

Approximations and Lipschitz continuity in p -adic semi-algebraic and subanalytic geometry

Raf Cluckers · Immanuel Halupczok

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Abstract It is known that a p -adic, locally Lipschitz continuous semi-algebraic function, is piecewise Lipschitz continuous, where finitely many pieces suffice and the pieces can be taken semi-algebraic. We prove that if the function has locally Lipschitz constant 1, then it is also piecewise Lipschitz continuous with the same Lipschitz constant 1 (again, with finitely many pieces). We do this by proving the following fine preparation results for p -adic semi-algebraic functions in one variable. Any such function can be well approximated by a monomial with fractional exponent such that moreover the derivative of the monomial is an approximation of the derivative of the function. We also prove these results in parameterized versions and in the subanalytic setting.

Keywords p -Adic semi-algebraic functions · p -Adic subanalytic functions · Lipschitz continuous functions · p -Adic cell decomposition

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R. Cluckers (✉)
Laboratoire Painlevé, CNRS, UMR 8524, Université Lille 1, Cité Scientifique,
59655 Villeneuve d'Ascq Cedex, France
e-mail: raf.cluckers@math.univ-lille1.fr
URL: <http://math.univ-lille1.fr/~cluckers>

R. Cluckers
Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B,
3001 Leuven, Belgium

I. Halupczok
Institut für Mathematische Logik und Grundlagenforschung, Universität Münster,
Einsteinstraße 62, 48149 Münster, Germany
e-mail: math@karimmi.de
URL: <http://www.immi.karimmi.de/en.math.html>

Introduction

In several senses, p -adic manifolds do not have curvature, they are flat. For example, any p -adic analytic submanifold of \mathbb{Q}_p^n is locally analytically isometric to an open ball in p -adic Euclidean space. In this paper, we show another phenomenon related to this absence of curvature, namely related to Lipschitz continuity. On the reals, a locally Lipschitz continuous function with constant ε , say on a sufficiently nice domain X in \mathbb{R}^n , can be expected to be globally Lipschitz continuous, but usually this happens only with a bigger Lipschitz constant $C > \varepsilon$; see for example [11]. For example, the function f on the unit circle S^1 around the origin in \mathbb{R}^2 which measures the distance to the point $(1, 0)$ along the circle is locally $(1 + \varepsilon)$ -Lipschitz for any $\varepsilon > 0$, but globally only $\frac{\pi}{2}$ -Lipschitz.

In the context of semi-algebraic or subanalytic sets on the p -adics, to encompass the total disconnectedness, one has to break the domain into finitely many pieces on each of which the function becomes Lipschitz continuous. Our main result is that in that case, one does not even need to increase the Lipschitz constant to pass from local to global. Note that in the reals, breaking the domain into finitely many pieces does not help to keep the same constant; see Example 1 below. Here is a precise formulation of our main result.

Theorem 1 *Let $f : X \subset \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ be a semi-algebraic function and let $\varepsilon > 0$ be a constant. Suppose that, locally around each point $x \in X$, the function f is Lipschitz continuous with Lipschitz constant ε . Then, there exists a finite partition of X into semi-algebraic parts A_i such that each of the restrictions $f|_{A_i}$ is Lipschitz continuous with constant ε (globally on the part A_i). The same property is true when one replaces semi-algebraic by subanalytic.*

In fact, we prove a parameterized version of Theorem 1, see Theorem 3.5. Theorem 1 is false if one would replace \mathbb{Q}_p by \mathbb{R} ; see Example 1 below.

Let us sketch the p -adic context. Let K be a finite field extension of the field of p -adic numbers \mathbb{Q}_p . One of the main results of [3] states that if $f : X \subset K^n \rightarrow K$ is a semi-algebraic (resp. subanalytic) function that is locally Lipschitz continuous with constant ε , then there exists a finite partition of X into semi-algebraic (resp. subanalytic) pieces A such that the restriction $f|_A$ of f to any of the pieces is globally Lipschitz continuous with possibly some bigger Lipschitz constant C ; this is the p -adic analogue of one of Kurdyka's results in [11]. Theorem 1 states that one can choose the finite partition in such a way that one can take $C = \varepsilon$. As in [3], this is proved by induction on n , and hence, a family (i.e., parameterized) version of this Lipschitz continuity result is more natural and flexible to work with, see Theorem 3.5.

The main ingredient in [3] is a preparation result stating that the domain of a definable function f can be cut into nice pieces (cells) on which f behaves nicely. The proof of our Theorem 3.5 is essentially the same, the main difference being that we need a finer piecewise preparation result for f . Indeed, the preparation result used in [3] is Proposition 3.12 of [3] (which is finer than the more classical cell decomposition results); this proposition in some sense gives a compatible cell decomposition of the domain of $f : X \subset \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ and its image, when $n = 1$. We refine that proposition to get a Preparation Theorem 3.3 that allows one to approximate f in a piecewise

way by monomials with fractional exponents such that at the same time the derivative of f is approximated by the derivatives of the monomials, still for $n = 1$. In fact, the Preparation Theorem 3.3 treats this property in semi-algebraic, resp. subanalytic, families.

The real situation is different, as is shown by the following example.

Example 1 Let $g: S^1 \rightarrow \mathbb{R}$ be the function on the unit circle in the real plane that sends $z \in S^1$ to the arc length between z and a fixed point on S^1 . Let X be the open annulus $\{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$ and let $f: X \rightarrow \mathbb{R}$ be the function that sends $x \in X$ to $g(z)/r$, where $x = r \cdot z$ with $z \in S^1$ and $1 < r < 2$. Then, f is globally subanalytic and locally Lipschitz continuous with constant 1, but there does not exist a finite partition of X into parts on which f is Lipschitz continuous with constant 1. One is obliged to replace 1 by e for some real $e > 1$ in order to find a finite partition of X into parts on which f is e -Lipschitz continuous.

The framework we will use in this article is essentially the same as in [3]. We will repeat all the necessary definitions, but we will give less details than in [3], and for some proofs that run exactly parallel to [3], we will only explain how to modify them. This means that a certain familiarity with [3] will be helpful when reading this article, and having a copy of [3] is indispensable to read some of the proofs.

1 Some general definitions

Let K be a valued field, with (additively written) value group Γ and valuation map $\text{ord}: K^\times \rightarrow \Gamma$; as usual, we extend ord to a map $K \rightarrow \Gamma \cup \{+\infty\}$ with $\text{ord}(0) = +\infty$. Put the valuation topology on K^n for $n \geq 1$. For $X \subset K$ open, a function $f: X \rightarrow K$ is called C^1 if f is differentiable at each point of X and the derivative $f': X \rightarrow K$ of f is continuous (this notion of C^1 is more naive than the one of Glöckner [10], but suffices for our purposes).

Write \mathcal{O}_K for the valuation ring of K , with maximal ideal \mathcal{M}_K . Further, write RV_K for the union of the quotient $K^\times/(1 + \mathcal{M}_K)$ with $\{0\}$, and $rv: K \rightarrow RV_K$ for the natural quotient map $K^\times \rightarrow K^\times/(1 + \mathcal{M}_K)$ extended by $rv(0) = 0$. Similarly, write $RV_{K,n}$ for the union of the quotient $K^\times/(1 + \mathcal{M}_K^n)$ with $\{0\}$, and $rv_n: K \rightarrow RV_{K,n}$ for the natural map when $n > 0$, and let rv_0 be ord .

A ball in K is by definition a set of the form $\{t \in K \mid \text{ord}(t - a) \geq z\}$ for some $a \in K$ and some $z \in \Gamma$.

Definition 1.1 (*Jacobian property*, [3, Definition 3.10]) Let $F: B_1 \rightarrow B_2$ be a function with $B_1, B_2 \subset K$. Say that F has the *Jacobian property* if the following conditions (a) up to (d) hold:

- (a) F is a bijection $B_1 \rightarrow B_2$ and B_1 and B_2 are balls;
- (b) F is C^1 on B_1 ;
- (c) $\text{ord}(\partial F/\partial x)$ is constant (and finite) on B_1 ;
- (d) for all $x, y \in B_1$ with $x \neq y$, one has

$$\text{ord}(\partial F/\partial x) + \text{ord}(x - y) = \text{ord}(F(x) - F(y)).$$

Definition 1.2 (*n-Jacobian property*) Let $F : B_1 \rightarrow B_2$ be a function with $B_1, B_2 \subset K$ and let $n > 0$ be an integer. Say that F has the *n-Jacobian property* if F has the Jacobian property, and moreover, the following stronger versions of conditions (c) and (d) hold:

- (c') $rv_n(\partial F/\partial x)$ is constant (and nonzero) on B_1 ;
- (d') for all $x, y \in B_1$ one has

$$rv_n(\partial F/\partial x) \cdot rv_n(x - y) = rv_n(F(x) - F(y)).$$

For convenience, 0-Jacobian property will just mean Jacobian property.

Definition 1.3 (*Local n-Jacobian property*) Let $f : X \rightarrow Y$ be a function with $X, Y \subset K$ and let $n > 0$ be an integer. Say that f has the *local n-Jacobian property* if f is a bijection and for each ball $B \subset X$ and for each ball $B' \subset Y$, the restrictions $f|_B : B \rightarrow f|_B(B)$ and $(f^{-1})|_{B'} : B' \rightarrow f|_{B'}^{-1}(B')$ have the *n-Jacobian property*.

Definition 1.4 (*Lipschitz continuity*) Given two metric spaces (X, d_X) and (Y, d_Y) , where d_X denotes the metric on the set X and d_Y the metric on Y , a function $f : X \rightarrow Y$ is called Lipschitz continuous if there exists a real constant $C \geq 0$ such that, for all x_1 and x_2 in X ,

$$d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2).$$

In the above case, we also call f Lipschitz continuous with Lipschitz constant C or just C -Lipschitz continuous. If there is a constant C such that each $x \in X$ has a neighborhood on which the function f is C -Lipschitz continuous, then f is called locally Lipschitz continuous with constant C or just locally C -Lipschitz continuous.

2 Definable sets over the p -adics

From now on, let K be a finite field extension of \mathbb{Q}_p with valuation ring \mathcal{O}_K and valuation map $\text{ord} : K \rightarrow \mathbb{Z} \cup \{+\infty\}$. We set $|x| := q_K^{-\text{ord}(x)}$ for $x \in K^\times$ and $|0| = 0$, where q_K is the cardinality of the residue field of K . Further we choose a uniformizer π_K of \mathcal{O}_K and for each integer $n > 0$, we let $\overline{\text{ac}}_n : K \rightarrow \mathcal{O}_K/(\pi_K^n)$ be the multiplicative map sending 0 to 0 and any nonzero x to $x\pi_K^{-\text{ord}(x)} \bmod (\pi_K^n)$.

We recall the notion of (globally) subanalytic subsets of K^n and of semi-algebraic subsets of K^n . Let $\mathcal{L}_{\text{Mac}} = \{0, 1, +, -, \cdot, \cdot\} \cup \{P_n\}_{n>0}$ be the ‘‘language of Macintyre,’’ where P_n stands for the set of n -th powers in K^\times . (Note that \mathcal{L}_{Mac} is not exactly the language introduced by Macintyre [12], but the variant used by Denef [7].) Let $\mathcal{L}_{\text{an}} = \mathcal{L}_{\text{Mac}} \cup \{-1\} \cup \bigcup_{m>0} K\{x_1, \dots, x_m\}$, where P_n stands for the set of n th powers in K^\times , where -1 stands for the field inverse extended to 0 by $0^{-1} = 0$, where $K\{x_1, \dots, x_m\}$ is the ring of restricted power series over K (i.e., formal power series over K converging on \mathcal{O}_K^m), and each element f of $K\{x_1, \dots, x_m\}$ is interpreted as

the restricted analytic function $K^m \rightarrow K$ given by

$$x \mapsto \begin{cases} f(x) & \text{if } x \in \mathcal{O}_K^m \\ 0 & \text{else.} \end{cases} \quad (2.0.1)$$

By subanalytic, we mean \mathcal{L}_{an} -definable, and by semi-algebraic, we mean \mathcal{L}_{Mac} -definable with parameters from K . Note that semi-algebraic, resp. subanalytic, sets can be given by a quantifier free formula with parameters from K in the language \mathcal{L}_{Mac} , resp. \mathcal{L}_{an} , by [7], resp. [8] and [9].

From now on we choose one of the two notions: semi-algebraic or subanalytic, and by *definable*, we will mean semi-algebraic, resp. subanalytic, according to our fixed choice.

As in [3, Section 3], for integers $m > 0$ and $n > 0$, we let $Q_{m,n}$ be the (definable) set

$$Q_{m,n} := \{x \in K^\times \mid \text{ord}(x) \in n\mathbb{Z}, \overline{\text{ac}}_m(x) = 1\}.$$

For $\lambda \in K$ let $\lambda Q_{m,n}$ denote $\{\lambda x \mid x \in Q_{m,n}\}$. The sets $Q_{m,n}$ are a variant of Macintyre's predicates P_ℓ of ℓ th powers; the corresponding notions of cells are slightly different but equally powerful and similar in usage, since any coset of P_ℓ is a finite disjoint union of cosets of some $Q_{m,n}$ and vice versa.

Definition 2.1 (*p -adic cells*, [3, Definition 3.1]) Let Y be a definable set. A cell $A \subset K \times Y$ over Y is a (nonempty) set of the form

$$A = \{(t, y) \in K \times Y \mid y \in Y', |\alpha(y)| \square_1 |t - c(y)| \square_2 |\beta(y)|, t - c(y) \in \lambda Q_{m,n}\}, \quad (2.1.1)$$

with $Y' \subset Y$ a definable set, constants $n > 0, m > 0, \lambda \in K, \alpha, \beta: Y' \rightarrow K^\times$ and $c: Y' \rightarrow K$ all definable functions, and \square_i either $<$ or no condition, and such that A projects surjectively onto Y' . We call c the center of the cell A , $\lambda Q_{m,n}$ the coset of A , α and β the boundaries of A , and Y' the base of A . If $\lambda = 0$ we call A a 0-cell, and if $\lambda \neq 0$ we call A a 1-cell. Call a 1-cell A unbounded if at least one of the \square_i in (2.1.1) is no condition.

Sometimes, one additionally requires that Y' is a K -analytic manifold or a cell, but for the purposes of this article, this is not necessary.

3 The main results

In the following two definitions, we introduce some new notions that will be needed to formulate our version of the Preparation Theorem. The first notion is that of a “fractional monomial”—the functions by which we will approximate arbitrary definable functions. A fractional monomial (“with center 0”) is supposed to be something like $t \mapsto et^q$ for some $e \in K$ and $q \in \mathbb{Q}$. The following definition makes this precise and moreover allows for parameters.

Definition 3.1 (*Fractional monomials*) Let $A \subset K \times Y$ be a cell over Y with center c , as in (2.1.1). A fractional monomial on A with center c is a continuous, definable function $m: A \rightarrow K$ such that there exist a definable map $e: Y \rightarrow K$ and coprime integers a and b with $b > 0$ such that for all $(t, y) \in A$

$$m(t, y)^b = e(y)(t - c(y))^a.$$

We use the conventions that $b = 1$ whenever $a = 0$, that $a = 0$ whenever A is a 0-cell, and that $0^0 = 1$. If furthermore $e(y)$ is nonzero for some y and some t with $(t, y) \in A$, then a/b is independent of any choices and we call a/b the exponent of m . In any case, we call e the coefficient of m .

Note that, although the center c of a cell A is usually not unique, we assume that cells and fractional monomials on the cells have the same (sometimes implicitly fixed) centers.

Now we define what it should mean for a function to be approximated by another function, for example a fractional monomial. This notion only makes sense on cells, and it only makes sense if both functions “are compatible” with that cell; our notion of 0-compatibility is essentially the same as the compatibility notion of Proposition 3.12 of [3].

Definition 3.2 Let $A \subset K \times Y$ be a 1-cell over Y , let $f: A \rightarrow K$ be definable, and let $n \geq 0$ be an integer. Write

$$f \times \text{id}: A \rightarrow K \times Y: (t, y) \mapsto (f(t, y), y)$$

and

$$A_f = (f \times \text{id})(A).$$

Say that f is n -compatible with the cell A if either A_f is a 0-cell over Y , or the following holds: A_f is a 1-cell over Y and for each $y \in Y$, the function $f_y: A_y \rightarrow f_y(A_y): t \mapsto f(t, y)$ has the local n -Jacobian property; here and later, $A_y = \{t \in K \mid (t, y) \in A\}$ denotes the fiber of A_y above y .

If $g: A \rightarrow K$ is a second function that is n -compatible with the cell A and if we have

$$A_f = A_g \text{ and } rv_n \left(\frac{\partial f(t, y)}{\partial t} \right) = rv_n \left(\frac{\partial g(t, y)}{\partial t} \right) \text{ on } A,$$

then we say that f and g are n -equicompatible with A .

If $A' \subset K \times Y$ is a 0-cell over Y (instead of a 1-cell), any definable function $h: A' \rightarrow K$ is said to be n -compatible with A' , and h and $k: A' \rightarrow K$ are n -equicompatible with A' if and only if $h = k$.

Theorem 3.3 (n -Preparation Theorem) Let $X \subset K \times Y$ and $f_j: X \rightarrow K$ be definable for $j = 1, \dots, r$ and let $n \geq 0$ be an integer. Then, there exists a finite partition

of X into cells A over Y such that the restriction $f_{j|A}$ is n -compatible with A for each cell A over Y and for each j , if one writes d_j for the center of A_{f_j} , then there exists a fractional monomial m_j on A such that the functions $d_j + m_j$ and $f_{j|A}$ are n -equicompatible with A .

If moreover A is unbounded, then the fractional monomials m_j are unique.

In the above theorem, the center d_j of A_{f_j} is identified with the function on A_{f_j} sending $(t, y) \in A_{f_j}$ to $d_j(y)$.

One can compare our results to the classically known cell decomposition theorem (due to Cohen [5], Denef [6, 7], and the first author [2]).

Theorem 3.4 (Classical p -adic Cell Decomposition) *Let $X \subset K \times Y$ and $f_j: X \rightarrow K$ be definable for $j = 1, \dots, r$ and let $n \geq 1$ be an integer. Then, there exists a finite partition of X into cells A over Y and for each occurring 1-cell A fractional monomials m_j on A such that*

$$rv_n(f_j(t, y)) = rv_n(m_j(t, y)) \quad \text{for each } (t, y) \in A.$$

We indicate how Theorem 3.4 follows from our new Theorem 3.3. (In fact, also the stronger Proposition 3.12 of [3] follows from Theorem 3.3.) Take a partition of X into cells A as in Theorem 3.3. By the definition of cells (applied to A_{f_j}), we may suppose that either d_j is identically zero or $rv_n(f_j) = rv_n(d_j)$ on A . In the second case, one is done since clearly d_j is a fractional monomial on A , and in the first case, one has $rv_n(f_j) = rv_n(m_j)$ on A and one is also done. Proposition 3.12 of [3] is as the Cell Decomposition Theorem 3.4 with $r = 1$ and $f = f_1$ and with the extra property that for each 1-cell A in the partition the restriction $f_{j|A}$ is 0-compatible with A . In fact, Proposition 3.12 of [3] and Theorem 3.4 will be used to prove Theorem 3.3.

Theorem 3.3 allows us to improve the main results (Theorems 2.1 and 2.3) of [3] to the following.

Theorem 3.5 (Piecewise Lipschitz continuity) *Let $\varepsilon > 0$ be given. Let Y be a definable set. Let $f: X \subset K^m \times Y \rightarrow K$ be a definable function such that for each $y \in Y$, the function $f(\cdot, y): x \mapsto f(x, y)$ is locally ε -Lipschitz continuous on $X_y = \{x \mid (x, y) \in X\}$. Then, there exists a finite definable partition of X into parts A_i such that for each $y \in Y$ and i the restriction of $f(\cdot, y)$ to A_{iy} is (globally on A_{iy}) ε -Lipschitz continuous.*

The main point in Theorem 3.5 is that the constant of the Lipschitz continuity does not change when passing from the local to the piecewise global property.

4 Proofs of the main results

We prove some auxiliary results first, after recalling the Banach Fixed Point Theorem in our setting (where we use the discreteness of the p -adic valuation to simplify its formulation). Lemmas 4.2 and 4.3 are key points in the proof of Theorem 3.3.

Lemma 4.1 (Banach Fixed Point Theorem) *Suppose that a function $f : \mathcal{O}_K^n \rightarrow \mathcal{O}_K^n$ is contracting in the sense that for any $x_1, x_2 \in \mathcal{O}_K^n$ with $x_1 \neq x_2$, $\text{ord}(f(x_1) - f(x_2)) > \text{ord}(x_1 - x_2)$, where the order of a tuple is defined as usual as the minimum of the orders of the entries of the tuple. Then, f has exactly one fixed point, that is, a point $x \in \mathcal{O}_K^n$ with $f(x) = x$.*

Lemma 4.2 *Suppose that $B, B_1, B_2 \subset K$ are balls, that $f_1 : B \rightarrow B_1$ and $f_2 : B \rightarrow B_2$ both satisfy the Jacobian property, and that $B_1 \cap B_2 \neq \emptyset$. Suppose moreover $\text{ord}(f'_1) \neq \text{ord}(f'_2)$. Then, there exists exactly one element $b_0 \in B$ such that $f_1(b_0) = f_2(b_0)$.*

Proof By the Jacobian property, we may without loss suppose that $\text{ord}(f'_1) < \text{ord}(f'_2)$ and thus $B_1 \supset B_2$. Consider the map $f_1^{-1} \circ f_2 : B \rightarrow B$. By the chain rule for differentiation and the Jacobian property, this map is contracting, and thus, by the Banach Fixed Point Theorem 4.1, it has exactly one fixed point b_0 , which is the point we are looking for. \square

Lemma 4.3 *Let $B \subset K$ be a ball containing 0 and let $f_1, f_2 : B \xrightarrow{\text{!}:\text{!}} B$ be definable functions satisfying the following for some integer $n \geq 1$:*

- (1) *both f_1 and f_2 have the n -Jacobian property;*
- (2) *$rv_n(f'_1) \neq rv_n(f'_2)$;*
- (3) *$f_1(a) - f_2(a) \in \mathcal{M}_K^{n-1}B$ for some $a \in B$.*

Then, there exists exactly one element $b_0 \in B$ such that $f_1(b_0) = f_2(b_0)$.

Proof First we prove uniqueness. Suppose that $f_1(b) = f_2(b)$ for some $b \in B$. Replacing f_i by $t \mapsto f_i(t + b) - f_i(b)$, we may suppose that $b = 0$ and that $f_i(0) = 0$ for $i = 1, 2$. By the n -Jacobian property,

$$rv_n(f_i(x)) = rv_n(f'_i) \cdot rv_n(x)$$

for all x in B and hence by (2), $f_1(x) \neq f_2(x)$ for nonzero x in B .

Next we prove existence. Suppose by contradiction that no such b_0 exists. By continuity of f_1 and f_2 , this implies that there exists an upper bound in \mathbb{Z} on the order of $f_1(x) - f_2(x)$ for $x \in B$; consider the minimal such upper bound γ and choose $b \in B$ with $\text{ord}(f_1(b) - f_2(b)) = \gamma$. In fact, we may suppose $b = a$, and replacing f_i by $t \mapsto f_i(t + b)$, we may moreover suppose that $b = 0$. Put $c := f_1(0) - f_2(0)$ and $d := f'_1(0) - f'_2(0)$. We have, by (1),

$$\text{ord}(f_i(x) - f'_i(0)x - f_i(0)) \geq \text{ord}(x) + n$$

for $i = 1, 2$ and all $x \in B$, and, subtracting,

$$\text{ord}(f_1(x) - f_2(x) - dx - c) \geq \text{ord}(x) + n \tag{4.3.1}$$

for all x in B . Take the unique $b' \in K$ with $db' + c = 0$. Since $\text{ord}(d) < n$ by (1) and (2) and since $\text{ord}(c) = \gamma$, one finds $\text{ord}(b') > \gamma - n$ and thus from (3) follows that $b' \in B$. Plugging in b' for x in (4.3.1) one finds that $\text{ord}(f_1(b') - f_2(b')) > \gamma$, a contradiction to the choice of γ . \square

We recall a Definition of [3].

Proposition-Definition 4.4 (Balls of cells, [3, Proposition-Definition 3.2]) Let Y be definable. Let $A \subset K \times Y$ be a 1-cell over Y . Then, for each $(t, y) \in A$, there exists a unique maximal ball $B_{t,y}$ containing t and satisfying $B_{t,y} \times \{y\} \subset A$, where maximality is under inclusion. For fixed $y_0 \in Y$, we call the collection of balls $\{B_{t,y_0}\}_{\{t|(t,y_0) \in A\}}$, the balls of the cell A above y_0 . Moreover, for each $(t, y) \in A$ one has

$$B_{t,y} = \{w \in K \mid \text{ord}(w - c(y)) = a, \overline{\text{ac}}_m(w - c(y)) = \overline{\text{ac}}_m(\lambda)\}$$

for some $a \in \mathbb{Z}$ depending on t and y , and where c , m , and λ are as in Definition 2.1.

Definition 4.5 Call a 1-cell A over Y thin if the collection of balls of A above any $y \in Y$ consists of at most one ball.

Theorem 3.3 relies on both Lemmas 4.2 and 4.3. The idea for the applications of these lemmas is that they suppose a certain bad behavior on a ball in view of the desired conclusion of Theorem 3.3, and then, they yield some isolated special points. In our definable setup, these isolated special points form a discrete definable subset; such sets are known to be finite, and moreover, their size is uniformly bounded in families; hence, these finite sets single out a finite collection of balls, each of which corresponds to a thin cell. On these thin cells, we prove Theorem 3.3 separately, directly from the Jacobian property.

Proof of Theorem 3.3 During the proof, we will cell-decompose X several times and treat each cell separately. A priori, if a function $f: A \rightarrow K$ on a cell A is n -compatible with A , then it does not need to be n -compatible with cells $B \subset A$ (unless the center of B is a restriction of the center of A). The following statement will be used as a remedy; it follows easily from the Cell Criterion 3.8 from [3].

- (\star) Suppose that A is a cell over Y and that $f: A \rightarrow K$ is a definable function which is n -compatible with A . Then for any cell $B \subset A$ over Y with center d , there exists a finite partition of B into cells B_i , whose centers are the natural restrictions of d , and such that $f|_{B_i}$ is n -compatible with B_i .

We now simplify the setting in several steps. Note that each time we decompose X into cells over Y , we may neglect the 0-cells, since for these, the result is trivial (using $m_j := 0$ for each j).

By Proposition 3.12 of [3], we may suppose that for each j separately, there is a finite partition of X into cells over Y such that the restriction of f_j to each cell in the partition is 0-compatible with that cell. In fact, by using the n -Jacobian property of [4, Section 6, Remark 6.3.16] instead of only the 0-Jacobian property, Proposition 3.12 also holds with n -compatibility instead of 0-compatibility, with the same proof as in [3]. Using (\star), we can combine the different partitions into a single one such that each f_j is n -compatible with each cell. From now on, let us assume that X is a single such cell.

Write c for the center of X and d_j for the center of X_{f_j} for each j .

By decomposing X using the Classical Cell Decomposition Theorem 3.4, we may suppose that there exist fractional monomials m_j on X for each j such that

$$rv_n(f_j - d_j) = rv_n(m_j) \tag{4.5.1}$$

holds on X . Here, we use (\star) to get back the n -compatibility of f_j with X . We may moreover exclude the simple cases that $f_j - d_j$ or m_j are constant.

By piecewise linearity results of Presburger functions on Presburger sets, see for example [1], we may suppose that for each j , $\text{ord}(\partial f_j(t, y)/\partial t)$ depends linearly on $\text{ord}(t - c(y))$.

By the simple form of fractional monomials (namely by taking a definable choice out of at most b roots with b as in Definition 3.1), we may further suppose that also the m_j are n -compatible with X .

We may suppose for each j that either X_{f_j} is included in $X_{d_j+m_j}$ or, vice versa, $X_{d_j+m_j}$ is included in X_{f_j} , by comparing the balls (in the sense of Definition 4.4) of the cells X_{f_j} and $X_{d_j+m_j}$ and possibly modifying the coefficients e_j of the m_j . Indeed, since X_{f_j} and $X_{d_j+m_j}$ have the same center, we can choose finitely many $r_{j,v} \in K^\times$ such that for any $y \in Y$ and any ball B of X above y , there exists a v such that one of the balls $f_j(B \times \{y\})$ and $(d_j + r_{j,v}m_j)(B \times \{y\})$ is contained in the other one.

Fix j and $y \in Y$, and set $A := \{t \in X_y \mid f_j(t, y) = d_j(y) + m_j(t, y)\}$. Suppose that B is a ball of X above y and with $f_j(B \times \{y\}) \neq (d_j + m_j)(B \times \{y\})$. Then $\text{ord}(\partial f_j(t, y)/\partial t) \neq \text{ord}(\partial m_j(t, y)/\partial t)$, so Lemma 4.2 yields an isolated point $b_0 \in A$. Since A can have only finitely many points, by decomposing X we can assume that either it is a thin cell, or the set X_{f_j} equals $X_{d_j+m_j}$ for each j .

Suppose that X is a thin cell. Then, by the n -Jacobian property, we know that, for each individual $y \in Y$ such that X_y is nonempty, $f_{jy} : X_y \rightarrow f_{jy}(X_y) : t \mapsto f_j(t, y)$ has the n -Jacobian property, and hence, there exists a linear function $\ell_{jy} : X_y \rightarrow f_{jy}(X_y) : t \mapsto a_{jy}t + b_{jy}$ such that ℓ_{jy} and f_{jy} are n -equicompatible with the ball X_y . By the definability of Skolem functions (also called definability of sections), we may suppose that the a_{jy} and the b_{jy} depend definably on y . Hence, we can take as our final $d_j + m_j$ the (linear) $(t, y) \mapsto a_{jy}t + b_{jy}$, that is, the definable functions $d_j(y) = b_{jy}$ on Y and $m_j(t, y) = a_{jy}t$ on X are as desired.

There only remains to treat the case that X_{f_j} equals $X_{d_j+m_j}$ for each j (and X is not thin). For this case, we will apply Lemma 4.3 in a similar way as we applied Lemma 4.2 before. More precisely, by Lemma 4.3 we can exclude finitely many thin cells (each of which can be treated as before), such that $rv_n(\partial f_j/\partial t) = rv_n(\partial m_j/\partial t)$ holds on the remaining part, and we are done. For this application of Lemma 4.3, it remains to ensure that condition (3) of that lemma holds. Condition (3) of Lemma 4.3 can indeed be ensured by further partitioning X if necessary and by slightly modifying the coefficients e_j of the monomials m_j , as we already did in the proof that one of $X_{f_j}, X_{d_j+m_j}$ is included in the other. \square

The following proposition specifies that one can take $C = 1$ for the Lipschitz constant in Proposition 2.4 of [3].

Proposition 4.6 (Cells with 1-Lipschitz continuous centers) *Let Y and $X \subset K^m \times Y$ be definable. Then, there exists a finite partition of X into definable parts A and for*

each part A a coordinate projection

$$\pi : K^m \times Y \rightarrow K^{m-1} \times Y$$

such that, over $K^{m-1} \times Y$ along this projection π , the set A is a cell with center $c : \pi(A) \rightarrow K$ and such that moreover the function

$$c(\cdot, y) : (t_1, \dots, t_{m-1}) \mapsto c(t_1, \dots, t_{m-1}, y)$$

is 1-Lipschitz continuous on $\pi(A)_y$ for each $y \in Y$.

As in [3], we prove Proposition 4.6 and Theorem 3.5 by a joint induction on m . More precisely, assuming both Proposition 4.6 and Theorem 3.3 for $\leq m - 1$, we first prove Proposition 4.6 for m and then Theorem 3.3 for m . Since Proposition 4.6 is trivial for $m = 1$, to start the induction, it suffices to prove Theorem 3.3 for $m = 1$.

We start by doing the induction step for Proposition 4.6, so that afterward, we can concentrate on Theorem 3.3.

Proof of the induction step for Proposition 4.6 This proof is exactly as the proof of Proposition 2.4 in [3], invoking our Theorem 3.5 for $m - 1$ instead of Theorem 2.3 of [3] to control the Lipschitz constants, and where in the second last sentence of the proof of Proposition 2.4 of [3], one replaces “are bounded in norm” by “have norms ≤ 1 ” and one further replaces C by 1.

Roughly, the idea is the following: start with any cell decomposition of X along any coordinate projection π . Consider a cell A with center $c : \pi(A) \rightarrow K$. If $|\partial c / \partial t_i| \leq 1$ for all $i \leq m - 1$, then we are done; otherwise, interchange the role of t_i and the projection coordinate. We would like to say that A is still a cell using this different coordinate projection and that the graph of the center is the same as before. Of course, this is not true in general, but by cutting A into pieces, this can be easily achieved if A was a 0-cell, and it can be achieved with some work if A was a 1-cell. \square

Proof of Theorem 3.5 for $m = 1$ We are given $\varepsilon > 0$, Y a definable set, and $f : X \subset K \times Y \rightarrow K$ a definable function such that for each $y \in Y$ the function $f(\cdot, y) : x \mapsto f(x, y)$ is locally ε -Lipschitz continuous on its natural domain $X_y := \{x \in K \mid (x, y) \in X\}$. Use Theorem 3.3 to partition X into finitely many cells X_i over Y . By working piecewise, we may suppose that $X = X_1$ and that X and X_f are 1-cells over Y . By the Jacobian property, $f(\cdot, y)$ is C^1 and by local ε -Lipschitz continuity,

$$|\partial f(x, y) / \partial x| \leq \varepsilon \tag{4.6.1}$$

for all $(x, y) \in X$. Write c for the center of X and d for the center of X_f . Since a function $g : A \subset K \rightarrow K$ is ε -Lipschitz continuous if and only if $A \rightarrow K : x \mapsto g(x + a) + b$ is ε -Lipschitz continuous for any constants $a, b \in K$, we may suppose, after translating, that c and d are identically zero. Thus, there exists a fractional monomial m such that f and m are n -equicompatible with X .

Now fix $y \in Y$. Take (x_1, y) and (x_2, y) in X . If x_1 and x_2 both lie in the same ball $B_{x_1, y}$ above y , then

$$|(\partial f(x_1, y)/\partial x) \cdot (x_1 - x_2)| = |f(x_1, y) - f(x_2, y)| \quad (4.6.2)$$

by the Jacobian property and we are done by (4.6.1).

Next suppose that $B_{x_1, y}$ and $B_{x_2, y}$ are two different balls. By our assumption that c and d are identically zero, there exist $\ell \in \mathbf{N}$, $\lambda \in K$ and $a_1, a_2 \in \mathbb{Z}$, $a_1 \neq a_2$ such that

$$B_{x_i, y} = \{x \in K \mid \text{ord}(x) = a_i, \overline{\text{ac}}_\ell(x) = \overline{\text{ac}}_\ell(\lambda)\} \quad (4.6.3)$$

for $i = 1, 2$; analogously, for the image of these balls under $f(\cdot, y)$, we can write

$$f(B_{x_i, y}, y) = \{z \in K \mid \text{ord}(z) = b_i, \overline{\text{ac}}_{\ell'}(z) = \overline{\text{ac}}_{\ell'}(\mu)\} \quad (4.6.4)$$

for some ℓ', μ, b_1, b_2 with $b_1 \neq b_2$. From these descriptions, we get the inequalities:

$$\text{ord}(f(x_1, y) - f(x_2, y)) = \min_{i=1,2} b_i$$

and

$$\text{ord}(x_1 - x_2) = \min_{i=1,2} a_i.$$

On the other hand by the Jacobian property (d) one finds, by comparing the sizes of the balls (4.6.3) and (4.6.4),

$$\ell + \text{ord}(\partial f(x_i, y)/\partial x) + a_i = \ell' + b_i$$

for $i = 1, 2$. Putting this together with (4.6.1) yields:

$$|f(x_1, y) - f(x_2, y)| = \max_{i=1,2} q_K^{-b_{x_i, y}} \leq \varepsilon q_K^{\ell' - \ell} \max_{i=1,2} q_K^{-a_{x_i, y}} = \varepsilon q_K^{\ell' - \ell} |x_1 - x_2|. \quad (4.6.5)$$

If $\ell' - \ell \leq 0$, then we are done by (4.6.5). Also if the exponent of the fractional monomial m is 1, then, by the n -equicompatibility of m and f and the linearity of m , one must have $\ell = \ell'$ and the statement follows from (4.6.5). Finally suppose that the exponent of m is unequal to 1 and that $\ell' - \ell > 0$. Then, $|\partial f(x, y)/\partial x|$ is not constant on X , since it is equal to $|\partial m(x, y)/\partial x|$. Hence, excluding finitely many thin cells from X , it follows that we can suppose that also

$$|\partial f(x, y)/\partial x| \leq \varepsilon q_K^{-(\ell' - \ell)} \quad (4.6.6)$$

holds for all $(x, y) \in X$. Now we are done by a similar calculation as in (4.6.5), using (4.6.6) instead of (4.6.1). Each of the remaining thin cells can be treated as separate part and the statement on such a part follows again by (4.6.2) and (4.6.1). \square

Proof of Theorem 3.5 for general $m > 1$ This proof is the same as the proof of [3, Theorem 2.3] for general $m > 1$ (see p. 83 of [3]), using our improved versions of some intermediary results used in [3]. We indicate the changes to be made; these changes are related to the Lipschitz constants when applying the induction hypothesis (both for Theorem 3.5 and Proposition 4.6). On page 83 of [3], one should invoke our Proposition 4.6 for $m - 1$ instead of Proposition 2.4 of [3]. Then, in the sentence of [3, p. 83] containing the statement (*), one should note that the mentioned bi-Lipschitz transformation, which replaces x_m by $x_m - c(\hat{x}, y)$ and which preserves the other coordinates, is an isometry. To show that this map is indeed an isometry one uses the fact that the Lipschitz constant for $c(\cdot, y)$ given by Proposition 4.6 equals 1. From that statement (*) of page 83 of [3] on, one takes and keeps $C = \varepsilon$. Further, the proof is the same as the proof of Theorem 2.3 in [3]. In a nutshell, the idea in [3] is to partition the domain into cells in which one can move along lower-dimensional subsets for which the induction hypothesis can be used. \square

References

1. Cluckers, R.: Presburger sets and p -minimal fields. *J. Symb. Logic* **68**, 153–162 (2003)
2. Cluckers, R.: Analytic p -adic cell decomposition and integrals. *Trans. Am. Math. Soc.* **356**, 1489–1499 (2004)
3. Cluckers, R., Comte, G., Loeser, F.: Lipschitz continuity properties for p -adic semi-algebraic and subanalytic functions. *Geom. Funct. Anal.* **20**(1), 68–87 (2010) arXiv:0904.3853
4. Cluckers, R., Lipshitz, L.: Fields with analytic structure. *J. Eur. Math. Soc.* **13**(4), 1147–1223 (2011) arXiv:0908.2376
5. Cohen, P.J.: Decision procedures for real and p -adic fields. *Comm. Pure Appl. Math.* **22**, 131–151 (1969)
6. Denef, J.: The rationality of the Poincaré series associated to the p -adic points on a variety. *Invent. Math.* **77**, 1–23 (1984)
7. Denef, J.: p -adic semi-algebraic sets and cell decomposition. *J. Reine Angew. Math.* **369**, 154–166 (1986)
8. Denef, J., van den Dries, L.: p -adic and real subanalytic sets. *Ann. Math.* **128**, 79–138 (1988)
9. van den Dries, L., Haskell, D., Macpherson, D.: One-dimensional p -adic subanalytic sets. *J. Lond. Math. Soc.* **59**(2), 1–20 (1999)
10. Glöckner, H.: Comparison of some notions of C^k -maps in multi-variable non-archimedean analysis. *Bull. Belg. Math. Soc. Simon Stevin* **14**(5), 877–904 (2007)
11. Kurdyka, K.: On a Subanalytic Stratification Satisfying a Whitney Property with Exponent 1. *Real Algebraic Geometry (Rennes, 1991)*, pp. 316–322, *Lecture Notes in Math.*, 1524. Springer, Berlin (1992)
12. Macintyre, A.: On definable subsets of p -adic fields. *J. Symb. Logic* **41**(3), 605–610 (1976)