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Loci of Integrability, Zero Loci, and Stability Under Integration for Constructible Functions on Euclidean Space with Lebesgue Measure

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We introduce and study loci of integrability. We prove a correspondence between zero loci and loci of integrability for constructible functions on Euclidean space, where a function is called constructible if it is a sum of products of globally subanalytic functions and of logarithms of globally subanalytic functions. We generalize the main result of the authors in Cluckers and Miller [Stability under integration of sums of products of real globally subanalytic functions and their logarithms. *Duke Mathematical Journal* 156, no. 2 (2011)] about the stability under integration of the class of constructible functions, by relaxing the conditions on integrability. Further, we give an interpolation result for constructible functions by constructible functions with maximal locus of integrability.

1 Introduction

This is the second paper in a series of papers on parameterized real integrals, following [7]. We define and study loci of integrability of certain (families of) functions. A recent

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insight into parameterized integrals is that, for functions f belonging to certain classes of functions on certain product measure spaces $E \times T$, a set of the form

$$\{x \in E \mid T \to \mathbb{C} : t \mapsto f(x, t) \text{ is measurable and integrable over } T\}, \tag{1.1}$$

is in fact equal to the zero locus of a function on E belonging to the same class of functions; see [5]. If we call the set in (1.1) the locus of integrability of f in E, then we can rephrase the recent insight as a link between loci of integrability and zero loci for certain kinds of functions.

In this paper, we give such a link for the class of constructible functions on Euclidean spaces with the Lebesgue measure; see Theorem 2.3. We follow the terminology of [7]: a constructible function is by definition a sum of products of globally subanalytic functions and of logarithms of globally subanalytic functions; see Section 2 below for more detailed definitions. The advantage of the class of constructible functions is that it is closed under integration. Indeed, in Cluckers and Miller [7] the authors prove that if f is constructible on $\mathbb{R}^n \times \mathbb{R}^m$ such that $y \mapsto f(x, y)$ is integrable over $\mathbb{R}^m$ for each $x \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^m} f(x, y)\, \mathrm{d}y$$

is constructible on $\mathbb{R}^n$, which generalizes results of [9]. We extend this stability result by relaxing the conditions on integrability; see Theorem 2.5. Further, we give an interpolation result, Theorem 2.4, of constructible functions by constructible functions with maximal locus of integrability.

2 The Main Results

Recall that a function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is called globally subanalytic if its graph is a globally subanalytic set, and a set $A \subset \mathbb{R}^n$ is called globally subanalytic if its image under the natural embedding of $\mathbb{R}^n$ into n-dimensional real projective space, namely $\mathbb{R}^n \to \mathbb{P}^n(\mathbb{R}) : (x_1, \ldots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$, is a subanalytic subset of $\mathbb{P}^n(\mathbb{R})$ in the classical sense; see Definition 3.1 below for a self-contained definition.

From now on in this paper, we write "subanalytic" instead of "globally subanalytic" (see again Definition 3.1).

Definition 2.1. For each subanalytic set X, let $\mathcal{C}(X)$ be the $\mathbb{R}$-algebra of real-valued functions on X generated by all subanalytic functions on X and all the functions

$x \mapsto \log f(x)$, where $f : X \to (0, +\infty)$ is subanalytic. Functions in $\mathcal{C}(X)$ are called constructible functions on X and $\mathcal{C}(X)$ is called the algebra of constructible functions on X. □

In the whole paper, we use the Lesbegue measure on $\mathbb{R}^n$. We introduce the locus of integrability of a function, as follows.

Definition 2.2. For E a set, and for $f : E \times \mathbb{R}^n \to \mathbb{C}$ a function, define the *locus of integrability of f in E* as the set

$$\mathrm{Int}(f, E) := \{x \in E \mid f(x, \cdot) \text{ is measurable and integrable over } \mathbb{R}^n\},$$

where $f(x, \cdot)$ is the function sending $y \in \mathbb{R}^n$ to $f(x, y)$, and where the Lebesgue measure is used on $\mathbb{R}^n$. □

The main results of this paper are the following three theorems, for which we will give relatively short and simple proofs.

Theorem 2.3. Let f be in $\mathcal{C}(E \times \mathbb{R})$ for some subanalytic set E. Then there exists h in $\mathcal{C}(E)$ such that

$$\mathrm{Int}(f, E) = \{x \in E \mid h(x) = 0\}. \tag{2.1}$$

Conversely, for every h in $\mathcal{C}(E)$ there exists f in $\mathcal{C}(E \times \mathbb{R})$ such that (2.1) holds. □

Theorem 2.3 thus gives a correspondence between loci of integrability and zero loci of constructible functions, at least when integration is in dimension 1, that is, over $\mathbb{R}$. One should not misunderstand Theorem 2.3: zero loci of constructible functions are much more general than, say, Zariski closed sets, and for example, a zero locus of $h \in \mathcal{C}(E)$ can easily be dense in E. Indeed, the characteristic function of any subanalytic subset of E lies in $\mathcal{C}(E)$. Note that when f in Theorem 2.3 is moreover subanalytic, then one can take h to be a subanalytic function as well by the main result of [9]. Theorem 2.3 implies Theorem 1.4 of [7]. In Cluckers and Miller [8] we treat a higher dimensional variant of Theorem 2.3, also treating L^p-integrability for various p.

Constructible functions allow an interpolation by constructible functions with maximal locus of integrability, as follows.

Theorem 2.4. Let f be in $\mathcal{C}(E \times \mathbb{R})$ for some subanalytic set E. Then there exists $g \in \mathcal{C}(E \times \mathbb{R})$ with

$$\mathrm{Int}(g, E) = E$$

and such that, for all $x \in \mathrm{Int}(f, E)$ and all $y \in \mathbb{R}$, one has

$$g(x, y) = f(x, y). \qquad \square$$

Finally, we can integrate in any dimension m to find the following generalization of the principal result, Theorem 1.3, of [7].

Theorem 2.5. Let f be in $\mathcal{C}(E \times \mathbb{R}^m)$ for some subanalytic set E and some $m > 0$. Then there exists $g \in \mathcal{C}(E)$ such that, for each $x \in \mathrm{Int}(f, E)$, one has

$$g(x) = \int_{y \in \mathbb{R}^m} f(x, y)\, \mathrm{d}y. \qquad \square$$

The above theorem is proved in Cluckers and Miller [7] under the extra condition that $\mathrm{Int}(f, E)$ equals E (which in turn generalized main results from [9, 15]). Note that integrals of constructible functions are related to what one could call families of periods; see [11–13]. In several special cases, explicit formulas for parameterized integrals of constructible functions are given in [1, 18]. Parameterized integrals of constructible functions are often used for the study of singularities, as in [2, 16, 17]. For context on subanalytic functions we refer the reader to [3, 10].

The present paper is inspired by Cluckers et al. [5] which contains several p-adic and motivic analogues of this paper, where [7] was more closely inspired on p-adic and motivic results of [4, 6]. The results and proofs of this paper can be used to replace some of the technical difficulties encountered in Cluckers and Miller [7].

3 Preparation Results in One Special Variable

In this section, we recall a basic form of the subanalytic preparation theorem from [14] (see also [19]), we fix some notation, and we give a new preparation result for constructible functions.

Definition 3.1. Call a function $f : X \subset \mathbb{R}^\ell \to \mathbb{R}^k$ analytic if it extends to an analytic function on an open neighborhood of X. A restricted analytic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that the restriction of f to $[-1, 1]^n$ is analytic and $f(x) = 0$ on $\mathbb{R}^n \setminus [-1, 1]^n$.

Call a set or a function subanalytic if and only if it is definable in the expansion of the real field by all restricted analytic functions. Thus in this paper, “subanalytic” is an abbreviation of “globally subanalytic”, and in this meaning, the natural logarithm $\log : (0, +\infty) \to \mathbb{R}$ is not subanalytic. □

For the rest of this section we fix an ordered list of variables $x_1, \ldots, x_{n+1}$, where $n \geq 0$, and we write x for $(x_1, \ldots, x_n)$ and write y for x_{n+1}, since the variable x_{n+1} will play a special role.

Definition 3.2 (Cells with a base). Consider subanalytic sets $A \subset \mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^n$ and an analytic subanalytic function $\theta : B \to \mathbb{R}$. Then A is called a 0-cell over $\mathbb{R}^n$ with base B if A equals the graph of an analytic subanalytic function $c : B \to \mathbb{R} : x \mapsto y = c(x)$.

Call A a 1-cell over $\mathbb{R}^n$ with base B and with center θ if there are analytic subanalytic functions $a : B \to \mathbb{R}$ and $b : B \to \mathbb{R}$, with $a < b$ on B, such that A is of the following form:

$$A = \{(x, y) \in B \times \mathbb{R} : a(x) \square_1 y \square_2 b(x)\},$$

with $\square_i$ either $<$ or no condition for each $i = 1, 2$, and such that the graph $\Gamma(\theta)$ of θ satisfies either

$$\Gamma(\theta) \subset \bar{A} \setminus A, \quad \text{or,} \quad \Gamma(\theta) \cap \bar{A} = \emptyset,$$

where $\bar{A}$ is the topological closure of A inside $\mathbb{R}^{n+1}$. In any case, A is called a cell over $\mathbb{R}^n$. □

Definition 3.3 (Strong functions). Let A be a 1-cell over $\mathbb{R}^n$ with base B and with center θ. A basic function with center θ is a function $\varphi : A \to \mathbb{R}^{N+2}$, for some $N \geq 0$, with bounded image and which is of the form

$$\varphi(x, y) = (a_1(x), \ldots, a_N(x), b_1(x)|y - \theta(x)|^{1/p}, b_2(x)|y - \theta(x)|^{-1/p}), \tag{3.1}$$

where $a_1, \ldots, a_N, b_1, b_2$ are analytic subanalytic functions from B to $\mathbb{R}$ and p is a positive integer. A strong function on A with center θ is a function $A \to \mathbb{R}$ of the form $F \circ \varphi$, where φ is a basic function with center θ and where the function F is given by a single power series that converges on an open neighborhood of the image of φ. Note that strong functions are automatically subanalytic functions. □

Theorem 3.4 (Preparation of subanalytic functions [14, 19]). Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $X \subset \mathbb{R}^{n+1}$. Then there exists a finite partition of X into cells over $\mathbb{R}^n$ such that the following holds for any 1-cell A over $\mathbb{R}^n$ in this partition:

There exists a center θ for A such that each $f \in \mathcal{F}$ can be written in the form

$$f(x, y) = g(x)|y - \theta(x)|^r S(x, y)$$

on A, where g is an analytic subanalytic function on the base of A, r is a rational number, and S is a strong function on A with center θ, and such that, moreover, $S > \epsilon$ on A for some $\varepsilon > 0$. □

The last part in the following corollary is new and simplifies the proofs concerning integration and integrability when compared with [7].

Corollary 3.5 (Preparation of constructible functions). Let $\mathcal{F}$ be a finite set of constructible functions on a subanalytic set $X \subset \mathbb{R}^{n+1}$. Then there exists a finite partition of X into cells over $\mathbb{R}^n$ such that for each 1-cell A over $\mathbb{R}^n$ with base B in this decomposition, there exists a center θ such that, the following holds for each $f \in \mathcal{F}$ and all $(x, y) \in A$, and with $\tilde{y} := y - \theta(x)$:

$$f(x, y) = \sum_{i=1}^{M} d_i(x) S_i(x, y) |\tilde{y}|^{\alpha_i} (\log|\tilde{y}|)^{\ell_i}, \tag{3.2}$$

for some $M \geq 0$, functions $d_i \in \mathcal{C}(B)$, rational numbers α_i, integers $\ell_i \geq 0$, and strong functions S_i on A with center θ. Moreover, one can ensure for each i that at least one of the following two conditions holds:

(1) $S_i(x, y) = 1$ on A;
(2) $y \mapsto |\tilde{y}|^{\alpha_i}$ is integrable over A_x for all $x \in B$. □

Proof. Let $\mathcal{F}'$ be a finite collection of subanalytic functions such that each $f \in \mathcal{F}$ is a finite sum of products of functions in $\mathcal{F}'$ and of logarithms of functions in $\mathcal{F}'$. Apply Theorem 3.4 to $\mathcal{F}'$. Note that $\log(S)$ is a strong function with center θ if S is a strong function with center θ satisfying $S > \epsilon$ for some $\epsilon > 0$. Hence, we are done with the first part of the statement by writing logarithms of products as sums of logarithms, and since the product of strong functions with center θ is a strong function with center θ.

Suppose now that, for some occurring term $S_i(x,y)d_i(x)|\tilde{y}|^{\alpha_i}$ on some cell A with center θ and base B, one has that $y \mapsto |\tilde{y}|^{\alpha_i}$ is not integrable over A_x for some (and hence for all) $x \in B$. Then, by the supposed presence of this nonintegrable term and by partitioning the cells slightly further, we may suppose that exactly one of the following two conditions holds:

(i) The graph of the center θ lies in $\bar{A}$ and A_x is bounded in $\mathbb{R}$ for each value of $x \in B$.

(ii) The graph of the center θ is disjoint from $\bar{A}$ and A_x is not contained in a compact subset of $\mathbb{R}$ for any value of $x \in B$.

Since the argument is completely similar in both cases, let us suppose (i) holds. Then, writing the strong function S_i as $F_i \circ \varphi$ with φ a basic function with center θ, as in (3.1), and F_i a converging power series, and by recalling that the image of φ is bounded, one sees that $b_2(x) = 0$ for all $x \in B$, with notation from (3.1). Moreover, $\tilde{y}$ is bounded on A_x for each x, and thus, $|\tilde{y}|^{q+\alpha_i}$ is integrable over A_x for all $x \in B$ as soon as $q \in \mathbb{Q}$ is sufficiently large. For any $s > 0$ we can develop finitely many terms of F_i in $|\tilde{y}|^{1/p}$ plus the remaining series in $|\tilde{y}|^{1/p}$, as follows:

$$S_i(x,y) = \left(\sum_{j=0}^{s-1} c_j(x)|\tilde{y}|^{j/p}\right) + \left(\sum_{j \geq s} c_j(x)|\tilde{y}|^{j/p}\right). \tag{3.3}$$

By pulling out the factor $|\tilde{y}|^{s/p}$ from the last term, by writing out $S_i(x,y)d_i(x)|\tilde{y}|^{\alpha_i}$ using distributivity and (3.3), and by taking s large enough, the first s such terms will be as in part (1) of the corollary, and the last term will be integrable as in (2). This completes the proof. ■

4 Proofs of the Main Results

Proofs of Theorems 2.3 and 2.4. Let f be in $\mathcal{C}(E \times \mathbb{R})$, with $E \subset \mathbb{R}^n$ for some n. Apply Corollary 3.5 to the collection of functions consisting only of f. Consider a 1-cell A over $\mathbb{R}^n$ in the obtained partition, with center θ, and write f as in (3.2). By regrouping the terms and using the notation of (3.2), we may suppose, for each i, that either $|\tilde{y}|^{\alpha_i}$ is integrable over A_x, or that (α_i, ℓ_i) is different from the (α_j, ℓ_j) for all $j \neq i$. Let I be those indices i such that $|\tilde{y}|^{\alpha_i}$ is not integrable over A_x. Now define Q_A as the set $\{x \in B \mid d_i(x) = 0$

for $i \in I\}$ and define, for $(x, y) \in A$, the constructible function

$$g(x, y) := \sum_{i \notin I} d_i(x) S_i(x, y) |\tilde{y}|^{\alpha_i} (\log |\tilde{y}|)^{\ell_i}.$$

Note that

$$\{x \in B : f(x, \cdot) \text{ is integrable over } A_x\} = Q_A,$$

because of condition (1) in Corollary 3.5, and because we have taken the exponent pairs (α_i, ℓ_i) mutually different for nonintegrable terms. Do the above construction for each occurring 1-cell A over $\mathbb{R}^n$. On any 0-cell A' over $\mathbb{R}^n$ in our partition, define $g(x, y)$ as $f(x, y)$. Then g is as desired by Theorem 2.4. Now note that a finite union of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the product of the h_i. Similarly, a finite intersection of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the sum of the squares of the h_i. Now one is done for $\mathrm{Int}(f, E)$. Indeed, $\mathrm{Int}(f, E)$ equals the finite intersection

$$\cap_A Q'_A,$$

where A runs over all 1-cells over $\mathbb{R}^n$ in the partition, and where, for any such 1-cell A, Q'_A equals the set $Q_A \cup (E \setminus B)$. Note that $E \setminus B$ is a subanalytic set and each of the Q'_A equals thus the zero locus of a constructible function on E. For the converse statement of Theorem 2.3, given h, it suffices to put $f(x, y) = h(x) y$ for all $(x, y) \in E \times \mathbb{R}$. ■

Proof of Theorem 2.5. Consider f in $\mathcal{C}(E \times \mathbb{R}^m)$ for some $m > 0$. If $m = 1$, then apply Theorem 2.4 to f to find g_0 in $\mathcal{C}(E \times \mathbb{R})$ with $\mathrm{Int}(g_0, E) = E$ and such that $g_0(x, y) = f(x, y)$ for all $x \in \mathrm{Int}(f, E)$ and all $y \in \mathbb{R}$. Now Apply Theorem 1.3 of [7] to g_0, which states that, if one defines, for $x \in E$,

$$g(x) := \int_{\mathbb{R}} g_0(x, y)\, \mathrm{d}y,$$

then g lies in $\mathcal{C}(E)$. Then this g is as desired. The result for general m now follows from Fubini's Theorem. ■

Alternatively to deriving Theorem 2.5 for $m = 1$ from Theorem 1.3 of [7], one can also derive the case $m = 1$ from Corollary 3.5 by the integration procedure by Lion and Rolin of [15], which is also used and explained in Cluckers and Miller [7]. This self-contained approach for obtaining Theorem 2.5 is simpler than the approaches of [7, 9, 15], which moreover only yielded special forms of Theorem 2.5.

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