$ is finite.

In section 1.4 and Proposition 4, we show that in the particular case where K is a p-adically closed field, the semi-algebraic subsets of K are precisely the $\mathscr{L}_M$-definable subsets of K (where definability is with parameters from K). Macintyre raised the question to us whether p-adically closed fields are the only fields in which this property is true; we leave this question to the future.

We will now explain the symbols used in our language. Fix a $(\mathbb{F}_q, \mathbb{Z})$-field K, fix an element π_K with smallest positive order, and write $\mathscr{M}_K$ for the maximal ideal of the valuation ring $\mathscr{O}_K$ of K. For each integer $n > 0$, let P_n be the set of nonzero n-th powers in K. For each $m > 0$, we will define sets $Q_{n,m}$ using so-called angular component maps. The following lemma shows that such maps exist and that they can be defined in a unique way.

1.3. Lemma. *For each integer $m > 0$, there is a unique group homomorphism*

$$\overline{\mathrm{ac}}_m \colon K^\times \to (\mathscr{O}_K \bmod \pi_K^m)^\times$$

such that $\overline{\mathrm{ac}}_m(\pi_K) = 1$ and such that $\overline{\mathrm{ac}}_m(u) \equiv u \bmod (\pi_K)^m$ for any unit $u \in \mathscr{O}_K$.

PROOF. Put $N_m := (q-1)q^{m-1}$ and let U be the set $P_{N_m} \cdot \mathcal{O}_K^\times$. Note that $K^\times$ equals the finite disjoint union of the sets $\pi_K^\ell \cdot U$ for integers ℓ with $0 \leqslant \ell \leqslant N_m - 1$. Hence, any element y of $K^\times$ can be written as a product of the form $\pi_K^\ell x^{N_m} u$, with $u \in \mathcal{O}_K^\times$, $\ell \in \{0, \dots, N_m - 1\}$, and $x \in K^\times$.

Since $\overline{\mathrm{ac}}_m$ is required to be a group homomorphism to a finite group with N_m elements, it must send P_{N_m} to 1. Also note that the projection $\mathcal{O}_K \to \mathcal{O}_K \bmod \pi_K^m$ (which is a ring homomorphism), induces a natural group homomorphism $p\colon \mathcal{O}_K^\times \to (\mathcal{O}_K \bmod \pi_K^m)^\times$. Now if we write $y = \pi_K^\ell x^{N_m} u$, we see that $\overline{\mathrm{ac}}_m$ must satisfy

$$\overline{\mathrm{ac}}_m(y) = p(u), \tag{1.3.1}$$

which implies that the map $\overline{\mathrm{ac}}_m$ is uniquely determined if it exists. Moreover, we claim that we can use (1.3.1) to define $\overline{\mathrm{ac}}_m$. This is certainly a well defined group homomorphism: if one writes $y = \pi_K^\ell \tilde{x}^{N_m} \tilde{u}$ for some other $\tilde{u} \in \mathcal{O}_K^\times$ and $\tilde{x} \in K^\times$, then clearly $p(u) = p(\tilde{u})$. It is also clear that this homomorphism sends π_K to 1 and satisfies our requirement that $\overline{\mathrm{ac}}_m(u) \equiv u \bmod (\pi_K)^m$ for any unit $u \in \mathcal{O}_K$. ⊣

Related results, on the existence of cross sections for formally p-adic fields (these are p-adically closed fields where the residue field is $\mathbb{F}_p$ and ord $p = 1$) were previously obtained by Scowcroft [15].

Using the angular component maps from Lemma 1.3 , we can define sets $Q_{n,m}$, for any $m, n > 0$, as follows:

$$Q_{n,m} := \{x \in P_n \cdot (1 + \mathcal{M}_K^m) \mid \overline{\mathrm{ac}}_m(x) = 1\}.$$

Note that $Q_{n,m}$ is an open subgroup of finite index of $K^\times$ (for the valuation topology), and that $Q_{n,m}$ is definable in the language of valued fields $(+, -, \cdot, |)$ by the above construction of $\overline{\mathrm{ac}}_m$.

For any element $\lambda \in K$, let $\lambda Q_{n,m}$ denote the set $\{\lambda t \mid t \in Q_{n,m}\}$. Now let $\mathcal{L}_M$ be the language consisting of predicates $R_{n,m}$ in three variables, with

$$R_{n,m}(x, y, z) \text{ if and only if } y - x \in zQ_{n,m}.$$

As is common notation, we write ord for the valuation map to the additively written value group, enriched with $+\infty$ for the image of 0. For each integer k, the three-variable predicate D_k is given by

$$D_k(x, y, z) \text{ holds if and only if } \operatorname{ord}(x - y) < \operatorname{ord}(z - y) + k.$$

Let $\mathcal{L}'_M$ be the language $\mathcal{L}_M$, extended with the predicates D_k.

By definable we always mean definable with parameters from the model. For the rest of the paper we will assume that K is a fixed $(\mathbb{F}_q, \mathbb{Z})$-field (unless it is explicitly stated otherwise.) The first main result of this paper is the following.

1. PROPOSITION. *The language $\mathcal{L}'_M$ is a definitional expansion of $\mathcal{L}_M$ in the sense that the structures $(K, \mathcal{L}'_M)$ and $(K, \mathcal{L}_M)$ have the same definable sets. Moreover, K admits quantifier elimination in the language $\mathcal{L}'_M$.*

We will use cell decomposition techniques to prove Proposition 1. The cells we consider will be sets of the following type:

1. Definition. An $\mathscr{L}'_M$-cell $A \subset K^k \times K$ is a set of the form

$$A = \{(x,t) \in D \times K \mid \\ \ell_1 + \operatorname{ord}(a_1 - c) \,\square_1\, \operatorname{ord}(t-c) \,\square_2\, \ell_2 + \operatorname{ord}(a_2 - c),\ t - c \in \lambda Q_{n,m}\},$$

where $x = (x_1, \dots x_k)$ and the ℓ_i are integers. The set D equals the image of A under the projection of A to K^k, and is a quantifier free $\mathscr{L}'_M$-definable subset of K^k. $\square_1$ and $\square_2$ denote '$<$' or 'no condition', $\lambda \in K$, the a_i and c are either one of the variables $x_1, \dots, x_k$ or a constant from K, and the $a_i - c$ do not vanish on D. We call c the center of A, and D the base of A.

Clearly, any $\mathscr{L}'_M$-cell is an $\mathscr{L}'_M$-definable set. Indeed, using the symbols from $\mathscr{L}'_M$, we can describe a cell A, where for example the $\square_i$ both denote '$<$', in the following way:

$$A = \{(x,t) \in D \times K \mid D_{-\ell_1}(a_1, c, t) \wedge D_{\ell_2}(t, c, a_2) \wedge R_{n,m}(c, t, \lambda)\}.$$

The following is the main technical result of the paper.

2. Proposition (Cell Decomposition). *Every $\mathscr{L}'_M$-definable subset X of $K^k \times K$ can be partitioned into finitely many $\mathscr{L}'_M$-cells.*

Note that, in a P-minimal structure, one has a cell decomposition result (in the form of [13]) if and only if one has definable Skolem functions, by the main result of [13]. (To our knowledge, it is not known whether every P-minimal structure automatically has definable Skolem functions.) The structure $(K, \mathscr{L}'_M)$ is different: it does not allow definable Skolem functions (see Section 3), and yet it has the cell decomposition result Proposition 2.

Observe that in our notion of cells, the boundaries on the left of $\square_1$ and on the right of $\square_2$ are in general not the order of definable functions (see Proposition 3), which contrasts with the definability of boundaries in the usual notions of p-adic cells, in the semi-algebraic and subanalytic languages and in the P-minimal context of [13]. This difference might (partly) explain the non-existence of Skolem-functions in our case, as the proof of the mentioned main result of [13] relies on the definability of the boundary functions.

The above cell decomposition result allows us to establish the triviality of $\mathscr{L}_M$-definable functions.

3. Proposition. *Any $\mathscr{L}_M$-definable function $f\colon X \subset K^k \to K$ is piecewise a coordinate projection or a constant function, where the pieces can be taken $\mathscr{L}_M$-definable.*

By providing two examples in Section 3 we show that $\mathscr{L}_M$ does not have definable Skolem functions, and that the classification of [2] for semi-algebraic and subanalytic sets does not analogously hold for $\mathscr{L}_M$-definable sets.

1.4. $\mathscr{L}_M$-minimality and P-minimality. For $K = \mathbb{Q}_p$, (and similarly for a finite field extension of $\mathbb{Q}_p$) the sets $Q_{n,m}$ are simply the sets

$$Q_{n,m} = \bigcup_{k \in \mathbb{Z}} p^{kn}(1 + p^m \mathbb{Z}_p).$$

Moreover, using Hensel's Lemma, it is easy to see that $Q_{n,m} \subseteq P_n$ for sufficiently large m. Since the sets $Q_{n,m}$ have finite index in $\mathbb{Q}_p^\times$, this implies that the sets P_n can be partitioned as finite unions of sets $\lambda Q_{n,m}$. By a similar argument one sees that $Q_{n,m}$ can be partitioned as a finite union of sets λP_N for suitable N. We can then use the cell decomposition result [7] of Denef to conclude that any semialgebraic subset of the line $\mathbb{Q}_p$ can be partitioned as a finite union of $\mathcal{L}'_M$-cells, which implies (because of Propositions 1 and 2), that the $\mathcal{L}_M$-definable subsets of $\mathbb{Q}_p$ coincide with the semi-algebraic subsets of $\mathbb{Q}_p$. (In the p-adic context, semi-algebraic means definable in the ring language, or equivalently, definable in Macintyre's language, cf. [11].)

Similar reasoning holds for any p-adically closed field. We therefore suggest the following candidate definition for a minimality notion on $(\mathbb{F}_q, \mathbb{Z})$-fields:

2. Definition. Let K be a $(\mathbb{F}_q, \mathbb{Z})$-field. Let $\mathcal{L}$ be any language on K expanding $\mathcal{L}_M$. The $\mathcal{L}$-structure K is called $\mathcal{L}_M$-minimal if and only if all $\mathcal{L}$-definable subsets of K are already $\mathcal{L}_M$-definable. The theory of $(K, \mathcal{L})$ is called $\mathcal{L}_M$-minimal if and only if all $\mathcal{L}$-definable subsets of K' are already $\mathcal{L}_M$-definable, where K' is any structure which is elementary equivalent to $(K, \mathcal{L})$.

This new minimality notion is then a natural generalization of P-minimality, since the cell decomposition result of [2] for subanalytic sets (or, alternatively, the P-minimality result for the subanalytic structure on $\mathbb{Q}_p$ of [8] and the cell decomposition result of [7]) imply the following result. (See [6] for more context on subanalytic sets.)

4. Proposition. *The subanalytic structure on $\mathbb{Q}_p$ (or on a finite field extension L of $\mathbb{Q}_p$) is $\mathcal{L}_M$-minimal and has $\mathcal{L}_M$-minimal theory. Hence, any intermediate structure between the structure $\mathcal{L}_M$ and the subanalytic structure on $\mathbb{Q}_p$ is $\mathcal{L}_M$-minimal and has $\mathcal{L}_M$-minimal theory.*

Furthermore, all P-minimal structures on $\mathbb{Q}_p$ (or on a finite field extension L of $\mathbb{Q}_p$) have $\mathcal{L}_M$-minimal theory (but not the other way around). Some of the intermediate structures between the structure $\mathcal{L}_M$ and the subanalytic structure on L are known to have cell decomposition, for example, the semi-affine expansion of $\mathcal{L}_M$ by Liu [10], or any of the analytic structures of [3] on L. Whether a general intermediate structure automatically has cell decomposition is actually an open question.

§2. The proofs.

The following is the main technical lemma. Recall that the field K is a fixed $(\mathbb{F}_q, \mathbb{Z})$-field.

2.1. Lemma. *Let C_1, C_2 be $\mathcal{L}'_M$-cells with centers c_1, resp. c_2, and with $C_1, C_2 \subset K^{e+1}$ for some $e \geq 0$. Then $C_1 \cap C_2$ can be partitioned into a finite union of $\mathcal{L}'_M$-cells A, each of which has a center which is either c_1 or c_2.*

Proof. By partitioning C_1 and C_2 further if necessary, we may suppose that they both use $Q_{n,m}$ with the same positive integers m, n, that is, that C_i consists of all $(x, t) \in D_i \times K$ satisfying

$$k_{1i} + \operatorname{ord}(a_{1i} - c_i) \,\square_{1i}\, \operatorname{ord}(t - c_i) \,\square_{2i}\, k_{2i} + \operatorname{ord}(a_{2i} - c_i),\ t - c_i \in \lambda_i Q_{n,m}$$

for $i = 1, 2$, where the symbols have their meaning as in Definition 1. We may suppose that $\lambda_i \neq 0$ for $i = 1, 2$, since otherwise the lemma is trivial. Up to a finite partition, we may suppose that on C_1, one of the following conditions holds for $k = 1 + m + n + \sum_{i,j=1,2} |k_{ij}|$, and some integer ℓ_1 with $-k \leq \ell_1 \leq k$:

$$\operatorname{ord}(t - c_1) > \operatorname{ord}(c_2 - c_1) + k, \tag{I$_k$}$$

$$\operatorname{ord}(t - c_1) < \operatorname{ord}(c_2 - c_1) - k, \tag{II$_k$}$$

$$\operatorname{ord}(t - c_1) + \ell_1 = \operatorname{ord}(c_2 - c_1). \tag{III$_{\ell_1}$}$$

Note that $(\mathrm{I})_k$ and $(\mathrm{II})_k$ imply respectively

$$\operatorname{ord}(t - c_1) > \operatorname{ord}(c_2 - c_1) + k = \operatorname{ord}(t - c_2) + k, \tag{i$_k$}$$

$$\operatorname{ord}(t - c_2) = \quad \operatorname{ord}(t - c_1) \quad < \operatorname{ord}(c_1 - c_2) - k. \tag{ii$_k$}$$

If $(\mathrm{I})_k$ holds on C_1, put

$$W = \{x \in D_2 \mid k_{12} + \operatorname{ord}(a_{12} - c_2)\square_{12} \operatorname{ord}(c_1 - c_2)\square_{22} k_{22} + \operatorname{ord}(a_{22} - c_2)\}.$$

Then one has

$$C_1 \cap C_2 = \{(x, t) \in (W \times K) \cap C_1 \mid c_1 - c_2 \in \lambda_2 Q_{n,m}\},$$

which is easily seen to be a finite disjoint union of $\mathscr{L}'_M$-cells of the desired form. If $(\mathrm{II})_k$ holds on C_1, then we may suppose, up to partitioning C_2, that either $C_1 \cap C_2$ is empty, or that $(\mathrm{ii})_k$ holds for all $(x, t) \in C_1$ and for all $(x, t) \in C_2$. If now the intersection $C_1 \cap C_2$ is nonempty then $C_1 \cap C_2$ consists of all points $(x, t) \in (D_1 \cap D_2) \times K$ satisfying the conditions

$$\max_{i \in I}\{k_{1i} + \operatorname{ord}(a_{1i} - c_i)\} < \operatorname{ord}(t - c_1) < \min_{i=1,2}\{k_{2i} + \operatorname{ord}(a_{2i} - c_i)\},$$

and

$$t - c_1 \in \lambda_1 Q_{n,m},$$

where I consists of i such that $\square_{1i}$ is the condition $<$, and where the maximum over the emptyset is $-\infty$. If $\#I \geq 1$ we have to show that the function $x \mapsto \max_{i \in I}\{k_{1i} + \operatorname{ord}(a_{1i} - c_i)\}$ is piecewise of the form $r + \operatorname{ord}(a - c_1)$, for some integer r and some a being either a constant from K or one of the variables x_i, and correspondingly for the minimum. From $(\mathrm{ii})_\mathrm{k}$ and the fact that $C_1 \cap C_2$ is nonempty, one can ensure that the relations

$$\operatorname{ord}(a_{jk} - c_2) = \operatorname{ord}(a_{jk} - c_1)$$

hold for $1 \leqslant j, k \leqslant 2$ and $(x, t) \in C_1 \cap C_2$. From this one can easily finish the case $(\mathrm{II})_k$.

We may suppose by symmetry (that is, reversing the role of C_1 and C_2) that, if $(\mathrm{III})_{\ell_1}$ holds on C_1, then also

$$\operatorname{ord}(t - c_1) + \ell_1 = \operatorname{ord}(c_2 - c_1) = \operatorname{ord}(t - c_2) + \ell_2 \tag{iii$_\ell$}$$

holds with $\ell = (\ell_1, \ell_2)$ and $-k \leq \ell_2 \leq k$. Suppose again that $C_1 \cap C_2$ is nonempty. If one now fixes the residue classes of $c_2 - c_1$ and of $t - c_1$ modulo $Q_{2kn,2kn}$, then the conditions

$$\operatorname{ord}(c_2 - c_1) = \operatorname{ord}(t - c_2) + \ell_2 \text{ and } t - c_2 \in \lambda_2 Q_{n,m}$$

follow automatically from $\operatorname{ord}(t - c_1) + \ell_1 = \operatorname{ord}(c_2 - c_1)$. Using this fact, one can easily partition $C_1 \cap C_2$ into finitely many $\mathscr{L}'_M$-cells. $\dashv$

The previous lemma has the following consequence, which forms a part of Proposition 2.

2.2. LEMMA. *Any quantifier-free $\mathscr{L}'_M$-definable subset of K^{k+1} can be partitioned as a finite union of $\mathscr{L}'_M$-cells.*

PROOF. By Lemma 2.1, it is enough to show that the sets (and complements of these sets)

$$\{(x,t)\in K^{k+1} \mid D_n(g_1,g_2,g_3)\} \quad \text{and} \quad \{(x,t)\in K^{k+1} \mid R_{n,m}(g_1,g_2,g_3)\},$$

with $g_i \in \{x_1,\dots,x_k,t\}\cup K$, can be partitioned into a finite union of $\mathscr{L}'_M$-cells, for integers $k\geq 0$ and $n,m\geq 1$. Since the $Q_{n,m}$ have finite index in $K^\times$, finding a partition for the relation $R_{n,m}$ is easy, as there exists a finite set $\Lambda_{n,m}\subset K$ such that

$$\{(x,y,t)\in K^3 \mid R_{n,m}(x,y,t)\} = \bigcup_{\lambda\in\Lambda_{n,m}} \{(x,y,t)\in K^3 \mid R_{nm}(x,y,\lambda)\wedge R_{nm}(t,0,\lambda)\}.$$

To treat the relations D_n, it suffices to show that the set

$$A := \{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) < n + \operatorname{ord}(t-c_2)\},$$

with $c_1,c_2\in\{x_1,\dots,x_k\}\cup K$, and $n\geq 1$, can be partitioned as a finite union of $\mathscr{L}'_M$-cells. We may suppose that $c_1\neq c_2$. Partition K^{k+1} in the following way:

$$\begin{aligned} K^{k+1} &= \{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) > \operatorname{ord}(c_1-c_2)\}\\ &\cup \{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) < \operatorname{ord}(c_1-c_2)\} \qquad (2.2.1)\\ &\cup \{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) = \operatorname{ord}(c_1-c_2)\}. \end{aligned}$$

By intersecting A with these three parts separately, we can finish the proof as follows. On

$$B_1 := A\cap\{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) > \operatorname{ord}(c_1-c_2)\},$$

one has that $\operatorname{ord}(t-c_2)=\operatorname{ord}(c_1-c_2)$, and that B_1 can be rewritten as

$$B_1 = \{(x,t)\in K^{k+1} \mid \operatorname{ord}(c_1-c_2) < \operatorname{ord}(t-c_1) < n+\operatorname{ord}(c_1-c_2)\}.$$

It is easy to see that this is a finite disjoint union of $\mathscr{L}'_M$-cells. The set

$$B_2 := \{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) < \operatorname{ord}(c_1-c_2)\},$$

is a subset of A, and is clearly a finite disjoint union of $\mathscr{L}'_M$-cells. Finally, let us consider the intersection

$$B_3 := A\cap\{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) = \operatorname{ord}(c_1-c_2)\}.$$

Then B_3 equals the intersection of

$$\{(x,t)\in K^{k+1} \mid \operatorname{ord}(c_1-c_2) < n+\operatorname{ord}(t-c_2)\}$$

with

$$\{(x,t)\in K^{k+1} \mid \operatorname{ord}(t-c_1) = \operatorname{ord}(c_1-c_2)\},$$

but these sets can both be partitioned as a finite disjoint unions of cells in a trivial way. Now use Lemma 2.1 to finish the proof. ⊣

PROOF OF PROPOSITION 1. The quantifier elimination statement follows from Lemma 2.2 and the definition of $\mathscr{L}'_M$-cells. Indeed, by the (partial) cell decomposition result from Lemma 2.2, it is sufficient to eliminate $\exists t$ from a condition $(x, t) \in A$, where $A \subset K^{k+1}$ is an $\mathscr{L}'_M$-cell. But the base of a cell is defined in a quantifier free way by Definition 1, and so we are done. It remains to show that $\mathscr{L}'_M$ is a definitional expansion of $\mathscr{L}_M$. Let Q_3 be the collection of nonzero elements of K with order divisible by 3. Write $\lambda \sim_3 \mu$ for $\lambda Q_3 = \mu Q_3$, where $\lambda Q_3 = \{\lambda t \mid t \in Q_3\}$. The relation $P(x, z)$ given by

$$(z - x) \sim_3 z \not\sim_3 x$$

is $\mathscr{L}_M$-definable and is equivalent with

$$\operatorname{ord}(z) < \operatorname{ord}(x) \wedge z \not\sim_3 x.$$

Hence, the relation $\operatorname{ord}(x) \leq \operatorname{ord}(y)$ for x and y in K is equivalent to

$$\forall z \big(P(x, z) \Longrightarrow z - y \sim_3 z\big).$$

Likewise, the relation $P(z - y, w - y)$ in y, z, w is $\mathscr{L}_M$-definable and is equivalent with

$$\operatorname{ord}(w - y) < \operatorname{ord}(z - y) \wedge w - y \not\sim_3 z - y.$$

We conclude that $\operatorname{ord}(z - y) \leqslant \operatorname{ord}(x - y)$ is equivalent with

$$\forall w \big(P(z - y, w - y) \Longrightarrow w - x \sim_3 w - y\big),$$

which is clearly $\mathscr{L}_M$-definable. Finally, for each integer k, $D_k(x, y, z)$ can easily be seen to be an $\mathscr{L}_M$-definable relation by using the above $\mathscr{L}_M$-definable relations and some quantifiers. ⊣

PROOF OF PROPOSITION 2. Follows from the quantifier elimination statement of Proposition 1 and Lemma 2.2. ⊣

PROOF OF PROPOSITION 3. Partition the graph of the definable function f in finitely many $\mathscr{L}'_M$-cells G of the form

$$\{(x, t) \in D \times K \mid \\ k_1 + \operatorname{ord}(a_1 - c) \,\square_1\, \operatorname{ord}(t - c) \,\square_2\, k_2 + \operatorname{ord}(a_2 - c),\ t - c \in \lambda Q_{n,m}\},$$

where the symbols have the same meaning as in Definition 1. For each $x \in D$ there is a unique t such that $(x, t) \in G$, since G is a part of the graph of f. This uniqueness condition implies that $\lambda = 0$ for each occurring part G, and thus each G is a cell of the following specific form:

$$\{(x, t) \in D \times K \mid t = c\}.$$

By Definition 1, the center c is either a constant from K or one of the variables x_i, so this finishes the proof. ⊣

§3. Some examples. The following lemma exhibits the fact that $\mathscr{L}_M$ does not have definable Skolem functions.

3.1. LEMMA. *Consider the $\mathscr{L}_M$-definable set*

$$A = \{(x, y) \in (K^\times)^2 \mid \operatorname{ord} y = 1 + \operatorname{ord} x\}.$$

Then there exists no $\mathscr{L}_M$-definable function $g\colon K^\times \to K$ such that the graph of g lies in A.

PROOF. Follows directly from Proposition 3. ⊣

Another natural question is whether there exists a reasonable classification of definable sets up to definable bijections. For semi-algebraic sets in $\mathbb{Q}_p$ or in a finite field extension of $\mathbb{Q}_p$, it is shown in [4] that there exists a semi-algebraic bijection between two infinite semi-algebraic sets if and only if they have the same dimension, and in [2] this is extended to the subanalytic setting. (See [16] for the definition and properties of the dimension of semi-algebraic sets, resp. [9] or [6] for subanalytic sets.) The next lemma gives an indication that a classification of $\mathscr{L}_M$-definable sets up to $\mathscr{L}_M$-definable bijections may be not so easy and will probably require new insights.

3.2. LEMMA. *Consider $\mathscr{L}_M$-definable sets given by*

$$A_i := \{x \in K \mid \operatorname{ord}(x) = k_i\}$$

for $i = 1, 2$ and some integers $k_1 \neq k_2$. Then there does not exist an $\mathscr{L}_M$-definable bijection between A_1 and A_2.

PROOF. After a finite partition of A_1 in $\mathscr{L}'_M$-cells, the bijection restricted to any of the parts would be of the form $x \mapsto x$ or $x \mapsto a$, for $a \in K$, by Propositions 2 and 3. But it is clear that none of these maps can have an image which is an infinite subset of A_2. Hence, such a bijection cannot exist. ⊣

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REFERENCES

[1] J. AX and S. KOCHEN, *Diophantine problems over local fields II. A complete set of axioms for p-adic number theory*, ***American Journal of Mathematics***, vol. 87 (1965), pp. 631–648.

[2] R. CLUCKERS, *Analytic p-adic cell decomposition and integrals*, ***Transactions of the American Mathematical Society***, vol. 356 (2003), no. 4, pp. 1489–1499.

[3] R. CLUCKERS and L. LIPSHITZ, *Fields with analytic structure*, ***Journal of the European Mathematical Society***, vol. 13 (2011), pp. 1147–1223.

[4] RAF CLUCKERS, *Classification of semi-algebraic p-adic sets up to semi-algebraic bijection*, ***Journal für die Reine und Angewandte Mathematik***, vol. 540 (2001), pp. 105–114.

[5] RAF CLUCKERS and FRANÇOIS LOESER, *b-minimality*, ***Journal of Mathematical Logic***, vol. 7 (2007), no. 2, pp. 195–227.

[6] J. DENEF and L. VAN DEN DRIES, *p-adic and real subanalytic sets*, ***Annals of Mathematics. Second Series***, vol. 128 (1988), no. 1, pp. 79–138.

[7] JAN DENEF, *p-adic semi-algebraic sets and cell decomposition*, ***Journal für die Reine und Angewandte Mathematik***, vol. 369 (1986), pp. 154–166.

[8] L. VAN DEN DRIES, D. HASKELL, and D. MACPHERSON, *One-dimensional p-adic subanalytic sets*, ***Journal of the London Mathematical Society***, vol. 59 (1999), no. 1, pp. 1–20.

[9] Deirdre Haskell and Dugald Macpherson, *A version of o-minimality for the p-adics*, this Journal, vol. 62 (1997), no. 4, pp. 1075–1092.

[10] Nianzheng Liu, *Semilinear cell decomposition*, this Journal, vol. 59 (1994), no. 1, pp. 199–208.

[11] A. Macintyre, *On definable subsets of p-adic fields*, this Journal, vol. 41 (1976), pp. 605–610.

[12] D. Macpherson and C. Steinhorn, *On variants of o-minimality*, ***Annals of Pure and Applied Logic***, vol. 79 (1996), no. 2, pp. 165–209.

[13] Marie-Hélène Mourgues, *Cell decomposition for P-minimal fields*, ***Mathematical Logic Quarterly***, vol. 55 (2009), no. 5, pp. 487–492.

[14] A. Prestel and P. Roquette, ***Formally p-adic fields***, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1984.

[15] Philip Scowcroft, *Cross-sections for p-adically closed fields*, ***Journal of Algebra***, vol. 183 (1996), no. 3, pp. 913–928.

[16] Philip Scowcroft and Lou van den Dries, *On the structure of semialgebraic sets over p-adic fields*, this Journal, vol. 53 (1988), no. 4, pp. 1138–1164.

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