Abstract
We prove the intersection of Igusa’s conjecture in [10, Introduction] and the Denef-Sperber conjecture of [7, page 56] on nondegenerate exponential sums modulo $p^m$.

1. Introduction
Let $f$ be a polynomial over $\mathbb{Z}$ in $n$ variables. Consider the “global” exponential sum

$$S_f(N) := \frac{1}{N^n} \sum_{x \in \{0, \ldots, N-1\}^n} \exp\left(2\pi i \frac{f(x)}{N}\right),$$

where $N$ varies over the positive integers. In order to bound $|S_f(N)|$ in terms of $N$, it is enough to bound

$$|S_f(p^m)|$$
in terms of $m > 0$ and prime numbers $p$. When $f$ is nondegenerate in several senses related to its Newton polyhedron, specific bounds that depend uniformly on $m$ and $p$ have been conjectured by Igusa and by Denef and Sperber.

We prove these bounds, thus solving a conjecture by Denef and Sperber from a 1990 manuscript [8] (published in 2001 as [7]) and the nondegenerate case of Igusa’s conjecture for exponential sums from [10, Introduction].

One of the main points in the study of global exponential sums is that while for finite field exponential sums like $S_f(p)$ one knows that the weights and Betti numbers have some uniform behavior for big $p$, for $p$-adic exponential sums $S_f(p^m)$ one does not yet completely know what the analogues of the weights and Betti numbers are, let alone that they have some uniform behavior in $p$. 
Notation. Let $f$ be a nonconstant polynomial over $\mathbb{Z}$ in $n$ variables with $f(0) = 0$. Write $f(x) = \sum_{i \in \mathbb{N}^n} a_i x^i$ with $a_i \in \mathbb{Z}$. The global Newton polyhedron $\Delta_{\text{global}}(f)$ of $f$ is the convex hull of the support $\text{Supp}(f)$ of $f$ with

$$\text{Supp}(f) := \{ i \mid i \in \mathbb{N}^n, a_i \neq 0 \}.$$

The Newton polyhedron $\Delta_0(f)$ of $f$ at the origin is

$$\Delta_0(f) := \Delta_{\text{global}}(f) + \mathbb{R}_+^n$$

with $\mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ and $A + B = \{ a + b \mid a \in A, b \in B \}$ for $A, B \subset \mathbb{R}^n$.

For a subset $I$ of $\mathbb{R}^n$, define

$$f_I(x) := \sum_{i \in I \cap \mathbb{N}^n} a_i x^i.$$

By the faces of $I$, we mean $I$ itself and each nonempty convex set of the form

$$\{ x \in I \mid L(x) = 0 \},$$

where $L(x) = a_0 + \sum_{i=1}^n a_i x_i$ with $a_i \in \mathbb{R}$ is such that $L(x) \geq 0$ for each $x \in I$. By the proper faces of $I$, we mean the faces of $I$ which are different from $I$. For $\mathcal{I}$ a collection of subsets of $\mathbb{R}^n$, call $f$ nondegenerate with respect to $\mathcal{I}$ when $f_I$ has no critical points on $(\mathbb{C}^\times)^n$ for each $I$ in $\mathcal{I}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. For $k \in \mathbb{R}_+^n$, put

$$\nu(k) = k_1 + k_2 + \cdots + k_n,$$

$$N(f)(k) = \min_{i \in \Delta_0(f)} k \cdot i,$$

$$F(f)(k) = \{ i \in \Delta_0(f) \mid k \cdot i = N(f)(k) \},$$

where $k \cdot i$ is the standard inproduct on $\mathbb{R}^n$. Denote by $F_0(f)$ the smallest face of $\Delta_0(f)$ which has nonempty intersection with the diagonal $\{(t, \ldots, t) \mid t \in \mathbb{R}\}$, and let $(1/\sigma(f), \ldots, 1/\sigma(f))$ be the intersection point. If there is no confusion, we write $\sigma$ instead of $\sigma(f)$, $N(k)$ instead of $N(f)(k)$, and $F(k)$ for $F(f)(k)$. Let $\kappa$ be the codimension of $F_0(f)$ in $\mathbb{R}^n$.

2. The main results

From here up to Section 9, let $f$ be a nonconstant polynomial over $\mathbb{Z}$ in $n$ variables with $f(0) = 0$. (The adaptation to the case where $f(0) \neq 0$ is easy.)

*When $f(0) \neq 0$, then there is no harm in replacing $f$ by $f - f(0)$: all corresponding changes in the article are easily made.
**Theorem 2.1**
Suppose that $f$ is homogeneous and nondegenerate with respect to the faces of $\Delta_0(f)$. Then there exist $c > 0$ and $M > 0$ such that

$$|S_f(p^m)| \leq c \cdot p^{-\sigma m} \cdot m^{\kappa - 1}$$

for all primes $p > M$ and all integers $m > 0$ with $\sigma = \sigma(f)$ and $\kappa$ as in our notation. Moreover, $c$ can be taken depending only on $\Delta_0(f)$.

One sees that the dependence on $p$ and $m$ is very simple. Moreover, since for each $p$ there exists $c_p > 0$ such that for all $m > 0$,

$$|S_f(p^m)| \leq c_p \cdot p^{-\sigma m} \cdot m^{n-1}$$

by [10], [9], or [3] and properties of toric resolutions, and since $\kappa \leq n$, one finds the following.

**Corollary 2.2**
With $f$ as in Theorem 2.1, there exists $c > 0$ such that for all primes $p$ and all integers $m > 0$,

$$|S_f(p^m)| \leq c \cdot p^{-\sigma m} \cdot m^{n-1}.$$

Denef and Sperber [7, proof of Theorem 1.2] prove Theorem 2.1 under the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$. Corollary 2.2 is the nondegenerate case of Igusa’s conjecture in [10] on exponential sums for toric resolutions of $f$. The exponent $\sigma$ in the bounds of Theorem 2.1 is conjectured to be optimal for infinitely many $p$ and $m$ by Denef and Sargos [5], [6] in analogy to conjectures on the real case. When no vertex of $F_0(f)$ belongs to $\{0, 1, 2\}^n$, the bounds in Theorem 2.1 are shown to be optimal for infinitely many $p$ and $m$ in [7, Theorem 1.3].

**3. Denef-Sperber formula for $S_f(p^m)$ for big $p$**
The following proposition has the same proof as that of [7, Proposition 2.1], but it is slightly more general. We give the proof for the convenience of the reader.

**Proposition 3.1**
Suppose that $f$ is nondegenerate with respect to (all) the faces of $\Delta_0(f)$. Then there exists $M > 0$ such that

$$S_f(p^m) = (1 - p^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} (A(p, m, \tau) + E(p, f_\tau)B(p, m, \tau)) \quad (3.1.1)$$
for all primes $p > M$ and all integers $m > 0$ with

$$A(p, m, \tau) := \sum_{k \in \mathbb{N}^n \atop N(f(k)) \geq m} p^{-v(k)},$$

$$B(p, m, \tau) := \sum_{k \in \mathbb{N}^n \atop N(f(k)) = m - 1} p^{-v(k)},$$

and

$$E(p, f_\tau) := \frac{1}{(p - 1)^n} \sum_{x \in \{1, \ldots, p - 1\}^n} \exp\left(\frac{2\pi i}{p} f_\tau(x)\right). \tag{3.1.2}$$

Proof

Writing

$$S_f(p^m) = \int_{\mathbb{Z}_p^n} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx|$$

with $|dx|$ the normalized Haar measure on $\mathbb{Q}_p^n$, we deduce

$$S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{k \in \mathbb{N}^n \atop F(f(k)) = \tau} \int_{\text{ord } x = k} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx|. \tag{3.1.3}$$

Put $x_j = p^k u_j$ for $k \in \mathbb{N}^n$. Then $|dx| = p^{-v(k)} |du|$, and

$$f(x) = p^{N(k)} (f_F(u) + p(\cdots)),$$

where the dots take values in $\mathbb{Z}_p$ and $N(k) = N(f)(k)$. Hence

$$S_f(p^m) = \sum_{\tau \text{ face of } \Delta_0(f)} \sum_{k \in \mathbb{N}^n \atop F(f(k)) = \tau} p^{-v(k)} \int_{u \in (\mathbb{Z}_p)^n} \exp\left(\frac{2\pi i}{p^{m-N(k)}} (f_\tau(u) + p(\cdots))\right) |du|, \tag{3.1.3}$$

where $\mathbb{Z}_p^\times$ denotes the group of $p$-adic units. Because of the nondegenerateness assumptions, for $\tau$ a face of $\Delta_0(f)$ and $p$ a big enough prime the reduction $f_\tau \mod p$ has no critical points on $(\mathbb{F}_p^n)^\times$. Hence the integral in (3.1.3) is zero whenever $m - N(f)(k) \geq 2$. When $m - N(k) \leq 0$, the integral over $(\mathbb{Z}_p^\times)^n$ in (3.1.3) is just the measure of $(\mathbb{Z}_p^\times)^n$ and thus equals $(1 - p^{-1})^n$. When $m - N(k) = 1$, the integral over $(\mathbb{Z}_p^\times)^n$ in (3.1.3) equals $p^{-n}(p - 1)^n E(p, f_\tau)$. The proposition now follows from (3.1.3).
4. Lower bounds for $\nu(k)$
The main result of this section is the following.

**Theorem 4.1**
Let $\tau$ be a face of $\Delta_0(f)$. Then for all $k$ with $F(f)(k) = \tau$, one has

$$\nu(k) \geq \sigma(f)\left(N(f)(k) + 1\right) - \sigma(f_i),$$  \hspace{1cm} (4.1.1)

where $\sigma(f)$ and $\sigma(f_i)$, as well as $\nu(k)$ and $N(f)(k)$, are as in our notation.

The main points are that one subtracts $\sigma(f_i)$ instead of $\sigma(f)$ and that $\sigma(f_i) \leq \sigma(f)$. Subtracting $\sigma(f)$ yields trivial bounds since one has $\nu(k) \geq \sigma(f)N(k)$ for all $k \in \mathbb{R}_+^n$.

The theorem’s proof is based on two facts.

**Lemma 4.2**
Let $\tau$ be a face of $\Delta_0(f)$, and let $R_j \in \mathbb{R}^n$ be finitely many points belonging to $\tau$. Let $\beta_j \geq 0$ satisfy

$$\sum_j \beta_j R_j \leq \left(\frac{1}{\sigma}, \ldots, \frac{1}{\sigma}\right),$$

where $a \leq b$ for $a, b \in \mathbb{R}^n$ means $a_i \leq b_i$ for all $i$. Then

$$\sum \beta_j \leq 1.$$

**Proof**
Clearly, there is no point $S$ in the interior of $\Delta_0(f)$ which satisfies $S \leq (1/\sigma, \ldots, 1/\sigma)$. When $\sum_j \beta_j > 1$, then $\sum_j \beta_j R_j$ lies in the interior of $\Delta_0(f)$. \hfill $\square$

**Corollary 4.3**
Let $\tau$, $R_j$, and $\beta_j$ be as in Lemma 4.2. Then

$$\sum \beta_j \leq \frac{\sigma(f_i)}{\sigma}.$$  \hspace{1cm} (4.3.1)

**Proof**
Since $\sum_j \beta_j R_j \leq (1/\sigma, \ldots, 1/\sigma)$, one has

$$\frac{\sigma}{\sigma(f_i)} \sum_j \beta_j R_j \leq \left(\frac{1}{\sigma(f_i)}, \ldots, \frac{1}{\sigma(f_i)}\right).$$

Lemma 4.2 thus implies that $(\sigma/(\sigma(f_i))) \sum_j \beta_j \leq 1$. \hfill $\square$
Proof of Theorem 4.1
Since \((1/\sigma, \ldots, 1/\sigma)\) lies in the interior of \(F_0(f)\), by convexity one can write
\[
\left(\frac{1}{\sigma}, \ldots, \frac{1}{\sigma}\right) = \sum_i \alpha_i P_i + \sum_j \beta_j R_j
\]
for some \(\alpha_i \geq 0\) and \(\beta_j \geq 0\) with \(\sum_i \alpha_i + \sum_j \beta_j = 1\), \(P_i\) finitely many integral points of \(F_0(f) \setminus \tau\), and \(R_j\) finitely many integral points of \(\tau\). For \(k \in \mathbb{N}^k\) with \(F(f)(k) = \tau\), calculate
\[
v(k) = \sigma \left(\frac{1}{\sigma}, \ldots, \frac{1}{\sigma}\right) \cdot k
\]
\[
= \sigma \left(\sum_i \alpha_i P_i + \sum_j \beta_j R_j\right) \cdot k
\]
\[
= \sigma \left(\sum_i \alpha_i P_i \cdot k + \sum j \beta_j R_j \cdot k\right)
\]
\[
\geq \sigma \left(\sum_i \alpha_i (N(k) + 1) + \sum j \beta_j N(k)\right)
\]
\[
= \sigma \left(\left(\sum_i \alpha_i + \sum j \beta_j\right)(N(k) + 1) - \sum j \beta_j\right)
\]
\[
= \sigma \left((N(k) + 1) - \sum j \beta_j\right)
\]
\[
\geq \sigma \left((N(k) + 1) - \frac{\sigma(f_\tau)}{\sigma}\right)
\]
where (4.3.5) follows from \(k \cdot R_j = N(k)\) and \(k \cdot P_i \geq N(k) + 1\), which is true by definition of \(N(k)\), and where (4.3.8) follows from Corollary 4.3.

5. Upper bounds for \(A(p, m, \tau)\) and \(B(p, m, \tau)\)
We recall one result from [7].

**Lemma 5.1 ([7, Lemma 3.3])**
Let \(C\) be a convex polyhedral cone in \(\mathbb{R}_+^n\) generated by vectors in \(\mathbb{N}^n\), and let \(L\) be a linear form in \(n\) variables with coefficients in \(\mathbb{N}\). We denote by \(C_{\text{int}}\) the interior of \(C\) in the sense of Newton polyhedra. Let \(\sigma > 0\) and \(\gamma \geq 0\) be real numbers satisfying
\[
v(k) \geq L(k)\sigma + \gamma \text{ for all } k \in C_{\text{int}} \cap \mathbb{N}^n.
\]
Put

\[ e = \dim\{ k \in C \mid \nu(k) = L(k)\sigma \}. \]

Then there exists a real number \( c > 0 \) such that for all \( m \in \mathbb{N} \) and for all \( p \in \mathbb{R} \) with \( p \geq 2 \),

\[
\sum_{k \in C \cap \mathbb{N}^n \atop \nu(k) = \lambda} p^{-\nu(k)} \leq cp^{-m\sigma - \gamma}(m + 1)^{\max(0, e-1)}.
\] (5.1.2)

From this lemma and from Theorem 4.1, we have the following.

**Corollary 5.2**

Let \( f, A(p, m, \tau), \) and \( B(p, m, \tau) \) be as in Proposition 3.1. Then there exists a real number \( c > 0 \) such that for all integers \( m > 0 \), for all faces \( \tau \) of \( \Delta_0(f) \), and for all big enough primes \( p \),

\[
A(p, m, \tau) \leq cp^{-m\sigma}m^{\kappa - 1}
\] (5.2.1)

and

\[
B(p, m, \tau) \leq cp^{-m\sigma + \sigma(f\tau)}m^{\kappa - 1}.
\] (5.2.2)

**Proof**

To derive (5.2.1) from Lemma 5.1, note that \( \nu(k) \geq N(k)\sigma \) for any \( k \in \mathbb{N}^n \) and that \( \kappa = \dim\{ k \in \mathbb{R}^n_+ \mid \nu(k) = N(k)\sigma \} \geq 1 \).

To derive (5.2.2) from Lemma 5.1 and Theorem 4.1, use for \( C \) the topological closure of the convex hull of \( \{0\} \cup \{ k \in \mathbb{N}^n \mid F(k) = \tau \} \), and note that \( \mathcal{C}^{\text{int}} \cap \mathbb{N}^n = \{ k \in \mathbb{N}^n \mid F(k) = \tau \} \). Clearly, \( \kappa \geq 1 \) and \( \kappa \geq \dim\{ k \in \mathcal{C}^{\text{int}} \mid \nu(k) = N(k)\sigma \} \). By (4.1.1), \( \nu(k) \geq N(k)\sigma + \sigma - \sigma(f\tau) \) for all \( k \in \mathcal{C}^{\text{int}} \cap \mathbb{N}^n \).

**6. Upper bounds for \( \sigma(f) \) and \( E(p, f) \)**

By Katz [11, Theorem 4], for \( f(x) \) a nonconstant homogeneous polynomial in \( n \) variables over \( \mathbb{Z} \) and \( d \) the dimension of \( \text{grad} f = 0 \) in \( A_{\mathbb{F}_p}^n \), there exists \( c \) such that for all big enough \( p \), one has

\[
\left| \sum_{x \in A^n(F_p)} \exp\left( \frac{2\pi i}{p} f(x) \right) \right| \leq cp^{(n+d)/2}.
\] (6.0.3)

Moreover, \( c \) can be taken depending on the degree of \( f \) only. This implies the following.
COROLLARY 6.1

Suppose that \( f(x) \) is homogeneous of degree at least 2, and let \( d \) be the dimension of \( \text{grad} f = 0 \) in \( \mathbb{A}^n_{\mathbb{C}} \). Then there exists \( c \) such that for all \( p \) big enough,

\[
\left| \sum_{x \in G^n_p(\mathbb{F}_p)} \exp\left( \frac{2\pi i}{p} f(x) \right) \right| < cp^{(n+d)/2}, \tag{6.1.1}
\]

and hence, for some \( c' \) one has, for all big enough \( p \),

\[
|E(p, f)| < c' p^{(-n+d)/2} \tag{6.1.2}
\]

with \( E(p, f) \) as defined by (3.1.2). Moreover, \( c \) and \( c' \) can be taken depending on \( \Delta_0(f) \) only.

Proof

Let \( f_0(x_2, \ldots, x_n) \) be the polynomial \( f(0, x_2, \ldots, x_n) \). Clearly, \( f_0 \) is homogeneous in \( n-1 \) variables. By Katz’s result (6.0.3), it is enough to show that \( n-1+d(f_0) \leq n+d \) with \( d(f_0) \) the dimension of \( \text{grad} f_0 = 0 \) in \( \mathbb{A}^{n-1}_{\mathbb{C}} \). This inequality follows from writing

\[ f(x) = x_1 g(x) + f_0(x_2, \ldots, x_n) \]

with \( g \) a polynomial in \( x \) and from comparing \( \text{grad} f \) with \( \text{grad} f_0 \). \( \square \)

6.2

Let \( \{(N_i, \nu_i)\}_{i \in I} \) be the numerical data of a resolution \( h \) of \( f \) with normal crossings (i.e., if \( \pi_f : Y \to \mathbb{A}^n_{\mathbb{C}} \) is an embedded resolution of singularities with normal crossings of \( f = 0 \), then for each irreducible component \( E_i \) of \( \pi_f^{-1} \circ f^{-1}(0) \), \( i \in I \), let \( N_i \) be the multiplicity of \( E_i \) in \( \text{div}(f \circ \pi_f) \), and let \( \nu_i - 1 \) be the multiplicity of \( E_i \) in the divisor associated to \( \pi_f^* (dx_1 \wedge \cdots \wedge dx_n) \); see [3]). The essential numerical data of \( \pi_f \) are the pairs \( (N_i, \nu_i) \) for \( i \in J \) with \( J = I \setminus I' \), where \( I' \) is the set of indices \( i \) in \( I \) such that \( (N_i, \nu_i) = (1, 1) \) and such that \( E_i \) does not intersect another \( E_j \) with \( (N_j, \nu_j) = (1, 1) \). Define \( \alpha(\pi_f) \) as

\[
\alpha(\pi_f) = -\min_{i \in J} \frac{\nu_i}{N_i} \tag{6.2.1}
\]

when \( J \) is nonempty, and define \( \alpha(\pi_f) \) as \(-2n\) otherwise.

It follows from [2, Theorem 5.1, Corollary 3.4] that

\[
\alpha(\pi_f) \geq \frac{-n + d}{2} \tag{6.2.2}
\]

with \( d \) the dimension of \( \text{grad} f = 0 \) in \( \mathbb{A}^n_{\mathbb{C}} \), where the empty scheme has dimension \(-\infty\).
LEMMA 6.3
Let $f$ be homogeneous of degree at least 2 and nondegenerate with respect to the faces of $\Delta_0(f)$. Let $d$ be the dimension of $\text{grad } f = 0$ in $\mathbb{A}^n_C$. Then

$$\sigma(f) \leq \frac{n - d}{2}. \quad (6.3.1)$$

Proof
By properties of a toric resolution $\pi_f$ of $f = 0$, one has

$$\sigma(f) = -\alpha(\pi_f)$$
with $\alpha(\pi_f)$ as defined by (6.2.1). Now, use (6.2.2).

From (6.3.1) and Corollary 6.1 applied to $f_\tau$, we have the following.

COROLLARY 6.4
Let $f$ be a homogeneous polynomial of degree at least 2 which is nondegenerate with respect to the faces of $\Delta_0(f)$. Then there exists $c$ such that for all faces $\tau$ of $\Delta_0(f)$ and all big enough primes $p$,

$$|E(p, f_\tau)| < cp^{-\sigma(f_\tau)} \quad (6.4.1)$$
with $E(p, f_\tau)$ as defined by (3.1.2). Moreover, $c$ can be taken depending on $\Delta_0(f)$ only.

7. Proof of the main theorem

Proof of Theorem 2.1
When the degree of $f$ is at least 2, use Proposition 3.1, Corollary 5.2, and (6.4.1). For linear $f$, the theorem is trivial.

8. Comparison with the Denef-Sperber approach
As mentioned in the paragraph following Corollary 2.2, Denef and Sperber [7] prove Theorem 2.1 under the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$. Key points in our proof of Theorem 2.1 are (4.1.1) (which implies Corollary 5.2) and (6.4.1). Instead of (4.1.1), Denef and Sperber use their result that for similar $k$ as in (4.1.1) but assuming the extra condition that no vertex of $F_0(f)$ belongs to $\{0, 1\}^n$,

$$\nu(k) \geq \sigma(f)(N(f)(k) + 1) - \frac{\dim \tau + 1}{2}. \quad (8.0.2)$$
This often fails if one omits the extra condition (see Examples (1), (2)). Instead of (6.4.1), they use the bounds of Adolphson and Sperber [1] and Denef and Loeser [4],

$$|E(p, f_\tau)| < cp^{(-\dim \tau - 1)/2},$$

(8.0.3)

which hold (in particular) under the same conditions as for (6.4.1) but which are sometimes not as good as the bounds (6.4.1).

We give two examples where our methods really make a difference with (8.0.2) and (8.0.3).

Examples
(1) First, for $f(x, y, z, u) = xy + zu$ and $\tau = F_0(f)$, one has $\dim \tau = 1$, $\sigma(f_\tau) = \sigma = 2$; (8.0.2) does not hold, and (8.0.3) is not optimal, while (6.4.1) yields the optimal

$$E(p, f_\tau) < cp^{-2}.$$ 

(2) Second, for $f(x, y, z, u) = xy + zu + xz + ayu$ with $a \in \mathbb{Z}, a \neq 1$, and $\tau = F_0(f)$, one has $\dim \tau = 2$, $\sigma(f_\tau) = \sigma = 2$; (8.0.2) does not hold, and (8.0.3) is not optimal, while (6.4.1) yields again the optimal $|E(p, f_\tau)| < cp^{-2}$ for big $p$. In this example, $E(p, f_\tau)$ can be calculated by performing a transformation on $G^4_m$ coming from an element of $\text{GL}_4(\mathbb{Z})$ transforming $f(x)$ into $f(x', y', z', u') = x' + y' + z' + ax'y'z'^{-1}$; the bounds for $E(p, f_\tau)$ are surprisingly sharp compared, for example, to bounds for the resembling Kloosterman sums.

9. Analogues over finite extensions of $\mathbb{Q}_p$ and over $\mathbb{F}_q((t))$
For any nonarchimedean local field $K$ with valuation ring $\mathcal{O}_K$, write $\psi_K$ for an additive character

$$\psi_K : K \rightarrow \mathbb{C}^\times$$

which is trivial on $\mathcal{O}_K$ but nontrivial on some element of $K$ of order $-1$. Write $\text{ord}_K : K^\times \rightarrow \mathbb{Z}$ for the valuation, write $|\cdot|_K : K \rightarrow \mathbb{R}$ for the norm on $K$, and write $\bar{K}$ for its residue field with $q_K$ elements. Let $k$ be a number field with ring of integers $\mathcal{O}_k$. In this section, $f$ is a nonconstant polynomial over $\mathcal{O}_k[1/N]$ in $n$ variables with $f(0) = 0$ and $N \in \mathbb{Z}$. For $K$ any nonarchimedean local field that is an algebra over $\mathcal{O}_k[1/N]$ and for $y \in K^\times$, consider the exponential integral

$$S_{f, K}(y) := \int_{\mathcal{O}_k} \psi_K(yf(x)) |dx|_K$$

with $|dx|_K$ the normalized Haar measure on $K^n$. Note that $K$ may be of positive characteristic. Then the following generalization of Theorem 2.1 holds.

*However, (8.0.3) is sometimes sharper than (6.4.1) in cases where it does not matter for our course.
THEOREM 9.1
Suppose that \( f \) is a homogeneous polynomial over \( \mathcal{C}_k[1/N] \) which is nondegenerate with respect to the faces of \( \Delta_0(f) \). Then there exist \( c > 0 \) and \( M > N \) such that
\[
|S_{f,K}(y)| \leq c \cdot |y|^\sigma \cdot |\text{ord}_K(y)|^{\kappa - 1}
\]
for all nonarchimedean local fields \( K \) that are algebras over \( \mathcal{C}_k[1/N] \) and have residue characteristic greater than \( M \) and all \( y \in K^\times \) with \( \text{ord}_K(y) < 0 \), with \( |\cdot| \) the absolute value on \( \mathbb{R} \). Moreover, \( c \) can be taken depending on \( \Delta_0(f) \) only.

Proof
This is the same proof as that of Theorem 2.1, but using Proposition 9.2 instead of Proposition 3.1.

PROPOSITION 9.2
Suppose that \( f \) is nondegenerate with respect to (all) the faces of \( \Delta_0(f) \). Then there exists \( M > N \) such that
\[
S_{f,K}(y) = (1 - q_K^{-1})^n \sum_{\tau \text{ face of } \Delta_0(f)} \left( A(q_K, m, \tau) + E(\bar{K}, \tau, y)B(q_K, m, \tau) \right)
\]
for all nonarchimedean local fields \( K \) that are algebras over \( \mathcal{C}_k[1/N] \) and have residue characteristic greater than \( M \) and all \( y \in K^\times \) with \( \text{ord}_K(y) \leq 0 \).

In these formulas, \( A(q_K, m, \tau) \) and \( B(q_K, m, \tau) \) are as in Proposition 3.1, and
\[
E(\bar{K}, \tau, y) := \frac{1}{(q_K - 1)^n} \sum_{u \in G_\mu(\bar{K})} \psi_y(f_\tau(u))
\]
with \( \psi_y \) a nontrivial additive character on \( \bar{K} \) depending on \( y \) and \( \psi_K \).\n
Proof
This is the same proof as that of Proposition 3.1.

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\[\text{For such } K \text{ and for } y \in K^\times \text{ with } \text{ord}_K(y) \geq 0, \text{ one has } S_{f,K}(y) = 1.\]

\[\text{In fact, the character } \psi_y \text{ depends only on } \psi_K \text{ and on } \overline{\varphi}(y) \text{ for any multiplicative homomorphism } \overline{\varphi} : K^\times \to \bar{K}^\times \text{ extending the natural projection } \mathcal{C}_K^\times \to \bar{K}^\times.\]
References


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