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Analytic van der Corput Lemma for p-adic and $\mathbf{F}_q((t))$ oscillatory integrals, singular Fourier transforms, and restriction theorems

Raf Cluckers *

Université Lille 1, Laboratoire Painlevé, CNRS - UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France
Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium

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ABSTRACT

We give a non-archimedean analogue of the van der Corput Lemma on oscillating integrals, where the condition of sufficient smoothness for the phase in the real case is replaced by the condition that the phase is a convergent power series. This result allows us, in analogy to the real situation, to study singular Fourier transforms on suitably curved (p-adic analytic) manifolds. As an application we give a restriction theorem for Fourier transforms of L^q functions to suitably curved analytic manifolds over non-archimedean local fields, similar to a real restriction result by E.M. Stein. Several analogues of the van der Corput Lemma were already known when the phase is a polynomial.

1. Introduction

Harmonic analysis on non-archimedean local fields is often very similar to the one on Euclidean space. In the work by Taibleson [14], many results have been carefully implemented on non-archimedean local fields in close analogy to Euclidean results. Also, many results of Stein's book [13] on real-variable methods in harmonic analysis can be translated rather directly to results over p-adic fields. However, certain aspects of oscillatory integrals can be very different, involving other proof

* Corresponding address: Université Lille 1, Laboratoire Painlevé, CNRS - UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France.

E-mail address: Raf.Cluckers@math.univ-lille1.fr.

URL: http://math.univ-lille1.fr/~cluckers.

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techniques and insights than in the real case. A lemma by van der Corput on oscillating integrals, which is widely applied in various Euclidean settings, remained open in the non-archimedean local field setting until the recent emergence of analogues in various degrees of generality and uniformity. In this paper we study a non-archimedean analogue of the van der Corput Lemma of [7] on oscillating integrals, which is very clearly presented in [13], Chapter VIII, Proposition 2. For polynomial phase, non-archimedean analogues of the van der Corput Lemma have been obtained by Rogers [12], Breuillard and Gelander [1], and more recently by Wright [16]. We give an analogue of the van der Corput Lemma where the phase is a power series over a p-adic field or over $\mathbf{F}_q((t))$, and where the obtained bounds are uniform in the power series, as long as the Gauss norm of the power series remains bounded and the non-archimedean local field is fixed. (The Gauss norm is the size of the largest coefficient of the power series.) It is important that we allow analytic phases in order to get applications about p-adic analytic manifolds (it is not clear what it could mean for an atlas of a manifold to be of a polynomial nature). In fact, our applications to p-adic analytic manifolds use nowadays standard arguments, see [13,14], but they seem very hard to obtain without having analytic phases and some uniformity in the power series as in Proposition 3.3.

Let us recall the van der Corput Lemma on real oscillatory integrals of [7] in the form of [13], Chapter VIII, Proposition 2, where f is a real-valued C^k function on an open interval (a, b) such that $|f^{(k)}(x)| \geq 1$ for some $k \geq 1$ and all x in (a, b). If either $k \geq 2$ or $k = 1$ and f' is monotonic, then

$$R(y) := \int_a^b \exp(2\pi \mathrm{i} y \cdot f(x)) \mathrm{d}x$$

satisfies

$$|R(y)| \leq c_k |y|^{-1/k} \quad \text{for all nonzero } y,$$

where c_k is a constant only depending on k (and thus not on a, b, y, and neither on f).

Proposition 3.3 is a p-adic and $\mathbf{F}_q((t))$ analogue for analytic phase f of this real van der Corput Lemma, allowing us to develop the theory further in great analogy to Chapter VIII of [13]. Since Gauss norms are naturally bounded in many situations we will encounter no difficulties in proceeding to the higher dimensional setting, see Propositions 3.6 and 3.9. The analyticity of the phase yields applications concerning K-analytic manifolds with suitable curvature (namely manifolds of finite type), with K a non-archimedean local field, see Theorem 3.11 on singular Fourier transforms and Theorem 4.2 on restrictions of Fourier transforms.

A literal translation of the van der Corput Lemma to non-archimedean local fields would be false because of at least two reasons: firstly, the constants are not absolute but depend in particular on the local field; secondly and more fundamentally, the condition of being C^k for the phase is too general a condition on non-archimedean fields and one probably has to resort to more subtle (and difficult to handle) notions of differentiability as for example in [10].

In a way, the van der Corput Lemma on the reals is based on the Fundamental Theorem of Integral Calculus, namely on its basic corollary that for a real C^1 function $\phi : \mathbf{R} \to \mathbf{R}$, if $\phi(c) = 0$ and $|\phi'(x)| \geq \varepsilon > 0$ on $\mathbf{R}$, then $|\phi(x+c)| \geq \varepsilon|x|$ for all $x \in \mathbf{R}$. The Fundamental Theorem of Integral Calculus does not have an analogue over non-archimedean local fields, but if ϕ is the identity function $x \mapsto x$, its corollary trivially holds over K, and one might try to apply a change of variables to reduce to the identity function. This is a technique which is implicitly used in our approach. Further, in Stein's version of the proof of the van der Corput Lemma one divides the interval (a, b) into at most three sub-intervals: a small, bad interval where a trivial bound is used, and the remaining two larger and nice intervals where one can use induction on k. Later on, the size of the bad interval is optimized to find the desired bounds. A difficulty in adapting Stein's version of the proof of the van der Corput Lemma is that, while cutting away one bad sub-interval of (a, b) one is left with at most two remaining intervals in the real case, in the non-archimedean case if one cuts away a small (bad) ball out of a big ball, one is left with a possibly huge (but still finite) number of remaining sub-balls. Hence, one has to control not only what size of balls the induction hypothesis can be applied to, but also the number of balls in which one subdivides the bigger ball, before optimizing the size of the bad ball on which the trivial bound is used.

As in Stein's book [13], Chapter VIII, when one knows more about the phase of the oscillating integral (like for example non-degeneracy w.r.t. a Newton polyhedron), sharper bounds can often be found, especially in the higher dimensional case. For these sharper but less uniform bounds we refer to e.g. [8,4,5,2].

The study presented in this paper arose in the context of the study of groups with the Howe–Moore property in [6]. Theorem 3.11 is used in its full generality in [6] to give an alternative proof for the Howe–Moore vanishing theorem in the p-adic case. We would like to thank warmly Valette for inviting us cordially to work on the question addressed in Theorem 3.11. Further we thank Rogers for inspiring us to study the relation of our results to his work in [12], which led us to formulate Corollary 3.4, and we thank B. Lichtin for interesting discussions.

2. Preliminaries

Write K for a fixed non-archimedean local field and $\mathcal{O}_K$ for its valuation ring with maximal ideal $\mathcal{M}_K$. Let $q_K = p_K^{e_K}$ be the number of elements of the residue field $\mathcal{O}_K/\mathcal{M}_K$, where p_K is a prime number and $e_K \geq 1$. Write π_K for a uniformizer of $\mathcal{O}_K$ and fix the norm $|\cdot|$ on K by assigning the value q_K^{-1} to π_K, and write $\mathrm{ord} : K \to \mathbf{Z} \cup \{+\infty\}$ for the order which assigns the value 1 to π_K and sends 0 to $+\infty$. For x in K^n, $|x|$ stands for $\max_{i=1}^n |x_i|$. Let ψ be an additive character on K which is trivial on $\mathcal{M}_K$ and nontrivial on $\mathcal{O}_K$.

2.1. Convergent and special power series

For x a variable, resp. a tuple of variables $(x_1, \ldots, x_n)$, write $K\{\{x\}\}$ for the collection of power series in x over K which converge on $\mathcal{O}_K$, resp. on $\mathcal{O}_K^n$, that is, those power series $\sum_{i\in\mathbf{N}^n} a_i x^i \in K[[x]]$ satisfying that $|a_i|$ goes to zero when $|i| := i_1 + \cdots, +i_n$ goes to infinity. Likewise, write $\mathcal{O}_K\{\{x\}\}$ for power series in $K\{\{x\}\}$ which also lie in $\mathcal{O}_K[[x]]$. For $f(x) \in K\{\{x\}\}$, write $\|f\|$ for the Gauss norm of f, which is by definition $\sup_i |a_i|$. From now on until Section 3.5, x will always denote one variable.

The following definition of Special Power series, abbreviated by SP, is a one-variable p-adic and $\mathbf{F}_q((t))$ analogue of real C^1 functions $(a, b) \to \mathbf{R}$ with big derivative on a real interval (a, b).

Definition 2.2. A power series $\sum_{i\geq 0} a_i x^i$ in one variable is called SP if it lies in $K\{\{x\}\}$, $a_1 \neq 0$, and $a_j \in a_1\mathcal{M}_K$ for all $j > 1$. If f is SP, write $|f|_{\mathrm{SP}}$ for $|a_1|$, which is nothing else than the Gauss norm of $f - f(0)$.

Note that a convergent power series $f = \sum_{i\geq 0} a_i x^i$ is SP if and only if the higher order terms have small coefficients compared to the linear term in the sense that $|a_j| < |a_1|$ for each $j > 1$. Therefore, f can be approximated by $a_0 + a_1 x$ in the senses that for all $x \in \mathcal{O}_K$

$$|f(x) - a_0 - a_1 x| < |f(x)|,$$

and

$$|f'(x)| = |a_1|.$$

Although the definition of SP may seem very restrictive, the philosophy behind it is that power series often become SP after basic manipulations like zooming in to good parts of the domain or taking derivatives. Lemmas 2.4 and 2.7 exhibit this kind of phenomena. Lemma 2.8 describes the linear behaviour of $|f(x)|$ in more detail.

Definition 2.3. Let $f(x)$ be in $K\{\{x\}\}$. Define the SP-number of f as the smallest integer $r \geq 0$ such that for all nonzero $c \in \mathcal{M}_K^r$ and all $b \in \mathcal{O}_K$, the power series

$$f_{b,c}(t) := \frac{1}{c} f(b + ct)$$

is SP if such r exists, and define the SP-number of f as $+\infty$ otherwise.

Lemma 2.4. *Let $f(x) = \sum_{i\geq 0} a_i x^i$ be in $K\{\{x\}\}$. Suppose that $|f'(x)| \geq 1$ for all x in $\mathcal{O}_K$. Then the SP-number of f is an integer r satisfying*

$$q_K^{r-1} \leq \|f - f(0)\|.$$

Moreover, for all nonzero $c \in \mathcal{M}_K^r$ and all $b \in \mathcal{O}_K$, one has $|f_{b,c}|_{SP} \geq 1$.

Proof. Note that $\|f - f(0)\| \geq 1$. If f is already SP the statement is clear. Namely, f is SP if and only if its SP-number is 0. Now suppose that f is not SP. For nonzero c in $\mathcal{O}_K$ such that $|c| < \|f - f(0)\|^{-1}$ and for $b \in \mathcal{O}_K$, expand the power series $f_{b,c}(t)$ in t as

$$f_{b,c}(t) = \sum_{j\geq 0} b_j t^j.$$

The chain rule for differentiation implies that $|b_1| = |(f_{b,c})'(0)| = |f'(b)| \geq 1$. On the other hand, each b_j for $j > 1$ lies in $\mathcal{M}_K$ by the above choice of c. Concluding, $f_{b,c}(t)$ is SP and $|f_{b,c}|_{SP} = |b_1| \geq 1$. □

Example 2.5. Clearly a power series in $K\{\{x\}\}$ is SP if and only if it has SP-number 0. Let $f(x)$ be the polynomial $x + 2^{-k}x^{2^{k+1}}$ for some integer $k \geq 0$ and suppose that $K = \mathbf{Q}_2$, the field of 2-adic numbers. Then $|f'(x)| = 1$ for all $x \in \mathbf{Z}_2$, and although the Gauss norm of $f - f(0)$ is big, the SP-number of f is just 1.

Definition 2.6. Call $f(x)$ in $\mathcal{O}_K\{\{x\}\}$ (Weierstrass) regular of degree $d \geq 0$ if $f(x)$ is congruent to a monic polynomial of degree d modulo the ideal $\pi_K \cdot \mathcal{O}_K\{\{x\}\}$.

It is clear that for any nonzero $f \in K\{\{x\}\}$, one has for a unique $d \geq 0$ and a unique $c \in K^\times$ that cf is regular of degree d.

Lemma 2.7. *Let $f(x) \in K\{\{x\}\}$ be nonconstant. Let c be the unique element of $K^\times$ such that $c \cdot (f(x) - f(0))$ is regular of degree $d \geq 1$. If the characteristic of K is zero or larger than d, then $f^{(d-1)}$ is SP and $|f^{(d-1)}|_{SP} = |d!| \cdot |c|^{-1}$. Moreover, the condition on the characteristic of K is necessary.*

Proof. Clearly the condition on the characteristic of K is necessary. Now suppose that the characteristic of K is zero or $>d$ and write $f = \sum_{i\geq 0} a_i x^i$. The coefficient of the linear term of $f^{(d-1)}$ equals $d!a_d$ with $c^{-1} = a_d$, and is thus nonzero. For any $j > 1$, the jth coefficient of $f^{(d-1)}$ equals

$$a_{j+d-1} \prod_{i=1}^{d-1} (j + d - i).$$

Since for any $j \geq 1$ one has

$$\left| \prod_{i=1}^{d-1} (j + d - i) \right| \leq |d!|,$$

and since d equals the maximum of all integers j such that $|a_j| = \|f(x) - f(0)\|$, it follows that $f^{(d-1)}$ is SP. □

The following lemma gives a link between f being SP and a lower bound for $|f(x)|$. It is the analogue of the fact that for a real C^1 function $\phi : \mathbf{R} \to \mathbf{R}$, if $\phi(c) = 0$ and $|\phi'(x)| \geq \varepsilon > 0$ on $\mathbf{R}$, then $|\phi(x + c)| \geq \varepsilon|x|$ for all $x \in \mathbf{R}$. In the real case this follows of course from the Fundamental Theorem of Integral Calculus, but on K one has to proceed differently.

Lemma 2.8. *Let $f = \sum_{i\geq 0} a_i x^i$ be in $K\{\{x\}\}$. Suppose that f is SP. If there exists $d \in \mathcal{O}_K$ such that $f(d) = 0$, then*

$$|f(x)| = |f|_{\mathrm{SP}} \cdot |x - d|$$

for all $x \in \mathcal{O}_K$. If there exists no such d, then

$$|f(x)| = |a_0| > |f|_{\mathrm{SP}}$$

for all $x \in \mathcal{O}_K$. In general, if $e \in \mathcal{O}_K$ is such that $|f(e)|$ is minimal among the values $|f(x)|$ for x in $\mathcal{O}_K$, then one has for all $x \in \mathcal{O}_K$

$$|f(x)| \geq |f|_{\mathrm{SP}} \cdot |x-e|.$$

Proof. Clearly $|a_0| \leq |f|_{\mathrm{SP}}$ if and only if there exists $d \in \mathcal{O}_K$ such that $f(d) = 0$ (this follows for example from non-archimedean Weierstrass preparation). If $|a_0| > |f|_{\mathrm{SP}}$ then clearly $|f(x)| = |a_0|$ for all $x \in \mathcal{O}_K$ and this finishes the second case. If $f(d) = 0$ for some $d \in \mathcal{O}_K$, then, with $g(t) = f(t+d)$, one has that $g(0) = 0$ and that g is SP, which implies that $|g(t)| = |g|_{\mathrm{SP}}|t|$ for all $t \in \mathcal{O}_K$. This finishes the first case since $|g|_{\mathrm{SP}} = |f|_{\mathrm{SP}}$. For the final statement, in the second case, any $e \in \mathcal{O}_K$ can serve; in the first case, one has to take $e = d$. □

Corollary 2.9. *Suppose that K has characteristic zero. Let $f = \sum_{i\geq 0} a_i x^i$ be in $K\{\{x\}\}$. Suppose that f' is SP and that $|f'(x)| > 0$ for all $x \in \mathcal{O}_K$. Then the SP-number of f is at most equal to $\mathrm{ord}(p_K)$, the ramification degree of K.*

Proof. By the second case of Lemma 2.8, we find $|a_1| > |2a_2| > |ja_j|$ for all $j \geq 3$. Hence, if we take for r the smallest integer satisfying $r > \mathrm{ord}(p_K)/p_K$, any nonzero $c \in \mathcal{M}_K^r$, and any $b \in \mathcal{O}_K$, then $f_{b,c}(t)$ is SP, as one can see by expanding in t. □

In fact, the above proof of Corollary 2.9 yields the stronger bound $\left\lfloor \frac{\mathrm{ord}(p_K)}{p_K} + 1 \right\rfloor$ for the SP-number of f.

3. Oscillatory integrals

We present p-adic and $\mathbf{F}_q((t))$ analogues of Chapter VIII of [13], namely of what Stein calls the theory of oscillatory integrals of the first kind. We motivate some of our choices for the possible reader with a better background in the real setting than in the non-archimedean setting. For an oscillatory integral (of the first kind), typically of the form

$$I(y) = \int_{\mathcal{O}_K} \psi(y \cdot f(x)) g(x) |\mathrm{d}x|,$$

where ψ is the additive character on K as introduced at the beginning of Section 2 and $|\mathrm{d}x|$ is the Haar measure on K normalized so that $\mathcal{O}_K$ has measure 1, the function f is usually called the phase and g the amplitude of the integral. For the many variables analogue, x or y can be tuples of variables and f can be a tuple of K-valued functions, and then $y \cdot f$ is the standard inner product.

In the non-archimedean set-up, f takes values in K while g takes real or complex values. While in Stein's set-up f and g are usually assumed to be sufficiently smooth in the sense of sufficiently continuously differentiable, we will have to make choices on which functions f and g to focus: C^∞ conditions on f are too general because of the total disconnectedness of K (and the implicit function theorem can fail for C^k functions $K \to K$). We typically require that f is given by a convergent power series. One usually requires that $g : \mathcal{O}_K \to \mathbf{C}$ is C^∞. Any C^∞ function $g : \mathcal{O}_K \to \mathbf{C}$ is locally constant, and by the compactness of $\mathcal{O}_K$ it has finite image. Therefore, we will assume that g is constantly equal to 1; any C^∞ function can be brought back to this situation by taking finite partitions, scaling the parts by homotheties, and replacing g by a multiple.

By similar scaling arguments, one can usually reduce integrals over more general domains to integrals over $\mathcal{O}_K$ (or over Cartesian powers of $\mathcal{O}_K$), and conditions of the form $|f'(x)| \geq \varepsilon$ can be reduced to the simpler condition $|f_1'(x)| \geq 1$ where f_1 is a multiple of f. Hence several of the statements below, like e.g. the van der Corput style Proposition 3.3, are more general than they seem at first sight.

3.1. The one-variable theory

We first state an almost trivial variant of classically known results, Lemma 3.2, about arbitrarily quick decays at infinity if the phase of the oscillatory integral is nice enough, where in our set-up nice enough means SP and quick decay actually means identically zero for large y.

Lemma 3.2. *Let $f(x) = \sum_{i\geq 0} a_i x^i$ in $K\{\{x\}\}$ be SP. Then, for all $y \in K$ with $|y| \geq |a_1|^{-1}$ one has*

$$\int_{\mathcal{O}_K} \psi(y \cdot f(x))|dx| = 0$$

and, for y with $|y| < |a_1|^{-1}$ one has

$$\int_{\mathcal{O}_K} \psi(y \cdot f(x))|dx| = \psi(y \cdot a_0).$$

Combining, one has

$$\left|\int_{\mathcal{O}_K} \psi(y \cdot f(x))|dx|\right| \leq q_K^{-1}|a_1|^{-1}|y|^{-1} \quad \textit{for all nonzero } y.$$

Proof. There is no loss in replacing f by a multiple so that one has $|a_1| = 1$. The equalities follow from the fact that ψ is trivial on $\mathcal{M}_K$ and nontrivial on $\mathcal{O}_K$, and from the basic relation of character sums (namely, for a nontrivial character ω on a finite abelian group G, the sum $\sum_{g\in G} \omega(g)$ equals zero). The summarizing statement follows from the fact that the norm of $\int_{\mathcal{O}_K} \psi(y \cdot f(x))|dx|$ is always ≤ 1 and that if $|y| < |a_1|^{-1}$ then $q_K^{-1}|a_1|^{-1}|y|^{-1} \geq 1$. □

The van der Corput Lemma

Fix $f(x) = \sum_{i\geq 0} a_i x^i$ in $K\{\{x\}\}$ and write, for $y \in K$,

$$I(y) = \int_{\mathcal{O}_K} \psi(y \cdot f(t))|dt|.$$

Note that $I(y)$ is the non-archimedean analogue of the real integral $R(y)$ of the introduction. The following is the main technical result of the paper.

Proposition 3.3 (*Analytic, Non-Archimedean Van Der Corput Lemma*). *Suppose that for some $k \geq 1$ one has that $|f^{(k)}(x)| \geq 1$ for all x in $\mathcal{O}_K$. Then one has for all $y \in K^\times$*

$$|I(y)| \leq c_k |y|^{-\frac{1}{k}},$$

where c_k only depends on k, q_K, and on the Gauss norm of $f - f(0)$. Alternatively, if K has characteristic zero, then c_k can be taken only depending on k, q_K, the ramification degree $\mathrm{ord}(p_K)$ of K, and on the SP-number of $f^{(k-1)}$.

Proof. If $|y| < 1$ then in fact any $c_k \geq q_K^{-1}$ can do in the bound for $|I(y)|$. Hence, we may suppose that $|y| \geq 1$. First we work for general k. By Lemma 2.4 the SP-number of $f^{(k-1)}$ is an integer r. Let c be a generator of $\mathcal{M}_K^r$. Note that

$$|c|^{-1} \leq q_K \|f^{(k-1)} - f^{(k-1)}(0)\| \leq q_K \|f - f(0)\|$$

by Lemma 2.4. Let b_i be a set of representatives in $\mathcal{O}_K$ of $\mathcal{O}_K/c\mathcal{O}_K$, for $i = 1, \ldots, |c|^{-1}$. Write

$$f_{b_i,c,k}(t) = \frac{1}{c^k} f(b_i + ct).$$

Then each of the $f_{b_i,c,k}^{(k-1)}$ is SP and satisfies $|f_{b_i,c,k}^{(k-1)}|_{\mathrm{SP}} \geq 1$ by Lemma 2.4 and the chain rule for differentiation. After a linear change of variables and by the linearity of the integral we can write

$$I(y) = \sum_{i=1}^{|c|^{-1}} \int_{b_i + c\mathcal{O}_K} \psi(y \cdot f(x))|dx| = |c| \sum_{i=1}^{|c|^{-1}} \int_{\mathcal{O}_K} \psi((c^k y) \cdot f_{b_i,c,k}(t))|dt|.$$

If we abbreviate the ith term as follows,

$$I_i(y) := \int_{\mathcal{O}_K} \psi((c^k y) \cdot f_{b_i,c,k}(t))|dt|,$$

then

$$|I(y)| \leq |c| \sum_{i=1}^{|c|^{-1}} |I_i(y)|, \tag{3.3.1}$$

or in words, $|I(y)|$ is bounded by the average value of the $|I_i(y)|$.

We now focus on the case that $k = 1$. By Lemma 3.2, for each i,

$$|I_i(y)| \leq q_K^{-1} |f_{b_i,c,1}|_{\mathrm{SP}}^{-1} |cy|^{-1} \leq q_K^{-1} |cy|^{-1}$$

and thus

$$\left| \int_{\mathcal{O}_K} \psi(y \cdot f(x)) |\mathrm{d}x| \right| \leq q_K^{-1} |cy|^{-1}.$$

We are done by Lemma 2.4 in the case that $k = 1$.

Finally fix $k \geq 2$ and suppose that the proposition is proved for all values up to $k-1$. So we start from the condition that $|f^{(k)}| \geq 1$ on $\mathcal{O}_K$. Recall that $f_{b_i,c,k}^{(k-1)}$ is SP for each i and satisfies $|f_{b_i,c,k}^{(k-1)}|_{\mathrm{SP}} \geq 1$. Fix i and suppose that $|f_{b_i,c,k}^{(k-1)}(d)|$ is minimal for some $d \in \mathcal{O}_K$ among the values $|f_{b_i,c,k}^{(k-1)}(x)|$ for $x \in \mathcal{O}_K$. Up to translating by d, we may suppose that $d = 0$. Then, by Lemma 2.8,

$$|f_{b_i,c,k}^{(k-1)}(x)| \geq |x| \tag{3.3.2}$$

for all $x \in \mathcal{O}_K$. Take a nonzero $\gamma \in \mathcal{O}_K$. Partition $\mathcal{O}_K$ into the ball

$$B_0 := \gamma \mathcal{O}_K$$

and n balls of the form

$$B_j := d_j + n_j \mathcal{O}_K$$

for d_j with $|d_j| > |\gamma|$ and n_j a generator of the ideal $d_j \mathcal{M}_K$, $j = 1, \ldots, n$, and where necessarily $n = (q_K - 1)\mathrm{ord}(\gamma)$. The ball B_0 will serve as a bad ball where we will use a trivial bound (namely the volume of B_0), while on the remaining B_j we will use bounds coming from induction. At some point, we will optimize the choice of γ for any given value of y (which is similar to the proof of the van der Corput Lemma in [13]). In this optimization, it is important that there are not too many parts B_j, which is indeed achieved by our choice of rather big radii n_j. Finally we will combine again the terms for all the i by (3.3.1). We write by the linearity of the integral

$$I_i(y) = \sum_{j=0}^{n} I_{ij}(y) \tag{3.3.3}$$

with

$$I_{ij}(y) := \int_{B_j} \psi(c^k y \cdot f_{b_i,c,k}(x)) |\mathrm{d}x|.$$

Clearly

$$|I_{i0}(y)| \leq \int_{B_0} |\psi(c^k y \cdot f_{b_i,c,k}(x))| \, |\mathrm{d}x| = \int_{B_0} |\mathrm{d}x| = |\gamma|.$$

For $j = 1, \ldots, n$ we can write, after a linear change of variables,

$$I_{ij}(y) = |n_j| \int_{\mathcal{O}_K} \psi(c^k y \cdot g_j(t)) |\mathrm{d}t|,$$

where

$$g_j(t) := f_{b_i,c,k}(d_j + n_j t).$$

By the definition of the SP-number, the $g_j^{(k-1)}$ are SP and by the chain rule $|g_j^{(k)}(t)| \geq |n_j^k|$ for all t in $\mathcal{O}_K$. In fact, the g_j are even better than that, allowing us to use the induction hypothesis for each j. Indeed, by (3.3.2) and the chain rule one has $|g_j^{(k-1)}(t)| \geq |n_j^{k-1}d_j| > 0$ for all t in $\mathcal{O}_K$. Note also that the Gauss norm of $g_j - g_j(0)$ is bounded by the Gauss norm of $(f - f(0))/c^k$, and that, in the case that K has characteristic zero, the SP-number of $g_j^{(k-2)}$ is bounded by $\mathrm{ord}(p_K)$ by Corollary 2.9. Therefore, we can use the induction hypothesis in k to g_j to find

$$|I_{ij}(y)| = |n_j| \cdot \left| \int_{\mathcal{O}_K} \psi(c^k y \cdot g_j(t))|\mathrm{d}t| \right| \leq \widetilde{c}_{k-1} \cdot |d_j c^k y|^{-\frac{1}{k-1}} \leq c'_{k-1} \cdot |d_j y|^{-\frac{1}{k-1}},$$

where $c'_{k-1} \geq \widetilde{c}_{k-1}|c|^{-\frac{k}{k-1}}$, and where $\widetilde{c}_{k-1}$ only depends on k, q_K, and $\|f - f(0)\|$. If K has characteristic zero we can alternatively suppose that $\widetilde{c}_{k-1}$ only depends on q_K, $\mathrm{ord}(p_K)$, and k. Hence, we may by Lemma 2.4 suppose that c'_{k-1} only depends on k, q_K, and $\|f - f(0)\|$, or alternatively, if K has characteristic zero, that c'_{k-1} only depends on k, q_K, $\mathrm{ord}(p_K)$, and the SP-number of $f^{(k-1)}$. Summing up for $j = 1, \ldots, n$ as in (3.3.3) yields, still with $n = (q_K - 1)\mathrm{ord}(\gamma)$,

$$|I_i(y)| \leq |\gamma| + c'_{k-1}|y|^{-\frac{1}{k-1}} \sum_{j=1}^{n} |d_j|^{-1/(k-1)}. \tag{3.3.4}$$

For each ℓ with $0 \leq \ell < \mathrm{ord}(\gamma)$, there are exactly $q_K - 1$ different d_j with $\mathrm{ord}(d_j) = \ell$. Hence we can calculate:

$$\begin{aligned}
\sum_{j=1}^{n} |d_j|^{-1/(k-1)} &= (q_K - 1) \sum_{\ell \geq 0}^{\mathrm{ord}(\gamma)-1} (q_K^{1/(k-1)})^{\ell} \\
&= (q_K - 1) \frac{|\gamma|^{-1/(k-1)} - 1}{q_K^{1/(k-1)} - 1} \\
&\leq |\gamma|^{-1/(k-1)} \frac{q_K - 1}{q_K^{1/(k-1)} - 1}
\end{aligned}$$

Combining with (3.3.4) yields

$$|I_i(y)| \leq |\gamma| + c''_{k-1}|\gamma y|^{-\frac{1}{k-1}} \tag{3.3.5}$$

for some c''_{k-1} only depending on k, q_K, and $\|f - f(0)\|$, resp. in characteristic zero only depending on k, q_K, $\mathrm{ord}(p_K)$, and the SP-number of $f^{(k-1)}$. Recall that we are considering y with $|y| \geq 1$. Choose γ in $\mathcal{O}_K$ such that

$$q_K^{-1}|y|^{-1/k} \leq |\gamma| < |y|^{-1/k}. \tag{3.3.6}$$

Together with (3.3.5) this gives

$$|I_i(y)| \leq c'''_k |y|^{-\frac{1}{k}} \tag{3.3.7}$$

for some c'''_k only depending on k, q_K, and $\|f - f(0)\|$, resp. only depending on k, q_K, $\mathrm{ord}(p_K)$, and the SP-number of $f^{(k-1)}$. Putting the bounds (3.3.7) in (3.3.1) yields the desired bound for $|I(y)|$ in terms of some constant c_k only depending on k, q_K, and $\|f - f(0)\|$, resp. in characteristic zero only depending on k, q_K, $\mathrm{ord}(p_K)$, and the SP-number of $f^{(k-1)}$. □

As a corollary of Proposition 3.3, we make a link between Weierstrass regularity of some derivative of f and the conditions of the van der Corput Proposition 3.3, to find back a generalization of the main thrust (Lemma 3) of [12], from which Rogers derives in a beautiful and direct way all principal results of [12]. Rogers gives in Lemma 3 of [12], in the case that f is a polynomial over $\mathbf{Q}_p$ and only treating the case $j = 1$, explicit values for the $c_{m,\mathbf{Q}_p}$ of Corollary 3.4.

We still consider our fixed f in $K\{\{x\}\}$ and the corresponding oscillating integral $I(y)$ as just above Proposition 3.3.

Corollary 3.4. *Suppose that $f^{(j)}$ is (Weierstrass) regular of degree $m-j>0$ for some $j\geq 0$. If the characteristic of K is zero, then there exists $c_{m,K}$, only depending on m, q_K, and $\operatorname{ord}(p_K)$, such that, for all nonzero $y\in K$,*

$$|I(y)|\leq c_{m,K}|y|^{-1/m}.$$

Proof. Clearly on the one hand

$$|f^{(m)}(x)|\geq|(m-j)!|\text{ for all }x\in\mathcal{O}_K,$$

and on the other hand, the SP-number of $f^{(m-1)}$ is zero. Now apply Proposition 3.3. □

3.5. Several variables

Now that we have obtained a non-archimedean analogue of the van der Corput Lemma for analytic phases, we can grasp its rewards and develop the theory in great analogy to [13, Sections 2 and 3, Chapter VIII]. Note that in [3], decay rates for higher dimensional non-archimedean Fourier transforms have been obtained for L^1-functions of a certain constructible nature, related to subanalytic functions. Here we will find more explicit decay rates, in a different setting than in [3] which is in some ways more general and in other ways more restrictive.

From now on we will consider tuples of variables $x=(x_1,\dots,x_n)$, writing $K\{\{x\}\}$ for the collection of power series in the variables x over K which converge on $\mathcal{O}_K^n$, that is, those power series $\sum_{i\in\mathbf{N}^n}a_ix^i\in K[[x]]$ satisfying that $|a_i|$ goes to zero when $|i|:=\sum_{j=1}^n i_j$ goes to infinity. Likewise, we write $\mathcal{O}_K\{\{x\}\}$ for power series in $K\{\{x\}\}$ which also lie in $\mathcal{O}_K[[x]]$ and for $f(x)\in K\{\{x\}\}$, we write $\|f\|$ for the Gauss norm of f, which is $\sup_{i\in\mathbf{N}^n}|a_i|$.

The following is a non-archimedean analogue of [13, Proposition 5, Chapter VIII] for analytic phase in the oscillating integral (where [13] is for real, smooth phase); note that in our proposition the Gauss norm of $f-f(0)$ plays the role of the C^{k+1} norm of the phase in Proposition 5 of [13, Chapter VIII].

Proposition 3.6. *Let $f(x)$ be a power series in $K\{\{x\}\}$ in the variables $x=(x_1,\dots,x_n)$. Suppose that for some multi-index $\alpha\in\mathbf{N}^n$ with $|\alpha|>0$, one has*

$$|\partial_x^\alpha f(x)|\geq 1\text{ for all }x\in\mathcal{O}_K^n,$$

where $|\alpha|=\sum_j\alpha_j$ and $\partial_x^\alpha f=\left(\prod_j\frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}\right)f$. Suppose also that the characteristic of K is either 0 or $>|\alpha|$. Then, for all nonzero $y\in K$,

$$\left|\int_{\mathcal{O}_K^n}\psi(y\cdot f(x))|\mathrm{d}x|\right|\leq d_k|y|^{-\frac{1}{k}},$$

where d_k only depends on K, n, $k=|\alpha|$, and on $\|f-f(0)\|$.

Proof. Consider the K-vector space $V_{k,n}(K)$ of homogeneous polynomials of degree k over K in the n variables $x=(x_1,\dots,x_n)$. By Lemma 3.7, there are vectors $\xi_1,\dots\xi_d$ in K^n of length 1 (that is, $|\xi_i|=1$) such that the homogeneous polynomials

$$(\xi_i\cdot x)^k,\quad i=1,\dots,d$$

form a basis for this vector space, with d the dimension of $V_{k,n}(K)$. Express the monomial x^α in this basis as

$$x^\alpha=\sum_i e_i(\xi_i\cdot x)^k,\quad e_i\in K.$$

Then, for $x_0 \in \mathcal{O}_K^n$,

$$1 \leq |\partial_x^\alpha f(x_0)| = \left|\sum_i e_i(\xi_i \cdot \nabla)^k f(x_0)\right| \leq \max_i(|e_i(\xi_i \cdot \nabla)^k f(x_0)|)$$

and hence

$$|(\xi_i \cdot \nabla)^k f(x_0)| \geq |1/e_i|$$

for at least one i with $e_i \neq 0$. Note that this implies that $\|e_i(f - f(0))\| \geq 1$. Hence, for such i,

$$|(\xi_i \cdot \nabla)^k f(x)| \geq |1/e_i|$$

for all x in the ball $B(x_0) := x_0 + c\mathcal{O}_K^n$ around x_0 with $c \in \mathcal{M}$ satisfying $|c| = \|e_i(f - f(0))\|^{-1} q_K^{-1}$. Define $g(z)$ as $e_i c^{-k} f(x_0 + cz)$ with $z = (z_1, \ldots, z_n)$. Then $g(z)$ is in $K\{\{z\}\}$ and satisfies

$$|(\xi_i \cdot \nabla)^k g| \geq 1.$$

After a measure preserving affine change of variables on K^n such that x_1 lies along ξ_i, we may suppose that $\xi_i = (1, 0, \ldots, 0)$, and thus that

$$|(\partial^k/\partial z_1^k)g(z)| \geq 1 \quad \text{for all } z \in \mathcal{O}_K^n.$$

For each $a_2, \ldots, a_n$ in $\mathcal{O}_K$, the Gauss norm of $g(t, a_2, \ldots, a_n) - g(0, a_2, \ldots, a_n)$ (where this power series lies in $K\{\{t\}\}$), is bounded by $\|g - g(0)\|$. Hence, by Proposition 3.3, we find

$$\begin{aligned}\left|\int_{B(x_0)} \psi(y \cdot f(x))|dx|\right| &= |c^n| \cdot \left|\int_{\mathcal{O}_K^n} \psi\left(\frac{c^k y}{e_i} \cdot g(z)\right)|dz|\right| \\ &= |c^n| \cdot \left|\int_{\mathcal{O}_K^{n-1}} \left(\int_{\mathcal{O}_K} \psi\left(\frac{c^k y}{e_i} \cdot g(z)\right)|dz_1|\right)|dz_2 \ldots dz_n|\right| \\ &\leq |c^n| \cdot \left|\int_{\mathcal{O}_K^{n-1}} c_k |c|^{-1}|y|^{-\frac{1}{k}}|dz_2 \ldots dz_n|\right| = c_k|c|^{n-1}|y|^{-\frac{1}{k}}\end{aligned}$$

where c_k only depends on k, n, K, and on the Gauss norm of $g - g(0)$. Since the Gauss norm of $g - g(0)$ is bounded by $|e_i c^{-k}| \cdot \|(f - f(0))\|$, and since

$$\left|\int_{\mathcal{O}_K^n} \psi(y \cdot f(x))|dx|\right| \leq \sum_{i=1}^{|c|^{-n}} \left|\int_{B(b_i)} \psi(y \cdot f(x))|dx|\right|$$

for any set b_i of representatives of $\mathcal{O}_K^n$ modulo $c\mathcal{O}_K^n$, we are done. $\square$

The following elementary lemma and its proof are a close adaptation of [13, Chapter VIII, 2.2.1] to a slightly more general setting.

Lemma 3.7. *Let $k > 0$ be an integer and $x = (x_1, \ldots, x_n)$ variables. Let L be an infinite field of characteristic either 0 or $>k$. Then the polynomials of the form*

$$(\xi \cdot x)^k, \quad \xi \in L^n,$$

where $\xi \cdot x = \sum_i \xi_i x_i$, span the L-vector space $V_{k,n}(L)$ of homogeneous polynomials of degree k in x over L.

Proof. On this vector space $V_{k,n}(L)$, consider the inner product (that is, bi-linear mapping to L)

$$\langle P, Q \rangle = \alpha! a_\alpha b_\alpha,$$

where $P(x) = \sum a_\alpha x^\alpha$ and $Q(x) = \sum b_\alpha x^\alpha$ and where $\alpha! = \prod_j(\alpha_j!)$. Note that

$$\langle P, Q \rangle = (Q(\partial/\partial x))(P),$$

where the polynomials are differentiated formally and where $\partial/\partial x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Thus, if P were orthogonal to all the polynomials of the form $(\xi \cdot x)^k$, then

$$(\xi \cdot \nabla)^k(P) = 0, \quad \text{for all } \xi \in L^n.$$

In other words,

$$(\partial/\partial t)^k P(t\xi) = 0 \quad \text{for all } \xi \in L^n,$$

which can happen only if $P(x)$ is the zero polynomial. Indeed,

$$(\partial/\partial t)^k P(t\xi) = (\partial/\partial t)^k (t^k P(\xi)) = k! P(\xi).$$

From this we can draw our conclusions. Suppose that the space spanned by the $(\xi \cdot x)^k$ has strictly smaller dimension than $V_{k,n}(L)$. Let $P_1, \ldots, P_{d'}$ be a basis for this span. But then

$$\bigcap_{j=1,\ldots,d'} P_j^{\perp},$$

where $P_j^{\perp} = \{Q \in V_{k,n} \mid P_j \cdot Q = 0\}$, has dimension >0, since this intersection is the solution set of d' homogeneous linear equations on $V_{k,n}(L)$. We are done by contradiction. □

Remark 3.8. A complex-valued C^∞ function h with compact support defined on an open subset of K^n is automatically locally constant, and it is constant on each ball in a finite partition of the support of h into balls. Hence, we simplify notation by working with characteristic functions of balls to serve, for example, as amplitudes, instead of with complex-valued C^∞ functions with compact support (which are the so-called Schwartz–Bruhat functions). The adaptation in the following theorem with an amplitude which is a Schwartz–Bruhat function is trivial to make. We will simplify likewise in Section 3.10.

The following result for mappings is closely related to Theorem 3.11.

Proposition 3.9 (*Mappings in Several Variables*)**.** *Let $f_1, \ldots, f_n$ be power series in $K\{\{x\}\}$ in the variables $x = (x_1, \ldots, x_d)$ and let $k \geq 1$ be an integer. Suppose that for each $v \in K^n$ of length 1 and for each $x \in K^d$, there exists some multi-index $\alpha \in \mathbf{N}^d$ with $k \geq |\alpha| > 0$ and*

$$|v \cdot (\partial_x^\alpha f(x))| \geq 1,$$

where $|\alpha| = \sum_j \alpha_j$ and $\partial_x^\alpha f = \left(\left(\prod_j \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}\right) f_i\right)_i$. Suppose also that the characteristic of K is either 0 or $>k$. Then there exists a constant c such that

$$\left| \int_{\mathcal{O}_K^n} \psi(y \cdot f(x)) |dx| \right| \leq c|y|^{-\frac{1}{k}}$$

for all nonzero $y \in K^n$. Moreover, c only depends on K, n, k, and on $\|f - f(0)\|$.

Proof. Write y as λv, where $\lambda \in K^\times$ and $v \in K^n$ with $|v| = 1$. Then, for each such v,

$$\left| \int_{\mathcal{O}_K^n} \psi(\lambda v \cdot f(x)) |dx| \right| \leq d_k |\lambda|^{-\frac{1}{k}}$$

for some d_k only depending on K, n, k, and on $\|v \cdot f - v \cdot f(0)\|$, by Proposition 3.6. The Gauss norm $\|v \cdot f - v \cdot f(0)\|$ takes only finitely many values when v varies over all vectors of length 1, since this Gauss norm varies continuously in v and v runs over a compact. Even more, $\|v \cdot f - v \cdot f(0)\|$ takes only values between 1 and $\|f - f(0)\|$. Hence we are done. □

3.10. *K-analytic manifolds of finite type and singular Fourier transforms*

For an open $X \subset K^n$, $n \geq 0$, a function $f : X \to K^m$ is called K-analytic if there is an open cover of X such that for each open U in the cover the restriction of f to U is given by m power series which converge on U. Call a subset M of K^n for some $n \geq 0$ a K-analytic manifold of dimension d if there exists an open cover of M such that for each open U in the cover there exists a coordinate projection $p_U : K^n \to K^d$ onto d of the n standard coordinates on K^n, such that p_U induces a bijection $U \to U'$ onto an open $U' \subset K^d$, and there exist K-analytic functions $f_1, \ldots, f_n : U' \to K$ such that $f = (f_1, \ldots, f_n)$ is the inverse map of p_U on U. Note that for a K-analytic manifold M, the open cover and the coordinate projections p_U can be taken such that the p_U are isometries. Then the induced volume μ_M on M is by definition the pull-back of the standard normalized Haar measure on K^d via the isometries p_U. By a K-analytic manifold we mean a K-analytic manifold of some dimension d.

Say that a K-analytic manifold $M \subset K^n$ is of finite type at $x_0 \in M$, if, for each hyperplane H in K^n containing x_0, and for each open U in M around x_0, one has that U is not contained in H.[1] Equivalently, for $f = (f_1, \ldots, f_n)$ analytic coordinates on M mapping U' to $U \subset M$ as in the definition of K-analytic manifolds and such that $x_0 \in U$, one says that M is of finite type at $x_0 \in M$ if there exists $k > 0$ such that for each nonzero $v \in K^n$ there exists $\alpha \in \mathbf{N}^d$ with $k \geq |\alpha| := \sum_{i=1}^d \alpha_i > 0$ and such that

$$v \cdot (\partial^\alpha f / \partial x^\alpha)(x_0) \neq 0$$

and the least value for k with this property is called the type of M at x_0. If M is of finite type at all its points, then we call M of finite type. If there exists k such that M is of type $\leq k$ at each of its points, then the least such integer k is called the type of M.

Now let $M \subset K^n$ be a K-analytic manifold of finite type, and let S be a compact open in M. Then S is a K-analytic manifold of type k for some $k > 0$. Let μ_S be the induced measure on S. The Fourier transform of μ_S is defined for $y \in K^n$ by

$$\widehat{\mu}_S(y) := \int_{x \in S} \psi(y \cdot x) \mathrm{d}\mu_S(x),$$

and thus it is a complex-valued function on K^n.

The following theorem is a non-archimedean analogue of [13, Theorem 2, Chapter VIII].

Theorem 3.11. *Suppose that the characteristic of K is either 0 or $>k$. Then*

$$\lim_{|y| \to \infty} \widehat{\mu}_S(y) = 0,$$

and, more precisely, there exists a constant c such that

$$|\widehat{\mu}_S(y)| \leq c|y|^{\frac{-1}{k}} \quad \textit{for all nonzero } y \in K^n.$$

Proof. By using finitely many charts with analytic isometries for the maps p_U as in the definition of K-analytic manifolds given above, the theorem is translated into a finite sum of integrals as treated in Proposition 3.9. □

4. Restriction of the Fourier transform

The above non-archimedean van der Corput Lemma allows us to develop the theory in analogy to what follows on the real van der Corput Lemma in the last part of Chapter VIII of [13]. We will implement a non-archimedean analogue of a restriction result by Stein, Theorem 3 of [13, Section 4, Chapter VIII]. In fact, we will stay very close to loc. cit., sometimes transcribing rather directly from the real case to the non-archimedean case. For deeper restriction results in the Euclidean setting, for which we are yet unable to prove non-archimedean analogues, see [9] or [15].

To sketch some context we base ourselves on the introduction from [13, Section 4, Chapter VIII]. The Fourier transform of an $L^1(K^n)$-function is a continuous function, and hence is defined everywhere on K^n. On the other hand, the Fourier transform of an L^2 function is itself no better than an L^2-function, and so can be defined only almost everywhere, and is thus completely arbitrary on a set of measure zero.

1 This includes the case that M is open in K^n.

In addition, when $1 < p \leq 2$, the classical Hausdorff–Young theorem allows one to realize the Fourier transform of an L^p function as an element of $L^q(K^n)$, $1/p+1/q = 1$, and so, at first sight, is determined only almost everywhere. In view of this, it is a remarkable discovery by Stein and Fefferman in the real case and adapted here to the non-archimedean case, that when $n \geq 2$ and M is a submanifold of K^n that has appropriate curvature, there is a $p_0 = p_0(M)$, with $1 < p_0 < 2$, so that every function in $L^p(K^n)$, for any p with $1 \leq p < p_0$, has a Fourier transform that has a well-defined restriction to M.

4.1

Let us make the notion of restriction of the Fourier transform precise.

Suppose that $M \subset K^n$ is a K-analytic manifold with induced measure μ_M. Say that the L^p restriction property is valid for M if there exists a $q = q(p)$ so that the inequality

$$\left(\int_{M_0} |\widehat{f}(\xi)|^q \mu_M(\xi)\right)^{1/q} \leq A_{p,q}(M_0) \cdot \|f\|_{L^p} \tag{4.1.1}$$

holds for each Schwartz–Bruhat function f on K^n with Fourier transform $\widehat{f}$, whenever M_0 is a compact open subset of M. Because the space of Schwartz–Bruhat functions is dense in L^p we can, when (4.1.1) holds, define $\widehat{f}$ on M (almost everywhere for μ_M), for each f in $L^p(K^n)$.

Theorem 4.2. *Let $M \subset K^n$ be a K-analytic manifold of type k, and suppose that the characteristic of K is either* 0 *or $>k$. Then there exists a p_0 depending on M and with $p_0 > 1$, such that M has the L^p restriction property* (4.1.1) *for all p with $1 \leq p \leq p_0$ and $q = 2$.*

Remark 4.3. The analysis being very similar to the analysis of [13, Section 4, Chapter VIII], we will get the same value

$$p_0 = \frac{2nk}{2nk-1}$$

as in Theorem 3 of [13, Section 4, Chapter VIII].

Proof. It will suffice to prove that, for compact open $M_0 \subset M$,

$$\left(\int_{M_0} |\widehat{f}(\xi)|^2 \mu_M(\xi)\right)^{1/2} \leq A \cdot \|f\|_{L^p(K^n)}$$

for any Schwartz–Bruhat function f on K^n. Define μ as $\chi_{M_0}\mu_M$ with χ_{M_0} the characteristic function of M_0. Consider the operator R on Schwartz–Bruhat functions, where $Rf(\xi)$ is defined for $\xi \in M$ by the Fourier transform

$$Rf(\xi) = \int_{K^n} \psi(x\cdot\xi) f(x)|dx|$$

The question then is whether R can be seen as a bounded mapping from $L^p(K^n)$ to $L^2(M,\mu)$, and, in studying this, we consider also its formal adjoint R^*, given for f and $x \in K^n$ by

$$R^*f(x) = \int_M \psi(-x\cdot\xi) f(\xi)\mu(\xi).$$

We have

$$\langle Rf, Rf\rangle_{L^2(M,\mu)} = \langle R^*Rf, f\rangle_{L^2(K^n)},$$

so to prove that

$$R : L^p(K^n) \to L^2(M,\mu)$$

is bounded, it suffices, by Hölder's inequality, to see that

$$R^*R : L^p(K^n) \to L^{p'}(K^n)$$

is bounded, where p' is the exponent conjugate to p. One sees that

$$(R^*Rf)(x) = \int_{K^n}\int_M \psi(\xi\cdot(y-x))\mu(\xi)f(y)|dy|,$$

so $(R^*Rf)(x) = (f * L)(x)$ with

$$L(x) = \widehat{\mu}(-x),$$

where $\widehat{\mu}$ is as in Section 3.10. By Theorem 3.11, we have for all nonzero $x \in K^n$

$$|L(x)| \leq A|x|^{\frac{-1}{k}}$$

for some A and clearly L is bounded, thus there exists a constant A' such that for nonzero x

$$|L(x)| \leq A'|x|^{-\gamma}, \quad \text{whenever } 0 \leq \gamma \leq 1/k.$$

By the theorem of fractional integration (see the Hardy–Littlewood–Sobolev inequality below), the operator $f \mapsto f * (|x|^{-\gamma})$ is bounded from $L^p(K^n)$ to $L^q(K^n)$, whenever $1 < p < q < \infty$ and $1/q = 1/p - 1 + \gamma/n$. Then if $q = p'$, we have $1/q = 1 - 1/p$, so the relation among the exponents becomes $2 - 2/p = \gamma/n$, and the restriction $0 \leq \gamma \leq 1/k$ becomes $1 \leq p \leq 2nk/(2nk-1)$, completing the proof of the theorem (since the case $p = 1$ is trivial). □

4.3.1. Hardy–Littlewood–Sobolev inequality

In the real set-up, there are many proofs for the Hardy–Littlewood–Sobolev inequality, which in the non-archimedean case reads as the inequality

$$\|f * (|y|^{-\gamma})\|_{L^q(K^n)} \leq A_{p,q}\|f\|_{L^p(K^n)} \tag{4.3.1}$$

for

$$0 < \gamma < n,\ 1 < p < q < \infty, \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}, \tag{4.3.2}$$

where we have written $|y|^{-\gamma}$ for the function $y \mapsto |y|^{-\gamma}$ on $K^n \setminus \{0\}$, extended trivially on 0. Although related results can be found in [14], we give an outline for the convenience of the reader. We will work out the non-archimedean version of the proof given in [13, Section 4.2, Chapter VIII], which is based on Hedberg's proof in [11]. First define, for any complex-valued function f on K^n, the maximal function

$$(Mf)(x) := \sup_{x\in B}\frac{1}{\mathrm{Vol}(B)}\int_B |f(y)|\,|dy|,$$

where the supremum is taken over all balls in K^n containing x, $\mathrm{Vol}(B)$ denotes the volume of B, and where a ball is any subset of K^n of the form

$$a + b\cdot\mathcal{O}_K^n$$

for $a \in K^n$ and nonzero b in K. (Note that the distinction made in the real case between the maximal function and the so-called sharp maximal function, cf. M and $\tilde{M}$ on page 13 of [13], is irrelevant here since they coincide in the non-archimedean case.) Recall the Hardy–Littlewood–Wiener maximal theorem, with literally the same proof as in the archimedean case of [13, Theorem 1, Chapter 1], namely that for $f \in L^p(K^n)$, $1 < p \leq \infty$, one has $M(f) \in L^p(K^n)$ and

$$\|M(f)\|_{L^p(K^n)} \leq A_p\|f\|_{L^p(K^n)} \tag{4.3.3}$$

where the constant A_p only depends on K, n, and p. Now, write

$$(f * (|y|^{-\gamma}))(x) = \int_{K^n} f(x-y)|y|^{-\gamma}|dy| = \int_{|y|<R} + \int_{|y|\geq R}$$

for $R > 0$.

Note that for a characteristic function χ_B of a ball B containing zero, by the definition of M, one has

$$|(f * \chi_B)(x)| \leq (Mf)(x)\cdot\mathrm{Vol}(B).$$

Since one can approximate the function $y \mapsto |y|^{-\gamma}\chi_{B_R}(y)$ on K^n, with B_R the ball given by $|y| < R$, by finite expressions of the form

$$\sum_{i=1}^{N} a_i \chi_{B_i},$$

with the B_i balls around 0, it follows by the Lebesgue monotone convergence theorem that our first integral, which can be rewritten as the convolution

$$\int_{K^n} f(x-y)(|y|^{-\gamma}\chi_{B_R}(y))|\mathrm{d}y|,$$

is bounded in norm by

$$(Mf)(x)\cdot \int_{|y|<R} |y|^{-\gamma}|\mathrm{d}y| \leq \delta R^{n-\gamma}\cdot (Mf)(x),$$

for some constant δ. By Hölder's inequality, the second integral

$$\int_{|y|\geq R} f(x-y)|y|^{-\gamma}|\mathrm{d}y|$$

is dominated by

$$\|f\|_{L^p(K^n)}\cdot \| |y|^{-\gamma}\cdot (1-\chi_{B_R})\|_{L^{p'}(K^n)}.$$

Now $|y|^{-\gamma}\cdot(1-\chi_{B_R})$ is in $L^{p'}(K^n)$ when $-\gamma p' < -n$ and, in view of (4.3.2),

$$\gamma p' - n = \frac{np'}{q} > 0.$$

Thus

$$\| |y|^{-\gamma}\cdot (1-\chi_{B_R})\|_{L^{p'}(K^n)} \leq eR^{-n/q}$$

for some constant e. Summing the two integrals, we have

$$|(f*(|y|^{-\gamma}))(x)| \leq A((Mf)(x)R^{n-\gamma} + \|f\|_{L^p(K^n)}R^{-n/q})$$

for some constant A. Choose R so that both terms on the right side are equal, that is,

$$\frac{(Mf)(x)}{\|f\|_{L^p(K^n)}} = R^{-n+\gamma-n/q} = R^{-n/p}.$$

Substituting this in the above gives

$$|f*(|y|^{-\gamma})(x)| \leq 2\cdot A\cdot (Mf(x))^{p/q}\cdot \|f\|_{L^p(K^n)}^{1-p/q}.$$

The inequality (4.3.1) then follows from the usual L^p inequality for the maximal operator M, namely the inequality (4.3.3) of the Hardy–Littlewood–Wiener maximal theorem.

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