NACHDIPLOM LECTURES NOTES – MOTIVIC INTEGRATION AND TRANSFER PRINCIPLES

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Abstract. The main topic of the course is motivic integration, with a viewpoint which is close to p-adic integration, uniform in p. The goal of the series is to work towards the recent transfer principles for motivic integrals, which allow one to transfer several results from p-adic fields to local fields of positive characteristic and vice versa. We will develop the notions of motivic exponential functions, forming a natural class of functions which is stable under integration and under Fourier transformation. These functions, together with the transfer principles, have been recently used in applications to the Langlands program, for example (with Hales and Loeser) to derive the Fundamental Lemma in characteristic zero from the work of Ngô, and to show local integrability of Harish-Chandra characters in large positive characteristic, with Gordon and Halupczok. The subject combines techniques from algebraic/non-archimedean geometry, model theory, harmonic analysis and number theory.

1. Lecture 1: Colloquium style overview

Slides have been used to present an overview.

2. Model theory

Model theory is an extension of algebra as an approach to geometry. One not only takes solution sets (zero loci) seriously, but also their images under coordinate projections. First order languages are to model theory what polynomial rings are for algebra and geometry. Taking this analogy further, definable sets are to model theory what zero loci (and their Boolean combinations) are for algebraic geometry. We give a focused introduction to model theory, based on [7]. For more details and context we refer to [7].

2.1. **One-sorted first order languages.** A first order language consists of functions symbols, relation symbols, and constant symbols, together with information about the arity for the function symbols and relation symbols.

Example 2.1.1. The Preburger language, for ordered abelian groups, more specifically, for the ordered abelian group $\mathbb{Z}$, consists of:

Function symbols: $+, -$ with arity for the domain: 2 (and arity of the range 1).

Relation symbols: $<$, $\cdot \equiv \cdot \bmod n$ for each $n \geq 1$, each with arity 2.

Constant symbols: $0, 1$.

2.2. **First order structure / intended structures.** A structure for a language $\mathcal{L}$ consists of a universe (a nonempty set) M (which is given arity 1), and for each function symbol from $\mathcal{L}$ a function with the corresponding arity, for each relation symbol from $\mathcal{L}$ a relation (subset of M^n) with the corresponding arity n, and for each constant symbol from $\mathcal{L}$ an element of M.

If one wants to distinguish between the symbols from $\mathcal{L}$ and their meaning in a structure M, one can for example write f for a function symbol in $\mathcal{L}$ and f^M for the function in the structure M, and c^M for the meaning of a constant symbol c of $\mathcal{L}$ and similarly for relation symbols.

Usually, a language is not interpreted arbitrarily to form a structure on some universe, but instead there usually is some intended or expected meaning of the symbols.

For example, the intended meaning of the Presburger language is with universe $\mathbb{Z}$, the group operations for $+, -$, order relation $<$, congruence relation modulo n, and the constant symbols $0, 1$.

The intended meaning often can be captured by a list of properties in the language itself, see the notion of theory below.

2.3. **Formulas.** Model theory is a formalism to describe properties of structures on the one hand, and to do geometry in the structures on the other hand.

For both purposes, one uses formulas and sentences in the language. Formulas are built from the language symbols, variables v_i, and the logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (not), the quantifiers $\exists$, $\forall$, and the left and right parenthesis (and). We will define in detail what formulas are, but first we define what terms in a language $\mathcal{L}$ are.

2.3.1. *Terms.* Terms are compositions of function symbols evaluated in variables and constants. More precisely:

The set of $\mathcal{L}$-terms is the smallest collection of finite (ordered) lists of symbols containing the following strings:

- each constant symbol from $\mathcal{L}$;
- each variables v_i for $i = 1, 2, \ldots$;
- compositions; namely, if $t_1, \ldots, t_n$ are terms, and f is a function symbol with arity n, then $f(t_1, \ldots, t_n)$ is a term.

If t is an $\mathcal{L}$-term in which the variables v_i for $i \in I$ and I a finite set of natural numbers, and if M is an $\mathcal{L}$-structure, then one has the corresponding meaning (interpretation) of t in M as the obvious function

$$M^I \to M,$$

also denoted by t, or by t^M if one wants to distinguish. More precisely, one defines t^M inductively on the complexity of the term t, where if $I = \emptyset$, M^I is the one-element set $\{*\}$, as follows.

- if t is just a function symbol c, then $I = \emptyset$ and one lets t take the constant value c^M on M^I.
- if v_i is a variable then the corresponding function is the identity function from M to M.
- if t_j are terms in the variables v_i for $i \in I_j \subset I$ and for $j = 1, \ldots, n$, and if t is the term $f(t_1, \ldots, t_n)$ for some function symbol f with arity n, then t^M is the composition of the function f^M with the map

$$M^I \to M^n : (a_1, \ldots, a_n) \mapsto (t_1^M(a_i)_{i \in I_1}, \ldots, t_n^M(a_i)_{i \in I_n}).$$

Just as one does for compositions of polynommial maps, one often writes $f(t_1(a), \ldots_n(a))$ for $a \in M^n$, for abbreviating the composition

$$f^M(t_1^M(a_i)_{i \in I_1}, \ldots, t_n^M(a_i)_{i \in I_n}) \text{ for } a \in M^I.$$

Terms are the building blocks for formulas. Later on we will see a yet more general kind of functions than those corresponding to terms, namely definable functions.

2.3.2. *Formulas.* First order formulas are expressions of finite length, build up as follows. The collection of $\mathcal{L}$-formulas is the smallest collection of finite (ordered) lists of symbols such that:

- if $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and if R is a relation symbol with arity n, then both
$$t_1 = t_2$$
and
$$R(t_1, \ldots, t_n)$$
are formulas.
- if ϕ and ψ are formulas, then so are
$$\neg\phi,$$
$$(\phi \wedge \psi),$$
$$(\phi \vee \psi),$$
$$\exists v_i \phi, \text{ and } \forall v_i \psi, \text{ for any } i > 0.$$

In fact, one uses parenthesis mainly to help readability and to avoid ambiguity; the strict usage of parenthesis as imposed by the definition of formulas is not important.

A formula ϕ is called a sentence if each variable that appears in ϕ is bound by a quantifier (at each occurrence of the variable).

A variable v_i which appears in a formula ϕ and occurs at least once freely (namely, unbound by a quantifier), is said to occur freely in ϕ.

For a language $\mathcal{L}$, an $\mathcal{L}$-structure M, and an $\mathcal{L}$-sentence ϕ, it is intuitively clear what it means for ϕ to hold in M, where one has to keep in mind that the quantifiers all run over the universe M. More generally and more precisely, one inductively defines for a formula in free variables v_i with $i \in I$ what it means to hold in M at a tuple $(a_i)_{i\in I}$ with $a_i \in M$, as follows.

An $\mathcal{L}$-formula of the form $t_1 = t_2$ for terms t_j in free variables v_j for $j \in I_J \subset I$ is true in M at $(a_i)_{i\in I}$ if and only if the values $t_1^M((a_i)_{i\in I_1})$ and $t_2^M((a_i)_{i\in I_2})$ are equal as elements of M. An $\mathcal{L}$-formula of the form $R(t_1, \dots, t_n)$ where the t_i are terms and where R is an n-ary relation symbol of $\mathcal{L}$ holds in M if and only if the tuple $(t_1^M, \dots, t_n^M)$ lies in the subset R^M of M^n.

Exercise 2.3.3. Check, for each of the formation rules for formulas, that you understand what it means to hold at a certain tuple $(a_i)_i$ in a structure. As an extra example, suppose that ϕ is a formula in free variables v_i for $i \in I$. Then $\exists v_j \phi$ is a formula in the free variables v_i for $i \in J := I \setminus \{j\}$ and it holds in $(a_i)_{i\in J}$ if and only if there exists an element $a_j \in M$ such that ϕ holds at the extended tuple $(a_i)_{i\in I}$.

For a sentence φ and M a structure, one denotes by $M \models \varphi$ that φ is true in M. For a formula φ in free variables v_i for $i \in I$ and elements a_i of M, one denotes by $M \models \varphi((a_i)_{i\in I})$ that φ holds in M at the tuple $(a_i)_{i\in I}$.

Example 2.3.4. Here is a formula in the Presburger language, with free variables v_1 and v_2:

$$((v_1 \equiv v_2 \mod 2) \vee (v_1 \equiv v_2 + 1 \mod 2))$$

This sentence is true for at any pair of integers, with the intended structure of the Presburger language on $\mathbb{Z}$. To be true (in a structure, or, at a tuple, etc.) is also called to be valid, to be logically valid, or to hold. We will abbreviate $\mathbb{Z}$ together with the intended $\mathcal{L}_{\text{Pres}}$-structure $\mathbb{Z}$ on $\mathbb{Z}$ typically by $\mathbb{Z}$ itself, or by $\mathbb{Z}, +, <$. If ϕ is the above formula, and ψ is the sentence $\forall v_1 \forall v_2 \phi(v_1, v_2)$, then we can thus write $\mathbb{Z} \models \psi$ to denote that ψ holds in the structure $\mathbb{Z}$.

2.4. **Definable sets.** Formulas serve two purposes: to describe properties (via sentences), or to describe sets (via formulas with free variables). These sets come in fact with the extra structure as subsets of some ambient spaces, and are called definable sets. This name is a bit ambiguous, since a definable set may be set-like, but it may also be more scheme-like, coming with a collection of sets by taking points lying on it over various structures.

A variable v_i which appears in a formula ϕ and occurs at least once freely (namely, unbound by a quantifier), is said to occur freely in ϕ.

Let φ be an $\mathcal{L}$-formula, say, with free variables v_i for $i \in I$ and I a finite set of natural numbers.

Let M be an $\mathcal{L}$-structure. Then φ determines a subset $\varphi(M)$ of M^I, as follows.

If $I = \emptyset$ is the emptyset, then $\varphi(M)$ is defined to be empty if φ does not hold in M, and to be the one-point set M^I if φ hold in M. Often, $M^\emptyset$ is denoted as the one-element set $\{*\}$, or as $\{0\}$, depending on the context.

If I is nonempty, then one puts

$$\varphi(M) = \{a = (a_i)_{i\in I} \in M^I \mid M \models \varphi(a)\}.$$

We give some side comments on the notation $M \models \varphi(a)$. Let us denote by $\varphi(a)$ the expression obtained from φ by replacing each free occurrence of v_i by the element a_i of M. Note that this expression $\varphi(a)$ can naturally be seen as a formula in the enrichment of $\mathcal{L}$ obtained by joining a constant symbol for the elements a_i of M. This language is denoted by $\mathcal{L}(a)$ and is called $\mathcal{L}$ with parameters (or coefficients) from the tuple $a = (a_i)_i$. One extends the $\mathcal{L}$-structure on M naturally to an $\mathcal{L}(a)$-structure. Whether $\varphi(a)$ holds or not in the $\mathcal{L}(a)$-structure M has the same meaning as for φ to hold at a in the $\mathcal{L}$-structure M. More generally, for A any subset of M, one denotes by $\mathcal{L}(A)$ the language $\mathcal{L}$ together with constant symbols for each of the elements of A.

The set $\varphi(M)$ is called a definable set, and it comes with the extra structure as a subset of M^I. Now that we know what a definable set is in a structure M, we can say what definable functions between them are.

A function $g : A \to B$ between definable sets A and B is called a definable function, whenever its graph

$$\{(x, y) \in A \times B \mid y = g(x)\}$$

is a definable set. Note that we use the word definable often instead of $\mathcal{L}$-definable, when the presence of a language $\mathcal{L}$ is sufficiently clear. The usual terminology for sets goes also for definable sets, like Cartesian products, intersections, fiber products, and so on.

In fact, one uses the precise numbering and listing of the free variables mainly to avoid ambiguity, and is not so important in itself. For example, one usually denotes M^I by M^n when I has n elements.

A definable set $A \subset M^n$ can be described by many different formulas. To choose a formula ϕ corresponding to a given definable set is called to choose a formula behind A, or, to choose representatives.

In the other way around, a formula ϕ yields many sets, similar to the way polynomial equations $f_i(x) = 0$ in variables $x_1, \ldots, x_n$ over a ring R give a solution set in R'^n for any algebra R' over R, and more abstractly similar to taking rational points on schemes. Most generally, a formula ϕ in a language $\mathcal{L}$ in n free variables gives a definable subset $\phi(M)$ of M^n for any $\mathcal{L}$-structure M. Most often, one is interested only in certain $\mathcal{L}$-structure M considered natural or relevant in the respective contexts. On all such scales of generality, the collection of sets $\phi(M)$ where M runs over a collection of $\mathcal{L}$-structures, is called a definable set. Note that thus, a definable set may actually be a whole collection of sets. Note also that we often denote the structure and the universe with the same notation.

2.5. **Many sorted first order languages.** In a many-sorted language, there can be several different kinds of variables running over different sets. In a one-sorted language, all variables are of the same kind, the same 'sort'.

For example, in the context of henselian valued fields, it is natural to have at least three sorts of variables: one for the valued field, one for the residue field, and one for the value group. Also the quantifiers will be dedicated to particular sorts and the function symbols and relation symbols will have precise information not only about their arities, but also the specific kinds of variables they take as input and output (the latter only for the functions and constant symbols).

In fact, the whole set-up of multi-sorted languages can be reduced, via the right definitions, to the one-sorted case, so that general properties of model theory go through without any change in the multi-sorted case.

Let us now define formally the multi-sorted first order languages, and describe how they fall under the scope of one-sorted languages.

Let there be given an index set S to index the sorts. A multi-sorted language $\mathcal{L}$ with sorts indexed by S consists of functions symbols, relation symbols, and constant symbols, together with information about the arity and description of the input and output for the function symbols and relation symbols and constant symbols. More precisely, a function symbol f from the language with arity n comes with the extra information of n ordered elements of S for the input, and one element of S for the output; a relation symbol with arity n comes with the extra information of n ordered elements of S for the input, and a constant symbol comes with the extra information of one element of S.

Example 2.5.1. The Denef-Pas language $\mathcal{L}_{\rm DP}$ for valued fields is as follows:

There are three sorts, called (more formally, denoted by): valued field, VF, residue field, RF, and value group together with an infinity element, VG.

The function symbols are:

- ring operations $+, -, \cdot$ with arity 2, input $(\mathrm{VF}, \mathrm{VF})$ and and output VF;
- ring operations $\overline{+}, \overline{-}, \bar{\cdot}$ with arity 2, input $(\mathrm{RF}, \mathrm{RF})$ and and output RF;
- group operations $\tilde{+}, \tilde{-}$ with arity 2, input $(\mathrm{VG}, \mathrm{VG})$ and and output VG;
- a function $\overline{\mathrm{ac}}$, called the angular component map, with arity 1, input VF and output RF,
- a function symbol ord, called the valuation map, with arity 1, input VF and output VG.

The relation symbols are $<$ and

$\equiv \bmod n$ for each $n \geq 1$, all with arity 2 and all on VG.

Finally, one has in each of the sorts a copy of the constant symbols 0 and 1.

The intended meaning will be described below.

A structure $\mathcal{M}$ for a multi-sorted language $\mathcal{L}$ now comes with for each $s \in S$ a universe (nonempty set) M_s, together with functions, relations and elements for each of the symbols of $\mathcal{L}$, of course with the corresponding arity and input/output universes.

A formula in a many-sorted language is build up as above, but now each variable has a designated sort associated to it. Let us write $v_{s,i}$ for $s \in S$ and $i \geq 1$ to denote variables of sort $s \in S$. Terms are defined as above, where now terms also have input-sorts and an output-sort which are to be handled in the obvious way.

Example 2.5.2. For k a field, the laurent series field $k((t))$ can be turned into an $\mathcal{L}_{\mathrm{DP}}$-structure with sorts $k((t))$ as valued field VF-sort, k as residue field RF-sort, and $\mathbb{Z} \cup \{+\infty\}$ as VG-sort. One interprets the ring operations in the natural way in k and in $k((t))$, the Presburger language naturally on $\mathbb{Z}\cup\{+\infty\}$, ord becomes the t-adic valuation map on $k((t))^{\times}$ assigning 1 to t, extended by $+\infty$ on 0, and $\overline{\mathrm{ac}}$ sends nonzero Laurent series to its first nonzero coefficient and sends 0 to 0. Here, $a \pm \infty$ is $+\infty$ for any integer a, and none of the $a \equiv +\infty \bmod n$ and $+\infty \equiv a \bmod n$ holds by convention.

A definable subset for $\mathcal{L}_{\mathrm{DP}}$ in the structure $k((t))$ is a subset of

$$k((t))^n \times k^m \times (\mathbb{Z} \cup \{+\infty\})^r$$

for some integers $n \geq 0, m \geq 0$, and $r \geq 0$, and one typically writes $\{0\}$ for the one-point definable set $\{*\}$ corresponding to a true formula without free variables.

2.6. **Link with one-sorted structures.** Let us now explain how many-sorted languages can be seen as falling in the framework of one-sorted languages.

Let $\mathcal{L}_S$ be a many-sorted language, with sorts indexed by S. We now construct a corresponding one-sorted language $\mathcal{L}_1$.

First, let us put a dummy constant symbol $\bullet$ in $\mathcal{L}_1$ which is not yet occurring in $\mathcal{L}_S$. For each $s \in S$, let us put a relation symbol R_s in $\mathcal{L}_1$, and finally, let us put all function, relation and constant symbols of $\mathcal{L}_S$ in $\mathcal{L}_1$, with the same arity but with only one sort.

If one has an $\mathcal{L}_S$-structure $\mathcal{M}$ with universes M_s for the sorts $s \in S$, there is a corresponding $\mathcal{L}_1$-structure M_1 with universe $M = \cup_{s \in S} M_s$, the union being realized in a disjoint way. Any function symbol f from $\mathcal{L}_S$ with arity n taking sorts $s_1, \dots, s_n$ as input and s_{n+1} as output gets the meaning of the function

$$M^n \to M : a = (a_1, \dots, a_n) \mapsto \begin{cases} f(a) & \text{if } a_i \in M_{s_i} \text{ for each } i, \\ \bullet & \text{otherwise.} \end{cases}$$

A relation symbol R of $\mathcal{L}_S$ with arity n and taking sorts $s_1, \dots, s_n$ as input has the meaning on M_1 that $R(a)$ is true if and only if $a_i \in M_{s_i}$ for each i and $R(a)$ holds in the $\mathcal{L}_S$-structure $\mathcal{M}$. Finally, the relation symbols R_s for $s \in S$ have arity 1 and $R_s(a)$ holds in M_1 if and only if $a \in M_s$.

This way, there is a correspondence between $\mathcal{L}_1$-definable sets in M and $\mathcal{L}_S$-definable sets in $\mathcal{M}$. This correspondence goes further than just for definable sets, with the notion of theories that we introduce next.

2.7. **Theories.** Theories can capture/prescribe algebraic properties of a structure. They are useful to study properties of definable sets as well. For us, languages will always be first order, and the same for formulas and so on.

Definition 2.7.1. A collection $\mathcal{T}$ of sentences in a language $\mathcal{L}$ is called an $\mathcal{L}$-theory. An $\mathcal{L}$-structure in which all of the sentences of a theory $\mathcal{T}$ are true is called a model of $\mathcal{T}$.

Example 2.7.2. In the Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$ we can formulate the theory $\mathcal{T}_{\mathrm{DP}}$ of henselian valued fields of characteristic 0.

Note that this theory is an infinite collection of sentences, consisting of the axioms of valued fields, for each degree an axiom saying that if the conditions of hensel's lemma are fulfilled for a polynomial of that degree then so is the conclusion of hensel's lemma, and, for each prime number p, an axiom saying that $p \neq 0$ in the valued field sort. Here, p is an abbreviation of the $\mathcal{L}_{\mathrm{DP}}$-term being the iterate sum of p copies of 1. Natural abbreviations will often go unexplained.

Here are some more details, for notational convenience explained on an $\mathcal{L}_{\mathrm{DP}}$-structure with three sorts F, k_F and $\Gamma_F \cup \{+\infty\}$. The intended meaning of the theory is that, if the theory holds in our structure, then F is a henselian valued field of characteristic zero, with residue field k_F, and additively written value group Γ_F, and F is furthermore equipped with an angular component map $\overline{\mathrm{ac}} : F \to k_F$.

- To say that F and k_F are fields with $\mathcal{L}_{\mathrm{DP}}$-sentences is easy.
- Also to express that Γ_F is an ordered abelian group with $\mathcal{L}_{\mathrm{DP}}$-sentences is easy.
- Next one expresses, with $\mathcal{L}_{\mathrm{DP}}$-sentences, that $\mathrm{ord} = \mathrm{ord}_F : F \to \Gamma_F \cup \{+\infty\}$ is a surjective map, with $\mathrm{ord}(0) = +\infty$, $\mathrm{ord}(x + y) \geq \min(\mathrm{ord}\, x, \mathrm{ord}\, y)$ for all x, y and ord restricts to a group homomorphism $F^\times \to \Gamma_F$.
- By taking universal quantifiers over the coefficients of the polynomial of given degree, one expresses with a sentence that hensel's lemma holds in that degree.

- Finally one has the sentences $p \neq 0$ in the valued field sort for all prime numbers p.

Note that the above theory leaves the interpretation of $\overline{\mathrm{ac}}$ and the relations $\equiv \bmod n$ unspecified. The above theory is naturally extended to the theory by sentences saying that

- one has $x \equiv y \bmod n$ if and only if there exists $z \in \Gamma_F$ such that $nz = x - y$, where nz is the abbreviation of the iterated sum of z with itself, n times,

and saying that

- the function $\overline{\mathrm{ac}} : F \to k_F$ is multiplicative, $\overline{\mathrm{ac}}(x) = 0$ if and only if $x = 0$, the restriction of $\overline{\mathrm{ac}}$ to $\mathcal{O}_F^\times \cup \{0\}$ is surjective onto k_F and coincides with the restriction to $\mathcal{O}_F^\times \cup \{0\}$ of the projection $\mathcal{O}_F \to \mathcal{O}_F/\mathcal{M}_F$ under a field isomorphism $k_F \sim \mathcal{O}_F/\mathcal{M}_F$.

Here, $\mathcal{O}_F$ is the valuation ring $\{x \in F \mid \mathrm{ord}(x) \geq 0\}$ of F, with maximal ideal $\mathcal{M}_F = \{x \in F \mid \mathrm{ord}(x) > 0\}$. By the axioms of the theory, we may and do consider k_F as the residue field of F.

Exercise 2.7.3.
- Check that the conditions on $\overline{\mathrm{ac}}$ as described at the end of Example 2.7.2 are expressible by finitely many $\mathcal{L}_{\mathrm{DP}}$-formulas.

We will see the next time how to generalize the Denef-Pas language to a more powerful language, and we will go from the infinite to the finite, via model theoretical compactness, and similarly, via Gödel's completeness result.

2.8. **Conventions on valued fields.** Before passing to the generalized Denef-Pas language, let us recall some basic definitions on valued fields. A valuation ring is an integral domain R such that for every element x of its field of fractions F, at least one of x or x^{-1} belongs to D. The field of fractions of a valuation ring is called a valued field.

Equivalently, a field F together with a surjective map $\mathrm{ord} = \mathrm{ord}_F : F \to \Gamma_F \cup \{+\infty\}$, with $\mathrm{ord}(0) = +\infty$ and with Γ_F an ordered abelian group, is called a valued field if $\mathrm{ord}(x + y) \geq \min(\mathrm{ord}\, x, \mathrm{ord}\, y)$ for all x, y and ord restricts to a group homomorphism $F^\times \to \Gamma_F$. The map ord is called the valuation map and Γ_F the value group.

By a local field we mean a locally compact topological field with topology coming from a discrete valuation, that is, the value group is the ordered group $\mathbb{Z}$. The local fields are precisely the p-adic fields, their finite field extensions, and the fields of Laurent series $\mathbb{F}_q((t))$ for any prime power q.

For F a valued field, we write $\mathcal{O}_F := \{x \in F \mid \mathrm{ord}_F(x) \geq 0\}$ for the valuation ring with maximal ideal $\mathcal{M}_F := \{x \in F \mid \mathrm{ord}_F(x) > 0\}$, residue field $k_F = \mathcal{O}_F/\mathcal{M}_F$, and value group Γ_F. Further, we write F_n for the quotient ring $\mathcal{O}_F/(n\mathcal{M}_F)$, $\overline{\mathrm{proj}}_n$ for the projection $\mathcal{O}_F \to \mathcal{O}_F/(n\mathcal{M}_F)$, and $\overline{\mathrm{proj}}_{mn}$ for the projection $\mathcal{O}_F/(m\mathcal{M}_F) \to \mathcal{O}_F/(n\mathcal{M}_F)$ for each n, m with $m > 0$ a multiple of $n > 0$.

Let I be an ideal of $\mathcal{O}_F$. An angular component modulo I (on F) is a multiplicative map $\overline{\mathrm{ac}}_I : F \to \mathcal{O}_F/I$, such that $\overline{\mathrm{ac}}_I(x) = 0$ if and only if $x = 0$, and coinciding with $\overline{\mathrm{proj}}_n$ on the units of $\mathcal{O}_F$. A system of angular components on F is the data, for each $n > 0$, of an angular component modulo $n\mathcal{M}_F$, which make commuting diagrams with the $\overline{\mathrm{proj}}_{mn}$, namely, $\overline{\mathrm{ac}}_n(x) = \overline{\mathrm{proj}}_{mn}\overline{\mathrm{ac}}_m(x)$ for all $x \in F$.

A valued field F is called henselian if $\mathcal{O}_F$ satisfies Hensel's Lemma:

> **Hensel's Lemma** For all $P(x)$ in $\mathcal{O}_F[x]$ and $a \in \mathcal{O}_F$, if $\mathrm{ord}(P(a)) > 0$ and $\mathrm{ord}(P'(a)) = 0$, then there exists a unique $b \in O_F$ such that $P(b) = 0$ and $a \equiv b \bmod \mathcal{M}_F$.

A valued field F is called complete if $\mathcal{O}_F$ is complete in its I-adic topology for some nonzero ideal I contained $\mathcal{M}_F$. Here to be complete in the I-adic topology means that $\mathcal{O}_F$ equals its completion $\widehat{\mathcal{O}}_F = \varprojlim \mathcal{O}_F/I^n$, that is, the natural map $\mathcal{O}_F \to \widehat{\mathcal{O}}_F$ is a bijection.

2.9. **Generalized Denef-Pas language.** The generalized Denef-Pas language consists of the sorts VF for valued field, the R_n for $n \geq 1$ for the residue rings modulo the product ideal of the ideal generated by the integer n with the maximal ideal, and VG for the union of $\{+\infty\}$ with the additively written value group. It has as symbols the ring language on VF, the ring language on the R_n, the Presburger language on VG, the order map from VF to VG, and the angular component maps $\overline{\mathrm{ac}}_n : \mathrm{VF} \to \mathrm{R}_n$ for all $n \geq 1$.

Let us write $\mathcal{L}_{\mathrm{gDP}}$ for the generalized Denef-Pas language.

The corresponding theory $\mathcal{T}_{\mathrm{gDP}}$ of $\mathcal{L}_{\mathrm{gDP}}$ is given by $\mathcal{T}_{\mathrm{DP}}$ together with the axioms that where for each $n > 0$, $\overline{\mathrm{ac}}_n$ is an angular component map modulo the product ideal of (n) with the maximal ideal. (Note that this includes the property that F_n and $\mathrm{R}_n(F)$ are ring-isomorphic.)

Exercise 2.9.1.

- Check that the theory $\mathcal{T}_{\mathrm{gDP}}$ is indeed a first order theory.

  Hint: let F be a henselian valued field with value group Γ_F, and with residue rings $F_n = \mathcal{O}_F/n\mathcal{M}_F$ for $n \geq 1$. Check that the mentioned conditions on the symbols and sorts of the generalized Denef-Pas language form a $\mathcal{L}_{\mathrm{gDP}}$-theory. This amounts to show that one can express with sentences that the F_n are (isomorphic to) $\mathcal{O}_F/n\mathcal{M}_F$ and that the conditions on the $\overline{\mathrm{ac}}_n$ hold.

Note that, given a field F, angular component maps are not canonical (but neither are valuation maps on F), and in fact, on some valued fields angular component maps may not exist, see [10]. In our setting, the value group will always be $\mathbb{Z}$, and then angular component maps always exist and they can be fixed by choosing a uniformizer of $\mathcal{O}_F$.

3. Logical compactness

3.1. In this section we study logical compactness, which is very closely related to Gödel's Completeness result and which links the infinite to the finite. In this respect it is similar to the topological notion of compactness, with the difference that model theoretical compactness is not something one has to impose but comes for free for any first order theory. This is in fact an important result, called the compactness theorem.

If not specified otherwise, our theories and formulas will be in a first order language $\mathcal{L}$.

Definition 3.1.1.
- A theory T is called satisfiable if and only if T has at least one model.
- A sentence ϕ is called a logical consequence of T if $M \models \phi$ for all models M of T. One writes $T \models \varphi$.

If T is a theory, then a subset of T is of course also a theory. The following compactness theorem links infinite theories with finite subsets.

Theorem 3.1.2 (Compactness)**.** *A theory T has a model if and only if every finite subset of T has a model.*

3.2. **Some basic applications.** We give three examples indicating that compactness, syntactical arguments, and quantifier elimination often go hand in hand.

1) The following property is a very basic variant of the Ax-Kochen principle.

Let an $\mathcal{L}_{\mathrm{DP}}$-sentence ϕ be a logical consequence of $\mathcal{T}_{\mathrm{DP}}$, the theory of henselian valued fields of characteristic 0. Then, there is M such that for any henselian valued field K of characteristic $> M$ one has that ϕ holds in K.

Exercise 3.2.1. Show the above property using the compactness theorem.

2) A quantifier free variant of the Ax-Kochen principle.

Let ϕ be a sentence without quantifiers running over the valued field (that is, there are no quantifiers with variables of the valued field sort). Then, there is M such that for any henselian valued fields K, L with isomorphic residue fields of characteristic al least M and with isomorphic value groups (as ordered groups), one has

$$K \models \phi \text{ if and only if } L \models \phi.$$

Exercise 3.2.2. Show the above property using syntactical properties (namely, properties of the formulas of the described forms) and the compactness theorem.

The full variant of the Ax-Kochen principle is stronger since it combines 2) with a quantifier elimination result.

3) Full Ax-Kochen (transfer) principle.

The previous variant in 2) holds in fact for any sentence, as follows. Let ϕ be any $\mathcal{L}_{\mathrm{DP}}$-sentence. Then, there is M such that for any henselian valued fields K, L with isomorphic residue fields of characteristic al least M and with isomorphic residue fields and isomorphic value groups (as ordered groups), one has

$$K \models \phi \text{ if and only if } L \models \phi.$$

This full version of the Ax-Kochen principle is proved using Compactness and syntax as in 1) and 2), and a quantifier elimination result in the language $\mathcal{L}_{\mathrm{DP}}$. This quantifier elimination result states that, for any $\mathcal{L}_{\mathrm{DP}}$-sentence ψ, there exists an $\mathcal{L}_{\mathrm{DP}}$-sentence ϕ without valued field quantifiers such that in all models K of $\mathcal{T}_{\mathrm{DP}}$, one has that ψ holds in K if and only if ϕ holds in K. We will see the proof of this quantifier elimination result later, and derive the full Ax-Kochen principle from it quite easily.

3.3. **Henkin's proof of the Compactness Theorem.** Let us prove the compactness result (theorem 3.1.2), following Marker [7], based on Henkin's method.

Definition 3.3.1.

- Call a theory T finitely satisfiable if every finite subset of T is satisfiable.
- Say that an $\mathcal{L}$-theory T has the witness property if whenever ϕ is a formula with one free variable v, then there is a constant symbol c in $\mathcal{L}$ such that
$$T \models (\exists v \phi(v)) \to \phi(c).$$
- Call a theory T maximal if for all sentences ϕ, either $\phi \in T$ or $\neg\phi \in T$.

The symbol arrow $\to$ in the above definition means 'implies', and $p \to q$ stands for the obvious abbreviation $\neg p \vee q$.

Likewise, a double arrow $p \leftrightarrow q$ abbreviates $(p \wedge q) \vee (\neg p \wedge \neg q)$.

Lemma 3.3.2. *Suppose that T is maximal and finitely satisfiable. If $\Delta \subset T$ is finite and $\Delta \models \psi$ for some sentence ψ, then also $\psi \in T$.*

Proof. Exercice. □

Lemma 3.3.3. *Suppose that T is an $\mathcal{L}$-theory with the witness property and that T is maximal and finitely satisfiable. Then T has a model.*

Sketch of proof. Let $\mathscr{C}$ be the set of constant symbols of T. We put on $\mathscr{C}$ the equivalence relation $\sim$, defined by $c \sim d$ if and only if

$$T \models c = d.$$

Now let M be the quotient $\mathscr{C}/\sim$. We will indicate how to turn M into an $\mathcal{L}$-structure so that it becomes a model for T.

A constant symbol c of $\mathcal{L}$ is interpreted naturally as $c/\sim$ in M. The relation symbols and function symbols of $\mathcal{L}$ are interpreted in M by taking representatives in $\mathscr{C}$ of the elements of M, where one shows independency of the choice of representatives. For example, a 1-ary relation symbol R is interpreted as the subset

$$\{c/\sim \in M \mid T \models R(c)\}.$$

As a second example, for a 1-ary function symbol f, by the witness property, for each $c/\sim \in M$, there exists $d \in \mathscr{C}$ such that

$$T \models f(c) = d;$$

one then puts $f(c/\sim) := d/\sim$. Next one shows that terms behave as expected. This means in the simple case of a term t in one free variable v that $t(c/\sim) = d/\sim$ if and only if $T \models t(c) = d$. This behaviour is proved by induction on the complexity of terms.

In the same spirit one shows for any formmula ψ that it holds at $(c_i/\sim)_i$ if and only if $T \models \psi((c_i)_i)$, namely by induction on the formula ψ. In particular, one has

$$M \models T,$$

which we needed to prove. □

Exercise 3.3.4. Check the details of the proof of Lemma 3.3.3. Discover where maximality and the witness property are used. Hint: use Lemma 3.3.2.

There only remains to be shown that any finitely satisfiable theory T can be extended to a finitely satisfiable theory T^* which is maximal and which has the witness property. This is done by the two following lemmas, combined with Zorn's Lemma, and with the fact that if $T_1 \subset T_2$ are $\mathcal{L}$-theories such that T_1 has the witness property, then T_2 has the witness property too.

Lemma 3.3.5. *Suppose that T is a finitely satisfiable theory and ϕ is a sentence. Then, either $\{\phi\} \cup T$ or $\{\neg\phi\} \cup T$ is finitely satisfiable.*

Proof. Suppose that $T \cup \{\phi\}$ is not finitely satisfiable. Then there exists a finite set $\Delta \subset T$ such that

$$\Delta \models \neg\phi.$$

One checks that $\Sigma \cup \Delta \cup \{\neg\phi\}$ is satisfiable for any finite set $\Sigma \subset T$. □

Lemma 3.3.6. *Let T be a finitely satisfiable theory. Then there is a language $\mathcal{L}^*$ containing $\mathcal{L}$ and an $\mathcal{L}^*$-theory T^* with the witness property and containing T.*

Sketch of proof. We show that there is a languag $\mathcal{L}_1$ containing $\mathcal{L}$ and a finitely satisfiable $\mathcal{L}_1$-theory T_1 with the following partial witness property: for any $\mathcal{L}$-formula ϕ in one free variable, there exists an $\mathcal{L}_1$-constant c such that

$$T_1 \models (\exists v \phi(v)) \rightarrow \phi(c).$$

The lemma then follows by iterating the procedure to obtain $\mathcal{L}_2$ and T_2, $\mathcal{L}_3$ and so on, and taking the countable union $\cup_i \mathcal{L}_i$ as $\mathcal{L}^*$ and $\cup_i T_i$ as T^*.

Let $\mathcal{L}_1$ be the language $\mathcal{L}$ together with, for every $\mathcal{L}$-formula ϕ in one free variable, a constant symbol c_ϕ. Let T_1 be T together with, for every $\mathcal{L}$-formula ϕ in one free variable, the sentence $(\exists v \phi(v)) \to \phi(c_\phi)$. To conclude one checks that T_1 is finitely satisfiable. $\square$

3.3.7. *Completeness.* A similar statement to the compactness result is the completeness theorem. Let us explain the completeness theorem without details.

We will not define what a strict proof is. Let us just say that a strict proof based on a theory T can be checked step by step by a computer, if the computer can check whether a sentence belongs to the theory T or not. We give Gödel's completeness for its historical relevance. It can be proved along the same lines as the Compactness theorem.

Theorem 3.3.8 (Gödel's completeness)**.** *A sentence ϕ has a strict proof from T if and only if ϕ is a logical consequence of T.*

3.3.9. *Incompleteness.* We end this section with a word on the incompleteness Theorem.

Given a structure M, the theory of the structure, denoted by $\mathcal{T}(M)$, is the collection of all the sentences which hold in M.

Such a theory $\mathcal{T}(M)$ is called decidable if there exist a subtheory $\mathcal{T} \subset \mathcal{T}(M)$ and a computer algorithm which decides whether $\phi \in \mathcal{T}$ or $\phi \notin \mathcal{T}$ such that, for any sentence ϕ, there exists a strict proof of ϕ or of $\not\phi$ using only sentences from $\mathcal{T}$ (and the derivation rules allowed in strict proofs).

The incompleteness result by Gödel states that the theory of the integers in the ring language is undecidable.

That there exist structures with undecidable theory is not so surprizing in itself, it is just a property for a theory of being 'not being finitely generated'. What is more surprising is that such a basic and omnipresent structure like the ring of integers itself is undecidable.

Also $\mathbb{Q}$ in the ring language is undecidable. Also some important subsets of the theory $\mathcal{T}(\mathbb{Z}, +, \cdot)$ of the ring $\mathbb{Z}$ are undecidable: the solution to Hilbert's 10th problem shows that already the collection of those sentences from $\mathcal{T}(\mathbb{Z}, +, \cdot)$ which are diophantine (namely sentences which begin with some existential quantifiers, followed by a quantifier free formula), form an undecidable theory.

Quantifier elimination results, which we will encounter later, are often related to decidability. However, we will use quantifier elimination for other, more geometric reasons.

4. Quantifier elimination for $\mathcal{L}_{\mathrm{gDP}}$

4.1. For having quantifier elimination, we need to enlarge $\mathcal{L}_{\mathrm{gDP}}$ a bit, to a definitial expansion.

Let $\mathcal{L}'_{\rm gDP}$ be the expansion of $\mathcal{L}_{\rm gDP}$ obtained by adding the function symbols ord_n and $\overline{\mathrm{proj}}_{mn}$ for each n, m with $n > 0$ dividing $m > 0$, and relation symbols A_n for each $n > 0$.

The $\mathcal{L}'_{\rm gDP}$ theory $\mathcal{T}'_{\rm gDP}$ explains how these symbols are interpreted, namely as the induced order maps ord_n from $\mathrm{R}_n \setminus \{0\}$ to VG extended by $+\infty$ on 0, the natural projection maps $\overline{\mathrm{proj}}_{mn} : \mathrm{R}_m \to \mathrm{R}_n$ for n dividing m and, for $n \geq 1$, the 1-ary relation symbol A_n is the subset of R_n consisting of all nonzero elements which lift to an element in the valued field with $\overline{\mathrm{ac}}_n$ equal to one; in symbols in a structure with valued field F,

$$A_n(F) = \{x \in F_n \setminus \{0\} \mid \exists y \in \mathcal{O}_F(\overline{\mathrm{ac}}_n(y) = 1 \wedge x = \overline{\mathrm{proj}}_n(y))\}.$$

Exercise 4.1.1. Note that $\mathcal{L}'_{\rm gDP}$ is a definitial expansion of $\mathcal{L}_{\rm gDP}$, meaning that they have the same definable sets. More precisely, any $\mathcal{L}_{\rm gDP}$-structure can be naturally made into a $\mathcal{L}'_{\rm gDP}$-structure, and, a set is $\mathcal{L}'_{\rm gDP}$-definable if and only if it is $\mathcal{L}_{\rm gDP}$-definable.

One has the following form of quantifier elimination for the generalized Denef-Pas language $\mathcal{L}_{\rm gDP}$.

Theorem 4.1.2. *Let ψ be an $\mathcal{L}_{\rm gDP}$-formula. Then there exists an $\mathcal{L}'_{\rm gDP}$-formula ϕ without valued field quantifiers, such that*

$$\mathcal{T}_{\rm gDP} \models \psi \leftrightarrow \phi.$$

In other words, for any $\mathcal{T}_{\rm gDP}$-model K, one has

$$K \models \psi$$

if and only if one has

$$K \models \phi.$$

Before turning to the proof of Theorem 4.1.2, let us come back to Ax-Kochen principles.

Exercise 4.1.3. Show how the full Ax-Kochen Principle 3) of Section 3.2 follows from compactness together with 2) of Section 3.2 and Theorem 4.1.2.

4.2. **Proving Theorem 4.1.2.** Theorem 4.1.2 can be proved using Proposition 4.2.1 by J. Flenner, similarly in [6]. Alternatively it can be proved by the proof scheme going back to Cohen [4], see also [5], [8], [9], or by more recent techniques of A. Robinson for proving quantifier elimination in model theory as in [1]. In fact, in the presented form, the quantifier elimination result in the form of Theorem 4.1.2 appears in [12].

Here we give Flenner's lemma, to get a flavour of what is used to prove Theorem 4.1.2. Other input are hensel's lemma, induction on the degrees of the polynomials, and some transformations to reduce to a situation where hensel's lemma can be applied.

Proposition 4.2.1 (Proposition 3.4 of [6]). *Let an integer $d > 0$ be given. Then there exists an integer D such that the following holds for all henselian valued fields F of characteristic 0, all polynomials $f(x)$ over F in one variable and of degree d, and all elements $z \in F$. Let C_f be the collection of all zeros in F of the polynommial f and its derivatives f', up to $f^{(d-1)}$. There exist $c = c(z)$ in C_f such that if one rewrites the polynomial f as*

$$f(x) = \sum_{i=0}^{d} a_{i,c}(x-c)^i,$$

and if one puts

$$\delta = \delta_{f,c}(z) = \min_i \operatorname{ord}(a_{i,c}(z-c)^i).$$

then one has

$$\delta \leq \operatorname{ord} f(z) \leq \delta + \operatorname{ord}(D).$$

5. The integers

In this section we treat the results from Section 2 of [3]. The proofs therein are based on quantifier elimination in the Presburger language [11], and the parametric rectilinearization result, Theorem 3, from [2].

References

1. Z. Chatzidakis, *Notes on the model theory of finite and pseudo-finite fields*, http://www.logique.jussieu.fr/∼zoe/index.html, Notes d'un mini-cours donné à l'Université Autonome de Madrid en novembre 2005.
2. R. Cluckers, *Presburger sets and p-minimal fields*, J. of Symbolic Logic **68** (2003), 153–162,
3. R. Cluckers, J. Gordon, and I. Halupczok, *Integrability of oscillatory functions on local fields: transfer principles*, (To appear in Duke Mathematical Journal), arXiv:1111.4405.
4. P. J. Cohen, *Decision procedures for real and p-adic fields*, Comm. Pure Appl. Math. **22** (1969), 131–151.
5. J. Denef, *p-adic semialgebraic sets and cell decomposition*, Journal für die reine und angewandte Mathematik **369** (1986), 154–166.
6. J. Flenner, *Relative decidability and definability in henselian valued fields*, J. Symbolic Logic **76** (2011), no. 4, 1240–1260.
7. D. Marker, *Model theory: an introduction*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, 2002.
8. J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, Journal für die reine und angewandte Mathematik **399** (1989), 137–172.
9. ______, *Cell decomposition and local zeta functions in a tower of unramified extensions of a p-adic field*, Proc. London Math. Soc. (3) **60** (1990), no. 1, 37–67.
10. ______, *On the angular component map modulo p*, J. Symbolic Logic **55** (1990), 1125–1129.
11. M. Presburger, *On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation*, History and Philosophy of Logic **12** (1991), no. 2, 92–101.
12. S. Rideau, *Some properties of analytic difference fields*, http://arxiv.org/abs/1401.1765.

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