Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 272 (2004) 692–700
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Grothendieck rings of Laurent series fields

Raf Cluckers [1]

Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001 Leuven, Belgium

Received 22 November 2002

Communicated by Leonard Lipshitz

Abstract

We study Grothendieck rings (in the sense of model theory) of fields, extending previous work of Haskell and the author in [R. Cluckers, D. Haskell, Bull. Symbolic Logic 7 (2) (2001) 262–269]. We construct definable bijections from the line to the line minus one point in the language of rings for valued fields like fields of formal Laurent series over p-adic numbers and fields of formal Laurent series over local fields of strictly positive characteristic. It follows that the Grothendieck rings of these fields are trivial.

Keywords: Grothendieck rings; Model theory; Valued fields; Henselian rings

1. Introduction

Recently, the Grothendieck ring of a structure, in the sense of logic, has been introduced in [3] as well as in [4]. The Grothendieck ring of a model-theoretical structure is built up as a quotient of the definable sets by definable bijections (see below), and thus, depends both on the model and the language. For $(M, \mathcal{L})$ a structure with the signature of a language $\mathcal{L}$ we write $K_0(M, \mathcal{L})$ for the Grothendieck ring of $(M, \mathcal{L})$. In [2] and [7], the following explicit calculations of Grothendieck rings of fields are made:

$K_0(\mathbb{R}, \mathcal{L}_{\mathrm{ring}})$ is isomorphic to $\mathbb{Z}$,
$K_0(\mathbb{Q}_p, \mathcal{L}_{\mathrm{ring}})$ is trivial,
$K_0(\mathbb{F}_p((t)), \mathcal{L}_{\mathrm{ring}})$ is trivial,

E-mail address: raf.cluckers@wis.kuleuven.ac.be.
URL: http://www.wis.kuleuven.ac.be/algebra/raf/.
[1] Research Assistant of the Fund for Scientific Research—Flanders (Belgium) (F.W.O.).

0021-8693/$ – see front matter
doi:10.1016/S0021-8693(03)00397-1

with $\mathcal{L}_{\text{ring}}$ the language $(+, -, \cdot, 0, 1)$. In [3] and [4] it is shown that the Grothendieck ring $K_0(\mathbb{C}, \mathcal{L}_{\text{ring}})$ is extremely big and complicated; $K_0(\mathbb{C}, \mathcal{L}_{\text{ring}})$, and many other Grothendieck rings, are not explicitly known.

Any Euler characteristic (in the sense of [2] or [4]), defined on the definable sets, factors through the natural projection of definable sets into the Grothendieck ring, and, in this sense, to know a Grothendieck ring is to know a universal Euler characteristic. Nevertheless, it happens that a Grothendieck ring is trivial.

The triviality of a Grothendieck ring can be proven by constructing a definable bijection from X to $X \setminus \{a\}$, where X is a definable set and $\{a\}$ a point on X. In Section 2, we give two criteria for valued fields to have a trivial Grothendieck ring and for the existence of definable bijections $K \to K^\times = K \setminus \{0\}$, see Propositions 2 and 3. These criteria are extensions of criteria given in [2].

In this paper we define a subgroup $H(K, \mathcal{L})$ of $\mathbb{Z}$, associated to a field K and a language $\mathcal{L}$ (see Section 2); this group reflects elementary arithmetical properties of the indices of nth powers in $K^\times$ and of the number of nth roots of 1 in $K^\times$. Using the definition, it follows immediately that, for example,

$H(\mathbb{R}, \mathcal{L}_{\text{ring}})$ is $\mathbb{Z}$,
$H(\mathbb{Q}_p, \mathcal{L}_{\text{ring}})$ is $\mathbb{Z}$, and
$H(\mathbb{C}, \mathcal{L}_{\text{ring}})$ is $\{0\}$.

In Section 1.2 we explain iterated Laurent series fields. In the present paper we prove the following application of the above mentioned criteria:

Theorem 1. *Let L be either $\mathbb{Q}_p$, or a finite field extension of $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$, where $\mathbb{F}_q$ is the finite field with $q = p^l$ elements and p a prime. Let K be one of the fields L, $L((t_1))$, $L((t_1))((t_2))$, $L((t_1))((t_2))((t_3))$, and so on. Then $K_0(K, \mathcal{L}_{\text{ring}})$ is trivial and there exists a $\mathcal{L}_{\text{ring}}$-definable[2] bijection $K \to K^\times$.[3] Moreover, if* char $K \neq 2$, *then $H(K, \mathcal{L}_{\text{ring}})$ is $\mathbb{Z}$.*

1.1. Valued fields

Fix a field K. We call K a valued field if there is an ordered group $(G, +, \leqslant)$[4] and a surjective valuation map $v : K \to G \cup \{\infty\}$ such that

(i) $v(x) = \infty$ if and only if $x = 0$;
(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$;
(iii) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in K$.

We write R for the valuation ring $\{x \in K \mid v(x) \geqslant 0\}$ of K, M for its unique maximal ideal, k for the residue field R/M and $R \to k : x \mapsto \bar{x}$ for the natural projection. We call K

[2] Here, as always, definable means definable with parameters.
[3] For $K = L$, this result is proven in [2].
[4] Here, an ordered group is a totally ordered non-trivial abelian group G such that $x < y$ implies $x + z < y + z$ for all x, y, z in G.

a G-valued field. If G has a minimal strictly positive element, we call G discrete. Remark that a minimal strictly positive element in G necessarily is unique.

A valued field often carries an angular component map modulo M, (or angular component map for short); it is a group homomorphism $\mathrm{ac}: K^\times \to k^\times$, extended by putting $\mathrm{ac}(0) = 0$, and satisfying $\mathrm{ac}(x) = \bar{x}$ for all x with $v(x) = 0$ (see [6]).

If the value group of K is $\mathbb{Z}^n$, and $t_1, \ldots, t_n$ are field elements such that $v(t_1) = (1, 0, \ldots, 0), \ldots, v(t_n) = (0, \ldots, 0, 1)$, there is a natural angular component map $\mathrm{ac}: K \to k$ given by $\mathrm{ac}(x) = (x \prod_i t_i^{-r_i}) \bmod M$ for nonzero x with $v(x) = (r_1, \ldots, r_n)$. This angular component map is canonical up to the choice of t_i.

1.2. Iterated Laurent series fields

To provide the reader with examples of valued fields satisfying the conditions of several results in the paper, we define iterated Laurent series fields by induction. Let $L((t_1))$ be the field of (formal) Laurent series in the variable t_1 over L and let $L((t_1))\ldots((t_{n-1}))((t_n))$ be the field of (formal) Laurent series in the variable t_n over $L((t_1))\ldots((t_{n-1}))$. On a field $L((t_1))\ldots((t_n))$ we can put many valuations, for example the valuation v_n taking values in the lexicographically ordered n-fold product of $\mathbb{Z}$, defined as follows. If $n = 1$, then we put $v_1(x) = s \in \mathbb{Z}$ whenever $x = \sum_{i \geqslant s} a_s t_1^i$ with $a_s \neq 0$ and $a_i \in L$. For general n, and $x = \sum_{i \geqslant s} a_s t_n^i$, where $a_s \neq 0$ and $a_i \in L((t_1))\ldots((t_{n-1}))$, we put $v_n(x) = (s, v_{n-1}(a_s)) \in \mathbb{Z}^n$. Remark that the valuation ring with respect to the valuation v_n is Henselian.

1.3. Grothendieck rings

Let $\mathcal{L}$ be a language and let M be a model for $\mathcal{L}$ with at least two elements. For $\mathcal{L}$-definable sets $X \subset M^m$, $Y \subset M^n$, $m, n > 0$, a $\mathcal{L}$-definable bijection $X \to Y$ is called an $\mathcal{L}$-isomorphism and we write $X \cong_{\mathcal{L}} Y$, or $X \cong Y$ if the context is clear, whenever X and Y are $\mathcal{L}$-isomorphic. (By definable we mean definable with parameters.) For definable X and Y, we can choose disjoint definable sets $X', Y' \subset K^{m'}$ for some $m' > 0$, such that $X \cong X'$ and $Y \cong Y'$, and then we define the *disjoint union* $X \sqcup Y$ of X and Y up to isomorphism as $X' \cup Y'$. By the Grothendieck group $K_0(M, \mathcal{L})$ of the structure $(M, \mathcal{L})$ we mean the group generated by symbols $[A]$, for A a $\mathcal{L}$-definable set, with the relations $[A] = [A']$ if $A \cong_{\mathcal{L}} A'$ and $[A] = [B] + [C]$ if A is the disjoint union of B and C. The group $K_0(M, \mathcal{L})$ carries a multiplicative structure induced by $[A \times B] = [A][B]$, where $A \times B$ is the Cartesian product of definable sets. The so-obtained ring is called the Grothendieck ring and for a $\mathcal{L}$-definable set X we write $[X]$ for the image of X in $K_0(M, \mathcal{L})$.

2. Calculations of Grothendieck rings

Let K be a field and $\mathcal{L}$ an expansion of $\mathcal{L}_{\mathrm{ring}}$. We write $P_n(K)$ or P_n for the set of nth powers in $K^\times$. For $n > 1$ we put

$$r_n(K) = \sharp\{x \in K \mid x^n = 1\} \quad \text{and} \quad s_n(K) = \left[K^\times : P_n(K)\right]$$

which is either a positive integer or ∞.

Definition 1. For $n > 1$ we put

$$\lambda_n(K, \mathcal{L}) = \frac{s_n(K)}{r_n(K)}$$

if the following conditions are satisfied

(1) $s_n(K) < \infty$ and $\frac{s_n(K)}{r_n(K)} \in \mathbb{Z}$;
(2) there exists a $\mathcal{L}$-definable nth root function. This means that there exists a definable set $\sqrt[n]{P_n} \subset K^\times$ and a definable bijection

$$\sqrt[n]{\cdot} : P_n(K) \to \sqrt[n]{P_n}$$

such that $(\sqrt[n]{x})^n = x$ for each $x \in P_n(K)$.

If one of the above conditions is not satisfied, we put $\lambda_n(K, \mathcal{L}) = 1$. We define $H(K, \mathcal{L})$ as the subgroup of $\mathbb{Z}$ generated by the numbers

$$\lambda_n(K, \mathcal{L}) - 1$$

for all $n > 1$.

Remark that if $\mathcal{L}'$ is an expansion of $\mathcal{L}$, then there is a group inclusion $H(K, \mathcal{L}) \to H(K, \mathcal{L}')$. As an example, we remark that $H(\mathbb{R}((t)), \mathcal{L}_{\text{ring}}) = \mathbb{Z}$, and the same holds for fields of iterated Laurent series over $\mathbb{R}$.

Let $\mathcal{L}_{\text{v}} = (\mathcal{L}_{\text{ring}}, R)$ be the language of rings with an extra 1-ary relation symbol R which corresponds to a valuation ring inside the model. If the model is a valued field, we take the natural interpretations.

Proposition 1. *Let K be a field and $\mathcal{L}$ an expansion of $\mathcal{L}_{\text{ring}}$. For each positive number $m \in H(K, \mathcal{L})$, there exists a $\mathcal{L}$-definable bijection*

$$\bigsqcup_{i=1}^{m+1} K^\times \to K^\times,$$

and thus, in $K_0(K, \mathcal{L})$,

$$m\big[K^\times\big] = 0.$$

Moreover, if K is a valued field and $\mathcal{L}$ is an expansion of $\mathcal{L}_{\text{v}}$, there exists a $\mathcal{L}$-definable bijection

$$\bigsqcup_{i=1}^{m+1} \big(R \setminus \{0\}\big) \to R \setminus \{0\},$$

and thus, in $K_0(K,\mathcal{L})$,

$$m\big[R\setminus\{0\}\big]=0.$$

Proof. We first prove that $K^\times\cong\bigsqcup_{i=1}^{\lambda_n}K^\times$ for all $\lambda_n=\lambda_n(K,\mathcal{L})$, $n>1$. If $\lambda_n=1$ there is nothing to prove, so suppose $\lambda_n>1$. Remark that for each $x\in K^\times$ and each definable set $A\subset K$ there is a $\mathcal{L}$-isomorphism $xA\cong A$. With the notation of Definition 1, the sets $x\sqrt[n]{P_n}$ form a partition of $K^\times$ when x runs over the nth roots of unity. This gives $\bigsqcup_{i=1}^{r_n}\sqrt[n]{P_n}\cong K^\times$. Since $K^\times$ is the disjoint union of all cosets of P_n inside $K^\times$, we find $\bigsqcup_{i=1}^{s_n}P_n\cong K^\times$. Combining with the isomorphism $P_n\cong\sqrt[n]{P_n}$ we calculate:

$$K^\times\cong\bigsqcup_{i=1}^{s_n}P_n\cong\bigsqcup_{i=1}^{s_n}\sqrt[n]{P_n}\cong\bigsqcup_{i=1}^{\lambda_n}\left(\bigsqcup_{i=1}^{r_n}\sqrt[n]{P_n}\right)\cong\bigsqcup_{i=1}^{\lambda_n}K^\times,$$

where $s_n=s_n(K)$ and $r_n=r_n(K)$. Now let $m>0$ be in $H(K,\mathcal{L})$ and let $n>1$ and $s>0$ be integers. By what we just have shown, we can add λ_n-1 disjoint copies of $K^\times$ to $\bigsqcup_{i=1}^{s}K^\times$, in the sense that $\bigsqcup_{i=1}^{s}K^\times\cong\bigsqcup_{i=1}^{s+\lambda_n-1}K^\times$. Similarly, if $s>\lambda_n-1$, we can subtract λ_n-1 disjoint copies of $K^\times$ from $\bigsqcup_{i=1}^{s}K^\times$, to be precise, $\bigsqcup_{i=1}^{s}K^\times\cong\bigsqcup_{i=1}^{s-\lambda_n+1}K^\times$. The proposition follows since the numbers λ_n-1 generate $H(K,\mathcal{L})$.

If $\mathcal{L}$ is an expansion of $\mathcal{L}_{\mathrm{v}}$, we have the same isomorphisms and the same arguments for $R\setminus\{0\}$ instead of $K^\times$, working with $R\cap P_n$ and $R\cap\sqrt[n]{P_n}$ instead of P_n and $\sqrt[n]{P_n}$. $\square$

Definition 2. Let R be a valuation ring, M its maximal ideal and let $\mathrm{ac}: K\to k$ be an angular component map modulo M, where k is the residue field of R. We define the set $R^{(1)}$ as

$$R^{(1)}=\big\{x\in R\mid \mathrm{ac}(x)=1\big\}.$$

The set $R^{(1)}$ is not necessarily definable in the language $\mathcal{L}_{\mathrm{ring}}$, nevertheless, it is always definable in languages of Pas [5], since languages of Pas contain an angular component map. At any rate, if $R^{(1)}$ is definable in some language $\mathcal{L}$ we have the following criterion.

Proposition 2. *Let K be a valued field. Suppose that the value group is discrete and let $\pi\in K$ have minimal strictly positive valuation. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_{\mathrm{v}}$ and let* ac *be an angular component map $K\to k$. If $R^{(1)}$ is $\mathcal{L}$-definable and $H(K,\mathcal{L})=\mathbb{Z}$, then*

$$K_0(K,\mathcal{L})=0,\quad K\cong_{\mathcal{L}}K^\times,\quad \textit{and}\quad R\cong_{\mathcal{L}}R\setminus\{0\}.$$

Proof. We first prove that $K_0(K,\mathcal{L})=0$. We may suppose that $\mathrm{ac}(\pi)=1$, otherwise we could replace π by π/a where a is an arbitrary element with $v(a)=0$ and $\mathrm{ac}(a)=\mathrm{ac}(\pi)$. The following is a $\mathcal{L}$-isomorphism

$$R\sqcup R^{(1)}\to R^{(1)}:\begin{cases}x\in R\mapsto 1+\pi x,\\ x\in R^{(1)}\mapsto \pi x.\end{cases}$$

This implies, in $K_0(K,\mathcal{L})$, that $[R]+[R^{(1)}]=[R^{(1)}]$, and thus after cancellation, $[R]=0$. By Lemma 1 and because $H(K,\mathcal{L})=\mathbb{Z}$, also $[R\setminus\{0\}]=0$. The following calculation implies $K_0(K,\mathcal{L})=0$:

$$0=[R]=\bigl[R\setminus\{0\}\bigr]+\bigl[\{0\}\bigr]=\bigl[\{0\}\bigr]=1.$$

We have $[\{0\}]=1$ because $[\{0\}]$ is the multiplicative unit in $K_0(K,\mathcal{L})$.

Next we prove $R\cong R\setminus\{0\}$, by taking translates and applying homotheties to the occurring sets. We make all occurring disjoint unions explicit. Write f_1 for the isomorphism

$$f_1: 1+\pi^2\bigl(R\setminus\{0\}\bigr)\to\pi^2\bigl(R\setminus\{0\}\bigr)\cup 1+\pi^2\bigl(R\setminus\{0\}\bigr),$$

given by Lemma 1; it is an isomorphism from one copy of $R\setminus\{0\}$ onto two disjoint copies of $R\setminus\{0\}$. Define the function f_2 on $\pi^2R\cup\pi+\pi^2R^{(1)}$ by

$$f_2:\pi^2R\cup\pi+\pi^2R^{(1)}\to\pi+\pi^2R^{(1)}:\ \begin{cases}\pi^2x\mapsto\pi+\pi^2(1+\pi x),\\ \pi+\pi^2x\mapsto\pi+\pi^2(\pi x),\end{cases}$$

then f_2 is an isomorphism from the disjoint union of R and $R^{(1)}$ to a copy of $R^{(1)}$. Finally, we find $\mathcal{L}$-isomorphisms:

$$f:R\to R\setminus\{0\}:\ x\mapsto\begin{cases}f_1(x) & \text{if } x\in 1+\pi^2(R\setminus\{0\}),\\ f_2(x) & \text{if } x\in\pi^2R\cup\pi+\pi^2R^{(1)},\\ x & \text{else}\end{cases}$$

and

$$K\to K^\times:\ x\mapsto\begin{cases}f(x) & \text{if } x\in R,\\ x & \text{else.}\end{cases}\qquad\square$$

Proposition 2 immediately yields the triviality of the Grothendieck rings of $\mathbb{Q}_p$ and of $\mathbb{F}_q((t))$ with characteristic different from 2, see [2].

Proposition 2 can also be applied to fields of iterated Laurent series over $\mathbb{R}$, like $\mathbb{R}((t_1))\dots((t_n))$, together with for example a language of Pas [5].

In case that $H(K,\mathcal{L})$ is different from $\mathbb{Z}$, we formulate another criterion, namely Proposition 3; it is a generalization of Theorem 1 of [2] and the proof is similar to the proof of Theorem 1 of [2].

Proposition 3. *Let K be a valued field. Suppose that the value group is discrete and let $\pi\in K$ have minimal strictly positive valuation. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_{\mathrm{v}}$ and let* ac *be an angular component map $K\to k$. If $R^{(1)}$ is $\mathcal{L}$-definable, then*

$$K_0(K,\mathcal{L})=0,\quad K^2\cong_{\mathcal{L}}K^2\setminus\bigl\{(0,0)\bigr\},\quad \textit{and}\quad R^2\cong_{\mathcal{L}}R^2\setminus\bigl\{(0,0)\bigr\}.$$

Proof. We first prove that $K_0(K,\mathcal{L}) = 0$. As above we may suppose that $\mathrm{ac}(\pi) = 1$ and we have a $\mathcal{L}$-isomorphism

$$R \sqcup R^{(1)} \cong_{\mathcal{L}} R^{(1)}$$

which implies that $[R] = 0$ in $K_0(K,\mathcal{L})$.

We argument that the disjoint union of two copies of $(R \setminus \{0\})^2$ is $\mathcal{L}$-isomorphic to $(R \setminus \{0\})^2$ itself. Define the sets

$$X_1 = \{(x, y) \in (R \setminus \{0\})^2 \mid v(x) \leqslant v(y)\},$$
$$X_2 = \{(x, y) \in (R \setminus \{0\})^2 \mid v(x) > v(y)\},$$

then X_1, X_2 form a partition of $(R \setminus \{0\})^2$. The isomorphisms

$$(R \setminus \{0\})^2 \to X_1 : (x, y) \mapsto (x, xy),$$
$$(R \setminus \{0\})^2 \to X_2 : (x, y) \mapsto (\pi xy, y),$$

imply that $(R \setminus \{0\})^2 \sqcup (R \setminus \{0\})^2$ is isomorphic to $X_1 \cup X_2$ which is exactly $(R \setminus \{0\})^2$. After cancellation, it follows that $[(R \setminus \{0\})^2] = 0$.

Since $0 = [R] = [R \setminus \{0\}] + [\{0\}] = [R \setminus \{0\}] + 1$ we have $[R \setminus \{0\}] = -1$. Together with $0 = [(R \setminus \{0\})^2] = [R \setminus \{0\}]^2$ this yields $1 = 0$, so $K_0(K,\mathcal{L})$ is trivial.

Combining these isomorphisms and taking appropriate disjoint unions inside R^2, we can find an isomorphism from R^2 to itself minus a point. For details of this construction, we refer to the proof of [2, Theorem 1]. □

Proposition 3 can, for example, be applied to valued fields with angular component map of strictly positive characteristic, together with a language of Pas. Since such fields often allow a definable injection $K^2 \to K$, also a definable bijection $K \to K^\times$ can often be obtained (the proof below exhibits this technique).

The value group G of a valued field K is always torsion-free. One can also prove that if G is finitely generated, then G is isomorphic (as an ordered group) to $\mathbb{Z}^n$ with the lexicographical order, for some $n > 0$; to see this first prove that the valuation group is discrete, and then use induction on the number of generators, by modding out modulo $t\mathbb{Z}$ where t is the least strictly positive element. We prove the following generalisation of Theorem 1:

Theorem 2. *Let K be a Henselian valued field with a finitely generated value group and a finite residue field, then the valuation ring is $\mathcal{L}_{\mathrm{ring}}$-definable, the Grothendieck ring $K_0(K,\mathcal{L}_{\mathrm{ring}})$ is trivial, and there exist $\mathcal{L}_{\mathrm{ring}}$-definable bijections $K \to K^\times$ and $R \to R \setminus \{0\}$. Moreover, if* $\mathrm{char}\, K \neq 2$, *then $H(K,\mathcal{L}_{\mathrm{ring}})$ is $\mathbb{Z}$.*

Proof. We will give a proof in the case that the residue field of K is $\mathbb{F}_p$ with p a prime; the other cases are similar. By the remark above we may suppose that the valuation v on $K^\times$

takes values in $G=\mathbb{Z}^n$, $n>0$, with lexicographical order. Let $e_1,\dots,e_n$ be the $\mathbb{Z}$-module basis $(1,0,\dots,0),\dots,(0,\dots,0,1)$ of G. Let t_i be an element of K for $i=1,\dots,n$ such that $v(t_i)=e_i$. By Hensels lemma, the valuation ring R can be described by

$$R=\left\{x\in K \mid 1+t_1x^2\in P_2(K)\ \&\ 1+t_2x^2\in P_2(K)\ \&\cdots\&\ 1+t_nx^2\in P_2(K)\right\}$$

if $p\neq 2$ and by

$$R=\left\{x\in K \mid 1+t_1x^3\in P_3(K)\ \&\cdots\&\ 1+t_nx^3\in P_3(K)\right\}$$

if $p=2$. Hence, R is $\mathcal{L}_{\mathrm{ring}}$-definable. Write M for the maximal ideal of R. Let $\mathrm{ac}:K\to\mathbb{F}_p$ be the angular component defined by $\mathrm{ac}(x)=\prod_i t_i^{-r_i}x \bmod M$ for nonzero x with $v(x)=(r_1,\dots,r_n)$. The set $R^{(1)}=\{x\in R \mid \mathrm{ac}(x)=1\}$ is $\mathcal{L}_{\mathrm{ring}}$-definable since it is the union of the sets

$$\prod_i t_i^{r_i}P_{p-1}(K)$$

for $r_i=0,\dots,p-2$ and $i=1,\dots,n$.

Suppose now that $\mathrm{char}\,K\neq 2$. Using Hensel's lemma and the theory of finite fields, one can calculate the numbers $r_q(K)$ and $s_q(K)$ for each prime number q different from $\mathrm{char}\,K$, and one finds that $s_q(K)/r_q(K)$ is a positive power of q. Further, using the definable angular component and Hensels lemma, one can check that taking qth roots is $\mathcal{L}_{\mathrm{ring}}$-definable. It follows that the generator λ_2-1 of $H(K,\mathcal{L}_{\mathrm{ring}})$ is odd and λ_q-1 is even for each prime q different from $\mathrm{char}\,K$, and thus, $H(K,\mathcal{L}_{\mathrm{ring}})=\mathbb{Z}$. Now we can use Proposition 2, to find the desired $\mathcal{L}_{\mathrm{ring}}$-definable bijections and to find that $K_0(K,\mathcal{L}_{\mathrm{ring}})$ is trivial. This proves the theorem when $\mathrm{char}\,K\neq 2$.

If $\mathrm{char}\,K=2$, we use Proposition 3 to find that $K_0(K,\mathcal{L}_{\mathrm{ring}})$ is trivial and to find a $\mathcal{L}_{\mathrm{ring}}$-definable bijection

$$g_1:R^2\to R^2\setminus\left\{(0,0)\right\}.$$

The following is $\mathcal{L}_{\mathrm{ring}}$-definable:

$$g_2:R^2\to R:\ (x,y)\mapsto x^2+t_ny^2.$$

Moreover, g_2 is injective because it is a group homomorphism with trivial kernel, and hence, we can define the $\mathcal{L}_{\mathrm{ring}}$-isomorphism

$$g:R\to R^\times:\ x\mapsto\begin{cases} g_2g_1(g_2^{-1}(x)) & \text{if } x\in g_2(R^2),\\ x & \text{else.}\end{cases}$$

This finishes the proof. $\square$

Remark 1. In [1] it is proven that for any infinite $\mathcal{L}_{\mathrm{ring}}$-definable subset X of $\mathbb{Q}_p^n$ there is a $\mathcal{L}_{\mathrm{ring}}$-definable bijection $X \to \mathbb{Q}_p^l$ with l the dimension of X, and similarly for finite field extensions of $\mathbb{Q}_p$. The analogue questions are open for the other fields in the statement of Theorems 1 and 2.

References

[1] R. Cluckers, Classification of semi-algebraic p-adic sets up to semi-algebraic bijection, J. Reine Angew. Math. 540 (2001) 105–114.
[2] R. Cluckers, D. Haskell, Grothendieck rings of $\mathbb{Z}$-valued fields, Bull. Symbolic Logic 7 (2) (2001) 262–269.
[3] J. Denef, F. Loeser, Definable sets, motives and p-adic integrals, J. Amer. Math. Soc. 14 (2) (2001) 429–469.
[4] J. Krajíček, T. Scanlon, Combinatorics with definable sets: Euler characteristics and Grothendieck rings, Bull. Symbolic Logic 6 (2000) 311–330.
[5] J. Pas, Uniform p-adic cell decomposition and local zeta functions, J. Reine Angew. Math. 399 (1989) 137–172.
[6] J. Pas, On the angular component map modulo p, J. Symbolic Logic 55 (1990) 1125–1129.
[7] L. van den Dries, Tame Topology and o-Minimal Structures, in: Lecture Notes Ser., Vol. 248, Cambridge Univ. Press, Cambridge, UK, 1998.