IGUSA AND DENEF-SPERBER CONJECTURES ON NONDEGENERATE p-ADIC EXPONENTIAL SUMS

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Abstract

We prove the intersection of Igusa's conjecture in [10, Introduction] and the Denef-Sperber conjecture of [7, page 56] on nondegenerate exponential sums modulo p^m.

1. Introduction

Let f be a polynomial over $\mathbf{Z}$ in n variables. Consider the "global" exponential sum

$$S_f(N) := \frac{1}{N^n} \sum_{x \in \{0,\ldots,N-1\}^n} \exp\Big(2\pi i \frac{f(x)}{N}\Big),$$

where N varies over the positive integers. In order to bound $|S_f(N)|$ in terms of N, it is enough to bound

$$|S_f(p^m)|$$

in terms of $m > 0$ and prime numbers p. When f is nondegenerate in several senses related to its Newton polyhedron, specific bounds that depend uniformly on m and p have been conjectured by Igusa and by Denef and Sperber.

We prove these bounds, thus solving a conjecture by Denef and Sperber from a 1990 manuscript [8] (published in 2001 as [7]) and the nondegenerate case of Igusa's conjecture for exponential sums from [10, Introduction].

One of the main points in the study of global exponential sums is that while for finite field exponential sums like $S_f(p)$ one knows that the weights and Betti numbers have some uniform behavior for big p, for p-adic exponential sums $S_f(p^m)$ one does not yet completely know what the analogues of the weights and Betti numbers are, let alone that they have some uniform behavior in p.

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Notation. Let f be a nonconstant polynomial over $\mathbf{Z}$ in n variables with $f(0) = 0$.* Write $f(x) = \sum_{i \in \mathbf{N}^n} a_i x^i$ with $a_i \in \mathbf{Z}$. The *global Newton polyhedron* $\Delta_{\text{global}}(f)$ of f is the convex hull of the support $\text{Supp}(f)$ of f with

$$\text{Supp}(f) := \{i \mid i \in \mathbf{N}^n,\ a_i \neq 0\}.$$

The *Newton polyhedron* $\Delta_0(f)$ *of* f *at the origin* is

$$\Delta_0(f) := \Delta_{\text{global}}(f) + \mathbf{R}_+^n$$

with $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$ and $A + B = \{a + b \mid a \in A,\ b \in B\}$ for $A, B \subset \mathbf{R}^n$. For a subset I of $\mathbf{R}^n$, define

$$f_I(x) := \sum_{i \in I \cap \mathbf{N}^n} a_i x^i.$$

By the *faces* of I, we mean I itself and each nonempty convex set of the form

$$\big\{x \in I \,\big|\, L(x) = 0\big\},$$

where $L(x) = a_0 + \sum_{i=1}^{n} a_i x_i$ with $a_i \in \mathbf{R}$ is such that $L(x) \geq 0$ for each $x \in I$. By the *proper faces* of I, we mean the faces of I which are different from I. For $\mathscr{I}$ a collection of subsets of $\mathbf{R}^n$, call f *nondegenerate with respect to* $\mathscr{I}$ when f_I has no critical points on $(\mathbf{C}^\times)^n$ for each I in $\mathscr{I}$, where $\mathbf{C}^\times = \mathbf{C} \setminus \{0\}$. For $k \in \mathbf{R}_+^n$, put

$$\nu(k) = k_1 + k_2 + \cdots + k_n,$$

$$N(f)(k) = \min_{i \in \Delta_0(f)} k \cdot i,$$

$$F(f)(k) = \big\{i \in \Delta_0(f) \,\big|\, k \cdot i = N(f)(k)\big\},$$

where $k \cdot i$ is the standard inproduct on $\mathbf{R}^n$. Denote by $F_0(f)$ the smallest face of $\Delta_0(f)$ which has nonempty intersection with the diagonal $\{(t, \ldots, t) \mid t \in \mathbf{R}\}$, and let $(1/\sigma(f), \ldots, 1/\sigma(f))$ be the intersection point. If there is no confusion, we write σ instead of $\sigma(f)$, $N(k)$ instead of $N(f)(k)$, and $F(k)$ for $F(f)(k)$. Let κ be the codimension of $F_0(f)$ in $\mathbf{R}^n$.

2. The main results

From here up to Section 9, let f be a nonconstant polynomial over $\mathbf{Z}$ in n variables with $f(0) = 0$. (The adaptation to the case where $f(0) \neq 0$ is easy.)

*When $f(0) \neq 0$, then there is no harm in replacing f by $f - f(0)$: all corresponding changes in the article are easily made.

THEOREM 2.1
Suppose that f *is homogeneous and nondegenerate with respect to the faces of* $\Delta_0(f)$. *Then there exist* $c > 0$ *and* $M > 0$ *such that*

$$|S_f(p^m)| \leq c\, p^{-\sigma m}\, m^{\kappa-1}$$

for all primes $p > M$ *and all integers* $m > 0$ *with* $\sigma = \sigma(f)$ *and* κ *as in our notation. Moreover,* c *can be taken depending on* $\Delta_0(f)$ *only.*

One sees that the dependence on p and m is very simple. Moreover, since for each p there exists $c_p > 0$ such that for all $m > 0$,

$$|S_f(p^m)| \leq c_p p^{-\sigma m} m^{n-1}$$

by [10], [9], or [3] and properties of toric resolutions, and since $\kappa \leq n$, one finds the following.

COROLLARY 2.2
With f *as in Theorem 2.1, there exists* $c > 0$ *such that for all primes* p *and all integers* $m > 0$,

$$|S_f(p^m)| \leq cp^{-\sigma m} m^{n-1}.$$

Denef and Sperber [7, proof of Theorem 1.2] prove Theorem 2.1 under the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$. Corollary 2.2 is the nondegenerate case of Igusa's conjecture in [10] on exponential sums for toric resolutions of f. The exponent σ in the bounds of Theorem 2.1 is conjectured to be optimal for infinitely many p and m by Denef and Sargos [5], [6] in analogy to conjectures on the real case. When no vertex of $F_0(f)$ belongs to $\{0, 1, 2\}^n$, the bounds in Theorem 2.1 are shown to be optimal for infinitely many p and m in [7, Theorem 1.3].

3. Denef-Sperber formula for $S_f(p^m)$ for big p

The following proposition has the same proof as that of [7, Proposition 2.1], but it is slightly more general. We give the proof for the convenience of the reader.

PROPOSITION 3.1
Suppose that f *is nondegenerate with respect to (all) the faces of* $\Delta_0(f)$. *Then there exists* $M > 0$ *such that*

$$S_f(p^m) = (1 - p^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} \big(A(p, m, \tau) + E(p, f_\tau)B(p, m, \tau)\big) \qquad (3.1.1)$$

for all primes $p > M$ and all integers $m > 0$ with

$$A(p, m, \tau) := \sum_{\substack{k \in \mathbf{N}^n \\ F(f)(k)=\tau \\ N(f)(k) \geq m}} p^{-\nu(k)},$$

$$B(p, m, \tau) := \sum_{\substack{k \in \mathbf{N}^n \\ F(f)(k)=\tau \\ N(f)(k)=m-1}} p^{-\nu(k)},$$

and

$$E(p, f_\tau) := \frac{1}{(p-1)^n} \sum_{x \in \{1,\ldots,p-1\}^n} \exp\Big(\frac{2\pi i}{p} f_\tau(x)\Big). \tag{3.1.2}$$

Proof
Writing

$$S_f(p^m) = \int_{\mathbf{Z}_p} \exp\Big(\frac{2\pi i}{p^m} f(x)\Big)\, |dx|$$

with $|dx|$ the normalized Haar measure on $\mathbf{Q}_p^n$, we deduce

$$S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{\substack{k \in \mathbf{N}^n \\ F(f)(k)=\tau}} \int_{\operatorname{ord} x = k} \exp\Big(\frac{2\pi i}{p^m} f(x)\Big)\, |dx|.$$

Put $x_j = p^{k_j} u_j$ for $k \in \mathbf{N}^n$. Then $|dx| = p^{-\nu(k)}|du|$, and

$$f(x) = p^{N(k)}\big(f_{F(k)}(u) + p(\cdots)\big),$$

where the dots take values in $\mathbf{Z}_p$ and $N(k) = N(f)(k)$. Hence

$$S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{\substack{k \in \mathbf{N}^n \\ F(f)(k)=\tau}} p^{-\nu(k)} \int_{u \in (\mathbf{Z}_p^\times)^n} \exp\Big(\frac{2\pi i}{p^{m-N(k)}} \big(f_\tau(u) + p(\cdots)\big)\Big)\, |du|, \tag{3.1.3}$$

where $\mathbf{Z}_p^\times$ denotes the group of p-adic units. Because of the nondegeneratedness assumptions, for τ a face of $\Delta_0(f)$ and p a big enough prime the reduction $f_\tau \bmod p$ has no critical points on $(\mathbf{F}_p^\times)^n$. Hence the integral in (3.1.3) is zero whenever $m - N(f)(k) \geq 2$. When $m - N(k) \leq 0$, the integral over $(\mathbf{Z}_p^\times)^n$ in (3.1.3) is just the measure of $(\mathbf{Z}_p^\times)^n$ and thus equals $(1 - p^{-1})^n$. When $m - N(k) = 1$, the integral over $(\mathbf{Z}_p^\times)^n$ in (3.1.3) equals $p^{-n}(p-1)^n E(p, f_\tau)$. The proposition now follows from (3.1.3). □

4. Lower bounds for $\nu(k)$

The main result of this section is the following.

THEOREM 4.1

Let τ be a face of $\Delta_0(f)$. Then for all k with $F(f)(k) = \tau$, one has

$$\nu(k) \geq \sigma(f)\bigl(N(f)(k) + 1\bigr) - \sigma(f_\tau), \tag{4.1.1}$$

where $\sigma(f)$ and $\sigma(f_\tau)$, as well as $\nu(k)$ and $N(f)(k)$, are as in our notation.

The main points are that one subtracts $\sigma(f_\tau)$ instead of $\sigma(f)$ and that $\sigma(f_\tau) \leq \sigma(f)$. Subtracting $\sigma(f)$ yields trivial bounds since one has $\nu(k) \geq \sigma(f)N(k)$ for all $k \in \mathbf{R}^n_+$. The theorem's proof is based on two facts.

LEMMA 4.2

Let τ be a face of $\Delta_0(f)$, and let $R_j \in \mathbf{R}^n$ be finitely many points belonging to τ. Let $\beta_j \geq 0$ satisfy

$$\sum_j \beta_j R_j \leq \Bigl(\frac{1}{\sigma}, \dots, \frac{1}{\sigma}\Bigr),$$

where $a \leq b$ for $a, b \in \mathbf{R}^n$ means $a_i \leq b_i$ for all i. Then

$$\sum \beta_j \leq 1.$$

Proof

Clearly, there is no point S in the interior of $\Delta_0(f)$ which satisfies $S \leq (1/\sigma, \dots, 1/\sigma)$. When $\sum_j \beta_j > 1$, then $\sum_j \beta_j R_j$ lies in the interior of $\Delta_0(f)$. □

COROLLARY 4.3

Let τ, R_j, and β_j be as in Lemma 4.2. Then

$$\sum \beta_j \leq \frac{\sigma(f_\tau)}{\sigma}. \tag{4.3.1}$$

Proof

Since $\sum_j \beta_j R_j \leq (1/\sigma, \dots, 1/\sigma)$, one has

$$\frac{\sigma}{\sigma(f_\tau)} \sum_j \beta_j R_j \leq \Bigl(\frac{1}{\sigma(f_\tau)}, \dots, \frac{1}{\sigma(f_\tau)}\Bigr).$$

Lemma 4.2 thus implies that $\bigl(\sigma/(\sigma(f_\tau))\bigr) \sum_j \beta_j \leq 1$. □

Proof of Theorem 4.1
Since $(1/\sigma, \dots, 1/\sigma)$ lies in the interior of $F_0(f)$, by convexity one can write

$$\Big(\frac{1}{\sigma}, \dots, \frac{1}{\sigma}\Big) = \sum_i \alpha_i P_i + \sum_j \beta_j R_j$$

for some $\alpha_i \geq 0$ and $\beta_j \geq 0$ with $\sum_i \alpha_i + \sum_j \beta_j = 1$, P_i finitely many integral points of $F_0(f) \setminus \tau$, and R_j finitely many integral points of τ. For $k \in \mathbf{N}^k$ with $F(f)(k) = \tau$, calculate

$$\nu(k) = \sigma\Big(\frac{1}{\sigma}, \dots, \frac{1}{\sigma}\Big) \cdot k \tag{4.3.2}$$

$$= \sigma\Big(\sum_i \alpha_i P_i + \sum_j \beta_j R_j\Big) \cdot k \tag{4.3.3}$$

$$= \sigma\Big(\sum_i \alpha_i P_i \cdot k + \sum_j \beta_j R_j \cdot k\Big) \tag{4.3.4}$$

$$\geq \sigma\Big(\sum_i \alpha_i \big(N(k)+1\big) + \sum_j \beta_j N(k)\Big) \tag{4.3.5}$$

$$= \sigma\Big(\Big(\sum_i \alpha_i + \sum_j \beta_j\Big)\big(N(k)+1\big) - \sum_j \beta_j\Big) \tag{4.3.6}$$

$$= \sigma\Big(\big(N(k)+1\big) - \sum_j \beta_j\Big) \tag{4.3.7}$$

$$\geq \sigma\Big(\big(N(k)+1\big) - \frac{\sigma(f_\tau)}{\sigma}\Big), \tag{4.3.8}$$

where (4.3.5) follows from $k \cdot R_j = N(k)$ and $k \cdot P_i \geq N(k) + 1$, which is true by definition of $N(k)$, and where (4.3.8) follows from Corollary 4.3. □

5. Upper bounds for $A(p, m, \tau)$ and $B(p, m, \tau)$

We recall one result from [7].

LEMMA 5.1 ([7, Lemma 3.3])
Let C be a convex polyhedral cone in $\mathbf{R}^n_+$ generated by vectors in $\mathbf{N}^n$, and let L be a linear form in n variables with coefficients in $\mathbf{N}$. We denote by C^{int} the interior of C in the sense of Newton polyhedra. Let $\sigma > 0$ and $\gamma \geq 0$ be real numbers satisfying

$$\nu(k) \geq L(k)\sigma + \gamma \quad \textit{for all } k \in C^{\text{int}} \cap \mathbf{N}^n. \tag{5.1.1}$$

Put

$$e = \dim\bigl\{k \in C \bigm| \nu(k) = L(k)\sigma\bigr\}.$$

Then there exists a real number $c > 0$ such that for all $m \in \mathbf{N}$ and for all $p \in \mathbf{R}$ with $p \geq 2$,

$$\sum_{\substack{k \in C^{\mathrm{int}} \cap \mathbf{N}^n \\ L(k)=m}} p^{-\nu(k)} \leq cp^{-m\sigma-\gamma}(m+1)^{\max(0,e-1)}. \tag{5.1.2}$$

From this lemma and from Theorem 4.1, we have the following.

COROLLARY 5.2
Let f, $A(p,m,\tau)$, and $B(p,m,\tau)$ be as in Proposition 3.1. Then there exists a real number $c > 0$ such that for all integers $m > 0$, for all faces τ of $\Delta_0(f)$, and for all big enough primes p,

$$A(p,m,\tau) \leq cp^{-m\sigma} m^{\kappa-1} \tag{5.2.1}$$

and

$$B(p,m,\tau) \leq cp^{-m\sigma+\sigma(f_\tau)} m^{\kappa-1}. \tag{5.2.2}$$

Proof
To derive (5.2.1) from Lemma 5.1, note that $\nu(k) \geq N(k)\sigma$ for any $k \in \mathbf{N}^n$ and that $\kappa = \dim\{k \in \mathbf{R}_+^n \mid \nu(k) = N(k)\sigma\} \geq 1$.

To derive (5.2.2) from Lemma 5.1 and Theorem 4.1, use for C the topological closure of the convex hull of $\{0\} \cup \{k \in \mathbf{N}^n \mid F(k) = \tau\}$, and note that $C^{\mathrm{int}} \cap \mathbf{N}^n = \{k \in \mathbf{N}^n \mid F(k) = \tau\}$. Clearly, $\kappa \geq 1$ and $\kappa \geq \dim\{k \in C \mid \nu(k) = N(k)\sigma\}$. By (4.1.1), $\nu(k) \geq N(k)\sigma + \sigma - \sigma(f_\tau)$ for all $k \in C^{\mathrm{int}} \cap \mathbf{N}^n$. □

6. Upper bounds for $\sigma(f)$ and $E(p, f_\tau)$

By Katz [11, Theorem 4], for $f(x)$ a nonconstant homogeneous polynomial in n variables over $\mathbf{Z}$ and d the dimension of grad $f = 0$ in $\mathbf{A}^n_{\mathbf{C}}$, there exists c such that for all big enough p, one has

$$\Bigl| \sum_{x \in \mathbf{A}^n(\mathbf{F}_p)} \exp\Bigl(\frac{2\pi i}{p} f(x)\Bigr)\Bigr| \leq cp^{(n+d)/2}. \tag{6.0.3}$$

Moreover, c can be taken depending on the degree of f only. This implies the following.

COROLLARY 6.1
Suppose that $f(x)$ *is homogeneous of degree at least* 2*, and let* d *be the dimension of* grad $f = 0$ *in* $\mathbf{A}^n_{\mathbf{C}}$. *Then there exists* c *such that for all* p *big enough,*

$$\Big| \sum_{x \in \mathbf{G}_m^n(\mathbf{F}_p)} \exp\Big(\frac{2\pi i}{p} f(x)\Big)\Big| < cp^{(n+d)/2}, \tag{6.1.1}$$

and hence, for some c' *one has, for all big enough* p,

$$|E(p, f)| < c' p^{(-n+d)/2} \tag{6.1.2}$$

with $E(p, f)$ *as defined by (3.1.2). Moreover,* c *and* c' *can be taken depending on* $\Delta_0(f)$ *only.*

Proof
Let $f_0(x_2, \dots, x_n)$ be the polynomial $f(0, x_2, \dots, x_n)$. Clearly, f_0 is homogeneous in $n-1$ variables. By Katz's result (6.0.3), it is enough to show that $n-1+d(f_0) \leq n+d$ with $d(f_0)$ the dimension of grad $f_0 = 0$ in $\mathbf{A}^{n-1}_{\mathbf{C}}$. This inequality follows from writing

$$f(x) = x_1 g(x) + f_0(x_2, \dots, x_n)$$

with g a polynomial in x and from comparing grad f with grad f_0. □

6.2
Let $\{(N_i, \nu_i)\}_{i \in I}$ be the numerical data of a resolution h of f with normal crossings (i.e., if $\pi_f : Y \to \mathbf{A}^n_{\mathbf{C}}$ is an embedded resolution of singularities with normal crossings of $f = 0$, then for each irreducible component E_i of $\pi_f^{-1} \circ f^{-1}(0)$, $i \in I$, let N_i be the multiplicity of E_i in div$(f \circ \pi_f)$, and let $\nu_i - 1$ be the multiplicity of E_i in the divisor associated to $\pi_f^*(dx_1 \wedge \cdots \wedge dx_n)$; see [3]). The *essential numerical data* of π_f are the pairs (N_i, ν_i) for $i \in J$ with $J = I \setminus I'$, where I' is the set of indices i in I such that $(N_i, \nu_i) = (1, 1)$ and such that E_i does not intersect another E_j with $(N_j, \nu_j) = (1, 1)$. Define $\alpha(\pi_f)$ as

$$\alpha(\pi_f) = -\min_{i \in J} \frac{\nu_i}{N_i} \tag{6.2.1}$$

when J is nonempty, and define $\alpha(\pi_f)$ as $-2n$ otherwise.

It follows from [2, Theorem 5.1, Corollary 3.4] that

$$\alpha(\pi_f) \geq \frac{-n+d}{2} \tag{6.2.2}$$

with d the dimension of grad $f = 0$ in $\mathbf{A}^n_{\mathbf{C}}$, where the empty scheme has dimension $-\infty$.

LEMMA 6.3
Let f be homogeneous of degree at least 2 and nondegenerate with respect to the faces of $\Delta_0(f)$. Let d be the dimension of grad $f = 0$ *in $\mathbf{A}^n_{\mathbf{C}}$. Then*

$$\sigma(f) \leq \frac{n-d}{2}. \tag{6.3.1}$$

Proof
By properties of a toric resolution π_f of $f = 0$, one has

$$\sigma(f) = -\alpha(\pi_f)$$

with $\alpha(\pi_f)$ as defined by (6.2.1). Now, use (6.2.2). □

From (6.3.1) and Corollary 6.1 applied to f_τ, we have the following.

COROLLARY 6.4
Let f be a homogeneous polynomial of degree at least 2 which is nondegenerate with respect to the faces of $\Delta_0(f)$. Then there exists c such that for all faces τ of $\Delta_0(f)$ and all big enough primes p,

$$|E(p, f_\tau)| < cp^{-\sigma(f_\tau)} \tag{6.4.1}$$

with $E(p, f_\tau)$ as defined by (3.1.2). Moreover, c can be taken depending on $\Delta_0(f)$ only.

7. Proof of the main theorem

Proof of Theorem 2.1
When the degree of f is at least 2, use Proposition 3.1, Corollary 5.2, and (6.4.1). For linear f, the theorem is trivial. □

8. Comparison with the Denef-Sperber approach

As mentioned in the paragraph following Corollary 2.2, Denef and Sperber [7] prove Theorem 2.1 under the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$. Key points in our proof of Theorem 2.1 are (4.1.1) (which implies Corollary 5.2) and (6.4.1). Instead of (4.1.1), Denef and Sperber use their result that for similar k as in (4.1.1) but assuming the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$,

$$\nu(k) \geq \sigma(f)\big(N(f)(k) + 1\big) - \frac{\dim \tau + 1}{2}. \tag{8.0.2}$$

This often fails if one omits the extra condition (see Examples (1), (2)). Instead of (6.4.1), they use the bounds of Adolphson and Sperber [1] and Denef and Loeser [4],

$$|E(p, f_\tau)| < cp^{(-\dim \tau - 1)/2}, \tag{8.0.3}$$

which hold (in particular) under the same conditions as for (6.4.1) but which are sometimes not as good as the bounds (6.4.1).*

We give two examples where our methods really make a difference with (8.0.2) and (8.0.3).

Examples

(1) First, for $f(x, y, z, u) = xy + zu$ and $\tau = F_0(f)$, one has $\dim \tau = 1$, $\sigma(f_\tau) = \sigma = 2$; (8.0.2) does not hold, and (8.0.3) is not optimal, while (6.4.1) yields the optimal $|E(p, f_\tau)| < cp^{-2}$.

(2) Second, for $f(x, y, z, u) = xy + zu + xz + ayu$ with $a \in \mathbf{Z}$, $a \neq 1$, and $\tau = F_0(f)$, one has $\dim \tau = 2$, $\sigma(f_\tau) = \sigma = 2$; (8.0.2) does not hold, and (8.0.3) is not optimal, while (6.4.1) yields again the optimal $|E(p, f_\tau)| < cp^{-2}$ for big p. In this example, $E(p, f_\tau)$ can be calculated by performing a transformation on $\mathbf{G}_m^4$ coming from an element of $\mathrm{GL}_4(\mathbb{Z})$ transforming $f(x)$ into $f(x', y', z', u') = x' + y' + z' + ax'y'z'^{-1}$; the bounds for $E(p, f_\tau)$ are surprisingly sharp compared, for example, to bounds for the resembling Kloosterman sums.

9. Analogues over finite extensions of $\mathbf{Q}_p$ and over $\mathbf{F}_q((t))$

For any nonarchimedean local field K with valuation ring $\mathcal{O}_K$, write ψ_K for an additive character

$$\psi_K : K \to \mathbf{C}^\times$$

which is trivial on $\mathcal{O}_K$ but nontrivial on some element of K of order -1. Write $\mathrm{ord}_K : K^\times \to \mathbf{Z}$ for the valuation, write $|\cdot|_K : K \to \mathbf{R}$ for the norm on K, and write $\bar{K}$ for its residue field with q_K elements. Let k be a number field with ring of integers $\mathcal{O}_k$. In this section, f is a nonconstant polynomial over $\mathcal{O}_k[1/N]$ in n variables with $f(0) = 0$ and $N \in \mathbf{Z}$. For K any nonarchimedean local field that is an algebra over $\mathcal{O}_k[1/N]$ and for $y \in K^\times$, consider the exponential integral

$$S_{f,K}(y) := \int_{\mathcal{O}_K^n} \psi_K\big(yf(x)\big)\,|dx|_K$$

with $|dx|_K$ the normalized Haar measure on K^n. Note that K may be of positive characteristic. Then the following generalization of Theorem 2.1 holds.

*However, (8.0.3) is sometimes sharper than (6.4.1) in cases where it does not matter for our course.

THEOREM 9.1

Suppose that f is a homogeneous polynomial over $\mathcal{O}_k[1/N]$ which is nondegenerate with respect to the faces of $\Delta_0(f)$. Then there exist $c > 0$ and $M > N$ such that

$$|S_{f,K}(y)| \leq c\,|y|_K^{-\sigma}\,|\mathrm{ord}_K(y)|^{\kappa-1}$$

for all nonarchimedean local fields K that are algebras over $\mathcal{O}_k[1/N]$ and have residue characteristic greater than M and all $y \in K^\times$ with $\mathrm{ord}_K(y) < 0$, with $|\cdot|$ the absolute value on $\mathbf{R}$. Moreover, c can be taken depending on $\Delta_0(f)$ only.

Proof

This is the same proof as that of Theorem 2.1, but using Proposition 9.2 instead of Proposition 3.1. □

PROPOSITION 9.2

Suppose that f is nondegenerate with respect to (all) the faces of $\Delta_0(f)$. Then there exists $M > N$ such that

$$S_{f,K}(y) = (1 - q_K^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} \big(A(q_K, m, \tau) + E(\bar{K}, \tau, y)B(q_K, m, \tau)\big)$$

*for all nonarchimedean local fields K that are algebras over $\mathcal{O}_k[1/N]$ and have residue characteristic greater than M and all $y \in K^\times$ with $\mathrm{ord}_K(y) \leq 0$.**

In these formulas, $A(q_K, m, \tau)$ and $B(q_K, m, \tau)$ are as in Proposition 3.1, and

$$E(\bar{K}, \tau, y) := \frac{1}{(q_K - 1)^n} \sum_{u \in \mathbf{G}_m^n(\bar{K})} \psi_y\big(f_\tau(u)\big) \tag{9.2.1}$$

with ψ_y a nontrivial additive character on $\bar{K}$ depending on y and ψ_K.†

Proof

This is the same proof as that of Proposition 3.1. □

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*For such K and for $y \in K^\times$ with $\mathrm{ord}_K(y) \geq 0$, one has $S_{f,K}(y) = 1$.

†In fact, the character ψ_y depends only on ψ_K and on $\overline{\mathrm{ac}}(y)$ for any multiplicative homomorphism $\overline{\mathrm{ac}} : K^\times \to \bar{K}^\times$ extending the natural projection $\mathcal{O}_K^\times \to \bar{K}^\times$.

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