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Classification of semi-algebraic p-adic sets up to semi-algebraic bijection

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Abstract. We prove that two infinite p-adic semi-algebraic sets are isomorphic (i.e. there exists a semi-algebraic bijection between them) if and only if they have the same dimension. □

In real semi-algebraic geometry (as opposed to p-adic semi-algebraic geometry) the following classification is well-known [4]:

There exists a real semi-algebraic bijection between two real semi-algebraic sets if and only if they have the same dimension and Euler characteristic.

More generally L. van den Dries [4] gave such a classification for o-minimal expansions of the real field, using the dimension and Euler characteristic as defined for o-minimal structures. Since the semi-algebraic Euler characteristic χ is in fact the canonical map from the real semi-algebraic sets onto the Grothendieck ring (see [1]) of $\mathbb{R}$ (which is $\mathbb{Z}$), we see that the isomorphism class of a real semi-algebraic set only depends on its image in the Grothendieck ring and its dimension.

In this paper we treat the p-adic analogue of this classification. The Grothendieck ring of $\mathbb{Q}_p$ is recently proved to be trivial by D. Haskell and the author [1], so the analogue of the real case is a classification of the p-adic semi-algebraic sets up to semi-algebraic bijection using only the dimension. We give such a classification for the semi-algebraic sets of $\mathbb{Q}_p$ and for finite field extensions of $\mathbb{Q}_p$, using explicit isomorphisms of [1] and the p-adic Cell Decomposition Theorem of J. Denef [2]. The most difficult part in giving this classification is to prove that for any semi-algebraic set X there is a finite partition into semi-algebraic sets, such that each part is isomorphic to a Cartesian product of one dimensional sets, in other words semi-algebraic sets have a rectilinearization. Since all arguments hold also for finite field extensions of $\mathbb{Q}_p$, we work in this more general setting.

Let p denote a fixed prime number, $\mathbb{Q}_p$ the field of p-adic numbers and K a fixed

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finite field extension of $\mathbb{Q}_p$. For $x \in K$ let $v(x) \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of x. Let $R = \{x \in K \mid v(x) \geqq 0\}$ be the valuation ring, $K^\times = K \backslash \{0\}$ and for $n \in \mathbb{N}_0$ let P_n be the set $\{x \in K^\times \mid \exists y \in K\, y^n = x\}$. We call a subset of K^m *semi-algebraic* if it is a Boolean combination (i.e. obtained by taking finite unions, complements and intersections) of sets of the form $\{x \in K^m \mid f(x) \in P_n\}$ with $f(x) \in K[X_1, \ldots, X_m]$. The collection of semi-algebraic sets is closed under taking projections $K^m \to K^{m-1}$, even more: it consists precisely of Boolean combinations of projections of affine p-adic varieties (see [2] and [5]). Further we have that sets of the form $\{x \in K^m \mid v(f(x)) \leqq v(g(x))\}$ with $f(x), g(x) \in K[X_1, \ldots, X_m]$ are semi-algebraic (see [2], Lemma 2.1). A function $f\colon A \to B$ is semi-algebraic if its graph is a semi-algebraic set; if further f is a bijection, we call f an *isomorphism* and we write $A \cong B$.

Let π be a fixed element of R with $v(\pi) = 1$, thus π is a uniformizing parameter for R. For a semi-algebraic set $X \subset K$ and $k \in \mathbb{N}_0$ we write

$$X^{(k)} = \{x \in X \mid v(\pi^{-v(x)}x - 1) \geqq k, x \neq 0\},$$

which is semi-algebraic (see [2], Lemma 2.1); $X^{(k)}$ consists of those points $x \in X$ which have a p-adic expansion $x = \sum_{i=s}^{\infty} a_i \pi^i$ with $a_s = 1$ and $a_i = 0$ for $i = s+1, \ldots, s+k-1$. By *a finite* partition of a semi-algebraic set we mean a partition into finitely many semi-algebraic sets. Let $X \subset K^n$, $Y \subset K^m$ be semi-algebraic. Choose disjoint semi-algebraic sets $X', Y' \subset K^k$ for some k, such that $X \cong X'$ and $Y \cong Y'$, then we define the disjoint union of X and Y up to isomorphism as $X' \cup Y'$. In the introduction of [1] it is shown that we can take $k = \max(m, n)$, i.e. we can realize the disjoint union without going into higher dimensional affine spaces.

We recall some well-known facts.

Lemma 1 (Hensel). *Let $f(t)$ be a polynomial over R in one variable t, and let $\alpha \in R$, $e \in \mathbb{N}$. Suppose that $f(\alpha) \equiv 0 \bmod \pi^{2e+1}$ and $v(f'(\alpha)) \leqq e$, where f' denotes the derivative of f. Then there exists a unique $\bar{\alpha} \in R$ such that $f(\bar{\alpha}) = 0$ and $\bar{\alpha} \equiv \alpha \bmod \pi^{e+1}$.*

Corollary 1. *Let $n > 1$ be a natural number. For each $k > v(n)$, and $k' = k + v(n)$ the function*

$$K^{(k)} \to P_n^{(k')}\colon x \mapsto x^n$$

is an isomorphism.

The next theorem gives some concrete isomorphisms between one dimensional sets.

Proposition 1 ([1], Prop. 2). (i) *The union of two disjoint copies of $R\backslash\{0\}$ is isomorphic to $R\backslash\{0\}$.*

(ii) *For each $k > 0$ the union of two disjoint copies of $R^{(k)}$ is isomorphic to $R^{(k)}$.*

(iii) $R \cong R\backslash\{0\}$.

We deduce an easy corollary, also consisting of concrete isomorphisms.

Corollary 2. *For each k we have isomorphisms*

(i) $R^{(k)} \cong R\backslash\{0\}$,

(ii) $R\backslash\{0\} \cong K$.

Proof. (i) There is a finite partition $R\backslash\{0\} = \bigcup_{\alpha} \alpha R^{(k)}$ with $v(\alpha) = 0$, say with s parts. Then $R\backslash\{0\}$ is a fortiori isomorphic to the union of s disjoint copies of $R^{(k)}$, which is by Proposition 1(ii) isomorphic to $R^{(k)}$.

(ii) The map

$$(\{0\} \times R) \cup \big(\{1\} \times (R\backslash\{0\})\big) \to K\colon \begin{cases} (0,x) \mapsto x, \\ (1,x) \mapsto 1/(\pi x), \end{cases}$$

is a well-defined isomorphism. It follows that K is isomorphic to the disjoint union of R and $R\backslash\{0\}$. Now use (i) and (iii) of Proposition 1. Q.E.D.

Give K^m the topology induced by the norm $|x| = \max(|x_i|_p)$ with $|x_i|_p = p^{-v(x_i)}$ for $x = (x_1, \ldots, x_m) \in K^m$. P. Scowcroft and L. van den Dries [6] proved there exists no isomorphism from an open set $A \subset K^m$ onto an open set $B \subset K^n$ with $n \neq m$, so we can define the dimension of semi-algebraic sets as follows.

Definition 1 ([6])**.** The *dimension* of a semi-algebraic set $X \neq \emptyset$ is the greatest natural number n such that we have a semi-algebraic subset $A \subset X$ and an isomorphism from A to a nonempty semi-algebraic open subset of K^n. We put $\dim(\emptyset) = -1$.

P. Scowcroft and L. van den Dries [6] proved many good properties of this dimension, for example that it is invariant under isomorphisms.

Proposition 2 ([6])**.** *Let A and B be semi-algebraic sets, then the following is true*:

(i) *If* $A \cong B$ *then* $\dim(A) = \dim(B)$.

(ii) $\dim(A \cup B) = \max\big(\dim(A), \dim(B)\big)$.

(iii) $\dim(A) = 0$ *if and only if A is finite nonempty.*

We will prove the converse of (i) for infinite semi-algebraic sets.

Lemma 2. *For any semi-algebraic set X of dimension* $m \in \mathbb{N}_0$ *there exists a semi-algebraic injection* $X \to K^m$.

Proof. By [6], Cor. 3.1 there is a finite partition of X such that each part A is isomorphic to a semi-algebraic open $A' \subset K^k$ for some $k \leqq m$. Now realize the disjoint union of the sets A' without going into higher embedding dimension (see the introduction). Q.E.D.

We formulate the p-adic Cell Decomposition Theorem by J. Denef [2], [3], which is the analogue of the real semi-algebraic Cell Decomposition Theorem.

Theorem 1 (Cell Decomposition [2], [3]). *Let $f_i(\hat{x}, x_m)$, $i = 1, \dots, r$, be polynomials in x_m with coefficients which are semi-algebraic functions in the variables $\hat{x} = (x_1, \dots, x_{m-1})$ from K^{m-1} to K. Let $n \in \mathbb{N}_0$ be fixed. Then there exists a finite partition of K^m into sets A of the form*

$$A = \{x \in K^m \mid \hat{x} \in D \text{ and } v(a_1(\hat{x})) \,\square_1\, v(x_m - c(\hat{x})) \,\square_2\, v(a_2(\hat{x}))\},$$

such that

$$f_i(x) = u_i(x)^n h_i(\hat{x}) (x_m - c(\hat{x}))^{v_i}, \quad \text{for each } x \in A,\ i = 1, \dots, r,$$

with $u_i(x)$ a unit in R for each x, $D \subset K^{m-1}$ semi-algebraic, $v_i \in \mathbb{N}$, h_i, a_1, a_2, c semi-algebraic functions from K^{m-1} to K and $\square_1$, $\square_2$ either $\leqq, <$, or no condition.

The next lemma is also due to J. Denef [3].

Lemma 3 ([3], Cor. 6.5). *Let $b: K^m \to K$ be a semi-algebraic function. Then there exists a finite partition of K^m such that for each part A we have $e > 0$ and polynomials $f_1, f_2 \in R[X_1, \dots, X_m]$ such that*

$$v(b(x)) = \frac{1}{e} v\left(\frac{f_1(x)}{f_2(x)}\right), \quad \text{for each } x \in A,$$

with $f_2(x) \neq 0$ for each $x \in A$.

We give an application of the Cell Decomposition Theorem and Lemma 3, inspired by similar applications in [3]. For details of the proof we refer to the proof of [3], Thm. 7.4. By λP_n with $\lambda = 0$ we mean $\{0\}$.

Lemma 4. *Let $X \subset K^m$ be semi-algebraic and $b_j: K^m \to K$ semi-algebraic functions for $j = 1, \dots, r$. Then there exists a finite partition of X s.t. each part A has the form*

$$A = \{x \in K^m \mid \hat{x} \in D, v(a_1(\hat{x})) \,\square_1\, v(x_m - c(\hat{x})) \,\square_2\, v(a_2(\hat{x})), x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have

$$v(b_j(x)) = \frac{1}{e_j} v((x_m - c(\hat{x}))^{\mu_j} d_j(\hat{x})),$$

with $\hat{x} = (x_1, \dots, x_{m-1})$, $D \subset K^{m-1}$ semi-algebraic, $e_j > 0$, $\mu_j \in \mathbb{Z}$, $\lambda \in K$, c, a_i, d_j semi-algebraic functions from K^{m-1} to K and $\square_i$ either $<$, $\leqq$ or no condition.

Proof. By Lemma 3 we have a finite partition of X such that for each part A_0 we have $e_j > 0$ and polynomials $g_j, g_j' \in R[X_1, \dots, X_m]$ with

$$v(b_j(x)) = \frac{1}{e_j} v\left(\frac{g_j(x)}{g_j'(x)}\right), \quad \text{for each } x \in A_0,\ j = 1, \dots, r.$$

Let f_i be the polynomials which appear in a description of A as a Boolean combination of sets of the form $\{x \in K^m \mid f(x) \in P_n\}$. Apply now the Cell Decomposition Theorem as in the proof of [3], Thm. 7.4 to the polynomials f_i, g_j and g_j' to obtain the lemma. Q.E.D.

The proof of the next proposition is an application of both the Cell Decomposition Theorem and some hidden Presburger arithmetic in the value group of K; it is the technical heart of this paper. If $l = 0$ then $\prod_{i=1}^{l} R^{(k)}$ denotes the set $\{0\}$.

Definition 2. We say that a semi-algebraic function $f: B \to K$ satisfies condition (1) (with constants e, μ_i, β) if we have constants $e \in \mathbb{N}_0$, $\mu_i \in \mathbb{Z}$, $\beta \in K$ such that each $x = (x_i) \in B$ satisfies

$$v(f(x)) = \frac{1}{e} v\left(\beta \prod_i x_i^{\mu_i}\right). \tag{1}$$

Proposition 3 (Rectilinearization). *Let X be a semi-algebraic set and $b_j: X \to K$ semi-algebraic functions for $j = 1, \ldots, r$. Then there exists a finite partition of X such that for each part A we have constants $l \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\mu_{ij} \in \mathbb{Z}$, $\beta_j \in K$, and an isomorphism*

$$f: \prod_{i=1}^{l} R^{(k)} \to A,$$

such that for each $x = (x_1, \ldots, x_l) \in \prod_{i=1}^{l} R^{(k)}$ we have

$$v(b_j \circ f(x)) = v\left(\beta_j \prod_{i=1}^{l} x_i^{\mu_{ij}}\right).$$

Proof. We work by induction on $m = \dim(X)$. Let $\dim(X) = 1$ and $b_j: X \to K$ semi-algebraic functions, $j = 1, \ldots, r$. By Lemma 2 we may suppose that $X \subset K$. We reduce first to the case that X and b_j have the special form (2) (see below). By Lemma 4 there is a partition such that each part A is either a point or of the form

$$A = \{x \in K \mid v(a_1) \square_1 v(x - c) \square_2 v(a_2), x - c \in \lambda P_n\},$$

and such that for each $x \in A$ we have $v(b_j(x)) = \dfrac{1}{e_j} v(\beta_j (x - c)^{\mu_j})$, with $a_i, c, \lambda, \beta_j \in K$, $e_j \in \mathbb{N}_0$ and $\mu_j \in \mathbb{Z}$. We may assume that $\lambda \neq 0$, $a_1 \neq 0 \neq a_2$, $\square_i$ is either $\leqq$ or no condition and since the translation

$$\{x \in K \mid v(a_1) \square_1 v(x) \square_2 v(a_2), x \in \lambda P_n\} \to A: x \mapsto x + c$$

is an isomorphism, we may also assume that $c = 0$. If both $\square_1$ and $\square_2$ are no condition we can partition A into parts $\{x \in A \mid 0 \leqq v(x)\}$ and $\{x \in A \mid v(x) \leqq -1\}$. It follows that if $\square_1$ is no condition we may suppose that $\square_2$ is $\leqq$, then we can apply the isomorphism

$$\left\{x \in K \mid v\left(\frac{1}{a_2}\right) \leqq v(x), x \in \frac{1}{\lambda} P_n\right\} \to A: x \mapsto \frac{1}{x},$$

and replace μ_j by $-\mu_j$. This shows we can reduce to the case that X has the form

$$X = \{x \in K \mid v(a_1) \leqq v(x) \square_2 v(a_2), x \in \lambda P_n\}, \tag{2}$$

with $a_1 \neq 0 \neq a_2$, $\lambda \neq 0$, $\square_2$ either $\leqq$ or no condition and $v(b_j(x)) = \frac{1}{e_j} v(\beta_j x^{\mu_j})$ for each $x \in X$.

Case 1: $\square_2$ is $\leqq$ (in equation (2)). By Hensel's Lemma we can partition X into finitely many parts of the form $y + \pi^s R$ for some fixed $s > v(a_2)$ and with $v(a_1) \leqq v(y) \leqq v(a_2)$ for each y. For each such part there is a finite partition $y + \pi^s R = \bigcup_{\gamma \in \Gamma} A_\gamma \cup \{y\}$, with $A_\gamma = y + \pi^s \gamma R^{(1)}$ and $v(\gamma) = 0$ for each γ. The functions f_γ: $R^{(1)} \to A_\gamma$: $x \mapsto y + \pi^s \gamma x$ are isomorphisms which satisfy $v(b_j \circ f_\gamma(x)) = \frac{1}{e_j} v(\beta_j y^{\mu_j})$ for all $x \in R^{(1)}$. This last expression is independent of x, so there exists $\beta_j' \in K$ such that $v(b_j \circ f_\gamma(x)) = v(\beta_j')$ for all $x \in R^{(1)}$. This proves Case 1.

Case 2: $\square_2$ is no condition (in equation (2)). The map

$$f_1\colon R \cap \lambda' P_n \to X\colon x \mapsto a_1 x,$$

with $\lambda' = \lambda / a_1$ is an isomorphism. Let n' be a common multiple of $e_1, \ldots, e_r$ and n. Choose $k > v(n')$ and put $k' = k + v(n')$. Let $R \cap \lambda' P_n = \bigcup_\gamma B_\gamma$ be a finite partition, with $B_\gamma = \gamma(R \cap P_{n'}^{(k')})$ and $0 \leqq v(\gamma) < n'$. Now we have that the map f_γ: $R^{(k)} \to B_\gamma$: $x \mapsto \gamma x^{n'}$ is an isomorphism by Corollary 1. Let g_γ be the semi-algebraic function $f_1 \circ f_\gamma$, which is an isomorphism from $R^{(k)}$ onto a semi-algebraic set $A_\gamma \subset X$. The sets A_γ form a finite partition of X. Put $\mu_j' = \mu_j n'/e_j$, then we have for each $x \in R^{(k)}$ that

$$\begin{aligned} v(b_j \circ g_\gamma(x)) &= \frac{1}{e_j} v\big(\beta_j (a_1 \gamma x^{n'})^{\mu_j}\big) = \frac{1}{e_j} v\big(\beta_j (a_1 \gamma)^{\mu_j}\big) + v(x^{\mu_j'}) \\ &= v(\beta_j') + v(x^{\mu_j'}) = v(\beta_j' x^{\mu_j'}), \end{aligned}$$

with β_j' in K such that $v(\beta_j') = \frac{1}{e_j} v\big(\beta_j (a_1 \gamma)^{\mu_j}\big)$. This proves Case 2.

Now let $\dim(X) = m > 1$ and let b_j: $X \to K$ be semi-algebraic functions, $j = 1, \ldots, r$. By Lemma 2 we may suppose that $X \subset K^m$.

Claim. *We can partition X such that for each part A we have an isomorphism of the form $f\colon D_1 \times D_{m-1} \to A$, with $D_1 \subset K$ and $D_{m-1} \subset K^{m-1}$ semi-algebraic, such that the functions $b_j \circ f$ satisfy condition* (1), *i.e. there are constants $e_j \in \mathbb{N}_0$, $\mu_{ij} \in \mathbb{Z}$, $\beta_j \in K$ such that each $x = (x_i) \in D_1 \times D_{m-1}$ satisfies*

$$v(b_j \circ f(x)) = \frac{1}{e_j} v\left(\beta_j \prod_i x_i^{\mu_{ij}}\right).$$

If the claim is true, we can apply the induction hypotheses once to D_1 and the functions $x_1 \mapsto x_1^{\mu_{1j}}$ and once to D_{m-1} and the functions $(x_2, \ldots, x_m) \mapsto \beta_j \prod_{i=2}^{m} x_i^{\mu_{ij}}$ for

$j = 1, \ldots, r$. It follows easily that we can partition X such that for each part A there is an isomorphism $f\colon \prod_i R^{(k)} \to A$ such that all $f \circ b_j$ satisfy condition (1) with constants e'_j, μ'_{ij} and β'_j. Now we can proceed as in Case 2 for $m = 1$ to make all $e'_j \in \mathbb{N}_0$ occurring in condition (1) equal to 1. The proposition follows now immediately.

Proof of the claim. First we show we can reduce to the case described in equation (3) below. Using Lemma 4 and its notation we find a finite partition of X such that each part A has the form

$$A = \{x \in K^m \mid \hat{x} \in D, v(a_1(\hat{x})) \square_1 v(x_m - c(\hat{x})) \square_2 v(a_2(\hat{x})), x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have $v(b_j(x)) = \dfrac{1}{e_j} v((x_m - c(\hat{x}))^{\mu_{mj}} d_j(\hat{x}))$, with $\mu_{mj} \in \mathbb{Z}$. Similar as for $m = 1$, we may suppose that $c(\hat{x}) = 0$ for all $\hat{x}$. Apply now the induction hypotheses to the set $D \subset K^{m-1}$ and the functions a_1, a_2, d_j. We find a finite partition of A such that for each part A' we have an isomorphism $f\colon B \to A'$, where B is a set of the form

$$B = \left\{x \in K^m \mid \hat{x} \in D', v\left(\alpha_1 \prod_{i=1}^{l} x_i^{\eta_i}\right) \square_1 v(x_m) \square_2 v\left(\alpha_2 \prod_{i=1}^{l} x_i^{\varepsilon_i}\right), x_m \in \lambda P_n\right\},$$

with $D' = \prod_{i=1}^{l} R^{(k)}$, $l \leqq m - 1$, such that each $b_j \circ f$ satisfies condition (1). We will alternately partition further and apply isomorphisms to the parts which compositions with b_j will always satisfy condition (1). By the induction hypotheses we may suppose that $\lambda \neq 0$ and $\dim(D') = m - 1$, i.e. $D' = \prod_{i=1}^{m-1} R^{(k)}$. Analogously as for $m = 1$ we may suppose that $\alpha_1 \neq 0 \neq \alpha_2$, $\square_2$ is either $\leqq$ or no condition and $\square_1$ is the symbol $\leqq$ (possibly after partitioning or applying $x \mapsto (x_1, \ldots, x_{m-1}, 1/x_m)$).

Choose $\bar{k} > v(n)$ and put $k' = \bar{k} + v(n)$. We may suppose that $k' > k$, so we have a finite partition $B = \bigcup_\gamma B_\gamma$ with $\gamma = (\gamma_1, \ldots, \gamma_m) \in K^m$, $0 \leqq v(\gamma_i) < n$ and

$$B_\gamma = \{x \in B \mid x_i \in \gamma_i P_n^{(k')}\}.$$

Now we have isomorphisms

$$f_\gamma\colon C_\gamma \to B_\gamma\colon x \mapsto (\gamma_1 x_1^n, \ldots, \gamma_m x_m^n),$$

with

$$C_\gamma = \left\{x \in \left(\prod_{i=1}^{m-1} R^{(\bar{k})}\right) \times K^{(\bar{k})} \mid v\left(\alpha'_1 \prod_{i=1}^{m-1} x_i^{\eta_i}\right) \leqq v(x_m) \square_2 v\left(\alpha'_2 \prod_{i=1}^{m-1} x_i^{\varepsilon_i}\right)\right\},$$

for appropriate choice of $\alpha'_i \in K$. Put $v_i = \varepsilon_i - \eta_i$, $\beta = \alpha'_2/\alpha'_1$, then we have the isomorphism

$$\left\{x \in \prod_{i=1}^{m} R^{(\bar{k})} \mid v(x_m) \square_2 v\left(\beta \prod_{i=1}^{m-1} x_i^{v_i}\right)\right\} \to C_\gamma\colon x \mapsto \left(x_1, \ldots, x_{m-1}, \alpha'_1 x_m \prod_{i=1}^{m-1} x_i^{\eta_i}\right).$$

If $\square_2$ is no condition, the claim is trivial. It follows that we can reduce to the case that we have an isomorphism

$$f\colon E = \left\{ x \in \prod_{i=1}^{m} R^{(\bar{k})} \mid v(x_m) \leqq v\left(\beta \prod_{i=1}^{m-1} x_i^{v_i}\right)\right\} \to X \tag{3}$$

with $\beta \neq 0$, $\bar{k} \in \mathbb{N}_0$ and $v \in \mathbb{Z}$, such that each $b_j \circ f$ satisfies condition (1).

Suppose we are in the case described in (3). If $v_i \leqq 0$ for $i = 1, \ldots, m-1$ then we have a finite partition $E = \bigcup_s E^{=s}$, with $s \in \{0, 1, \ldots, v(\beta)\}$ and $E^{=s} = \{x \in E \mid v(x_m) = s\}$. Also, $E^{=s} = \{(x_1, \ldots, x_{m-1}) \mid \exists x_m\ (x_1, \ldots, x_m) \in E^{=s}\} \times \{x_m \in R^{(\bar{k})} \mid v(x_m) = s\}$ and the claim follows.

Suppose now that $v_1 > 0$ in (3). First we prove the proposition when $v_1 = 1$, using some implicit Presburger arithmetic on the value group. We can partition E into parts E_1 and E_2, with

$$E_1 = \left\{ x \in E \mid v(x_m) < v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right)\right\},$$

$$E_2 = \left\{ x \in E \mid v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right) \leqq v(x_m)\right\}$$

$$= \left\{ x \in \prod_{i=1}^{m} R^{(\bar{k})} \mid v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right) \leqq v(x_m) \leqq v\left(\beta x_1 \prod_{i=2}^{m-1} x_i^{v_i}\right)\right\}.$$

Since $v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right) \leqq v\left(x_1 \beta \prod_{i=2}^{m-1} x_i^{v_i}\right)$ for $x \in E_1$, it follows that

$$E_1 = R^{(\bar{k})} \times \left\{ (x_2, \ldots, x_m) \in \prod_{i=2}^{m} R^{(\bar{k})} \mid v(x_m) < v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right)\right\},$$

and the restrictions $b_j \circ f \mid E_1$ satisfy condition (1).

As for E_2, let D_{m-1} be the set

$$D_{m-1} = \left\{ (x_2, \ldots, x_m) \in \prod_{i=2}^{m} R^{(\bar{k})} \mid v\left(\beta \prod_{i=2}^{m-1} x_i^{v_i}\right) \leqq v(x_m)\right\}.$$

We may suppose that $\beta \in K^{(\bar{k})}$, then the map

$$R^{(\bar{k})} \times D_{m-1} \to E_2\colon x \mapsto \left(\frac{x_1 x_m}{\beta \prod_{i=2}^{m-1} x_i^{v_i}}, x_2, \ldots, x_m \right),$$

can be checked by elementary Presburger arithmetic to be an isomorphism. This proves the claim when $v_1 = 1$.

Suppose now that X is of the form described in (3) and $v_1 > 1$. We prove we can reduce to the case $v_1 = 1$ by partitioning and applying appropriate power maps. Choose $\tilde{k} > v(v_1)$ and put $\tilde{k}' = \tilde{k} + v(v_1)$. We may suppose that $\tilde{k} \geqq \bar{k}$, so we have a finite partition $E = \bigcup_\alpha E_\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_m) \in K^m$, $v(\alpha_1) = 0$, $0 \leqq v(\alpha_i) < v_1$ for $i = 2, \dots, m$ and

$$E_\alpha = \{x \in E \mid x_1 \in \alpha_1 R^{(\tilde{k})}, x_i \in \alpha_i P_{v_1}^{(\tilde{k}')} \text{ for } i = 2, \dots, m\}.$$

By Corollary 1 we have isomorphisms

$$f_\alpha\colon C_\alpha \to E_\alpha\colon x \mapsto (\alpha_1 x_1, \alpha_2 x_2^{v_1}, \dots, \alpha_m x_m^{v_1}),$$

with $C_\alpha = \left\{ x \in \prod_{i=1}^m R^{(\tilde{k})} \mid v(x_m) \leqq v(\beta' x_1 \prod_{i=2}^{m-1} x_i^{v_i}) \right\}$, where $\beta' \in K^\times$ depends on α. This reduces the problem to the case described in (3) with $v_1 = 1$ and thus the proposition is proved. Q.E.D.

Remark 1. Proposition 3 can be strengthened, adding conditions on the Jacobians of the isomorphisms, to become useful to p-adic integration theory.

Theorem 2. *Let X be a semi-algebraic set, then either X is finite or there exists a semi-algebraic bijection $X \to K^k$ with $k \in \mathbb{N}_0$ the dimension of X.*

Proof. We give a proof by induction on $\dim(X) = m$. Let $\dim(X) = 1$. Use Proposition 3 to partition X such that each part is isomorphic to either $R^{(k)}$ or a point. By combining the isomorphisms of Proposition 1 and Corollary 2, it follows that $X \cong K$.

Now suppose $\dim(X) = m > 1$. Proposition 3 together with the case $m = 1$ implies that we can finitely partition X such that each part is isomorphic to K^l, for some $l \in \{0, \dots, m\}$, with $K^0 = \{0\}$. By Proposition 2 at least one part must be isomorphic to K^m. Suppose that A and B are disjoint parts, such that $A \cong K^l$ and $B \cong K^m$, with $l \in \{0, \dots, m\}$. It is enough to prove that $A \cup B \cong K^m$. First suppose that $l = 0$, so A is a singleton $\{a\}$. Since $m > 1$ there exists an injective semi-algebraic function $i\colon R \to A \cup B$ such that $i(R \backslash \{0\}) \subset B$ and $i(0) = a$. It follows that $A \cup B \cong B \cong K^m$ since $R \cong R \backslash \{0\}$ (Proposition 1). If $1 \leqq l$ we have $A \cup B \cong K \times (A' \cup B')$, for some disjoint sets $A' \cong K^{l-1}$ and $B' \cong K^{m-1}$. By induction we find $A' \cup B' \cong K^{m-1}$ and thus $A \cup B \cong K^m$. This proves Theorem 2. Q.E.D.

We obtain as a corollary of Theorem 2 the following classification of the p-adic semi-algebraic sets.

Corollary 3. *Two infinite semi-algebraic sets are isomorphic if and only if they have the same dimension.*

Remark 2. The field $\mathbb{F}_q((t))$ of Laurent series over the finite field is often considered to be the characteristic p counterpart of the p-adic numbers. D. Haskell and the author [1] proved that the Grothendieck ring of $\mathbb{F}_q((t))$ in the language of rings is trivial, analogously as for $\mathbb{Q}_p$. In [1], proof of Thm. 2, it is also shown that there is a definable injection from the plane $\mathbb{F}_q((t))^2$ into the line $\mathbb{F}_q((t))$, which makes it plausible we cannot define an invariant for this field which behaves like a (good) dimension. It is an open question to classify the definable sets of $\mathbb{F}_q((t))$ up to definable bijection.

References

[1] *Cluckers, R., Haskell, D.*, Grothendieck rings of $\mathbb{Z}$-valued fields, Bull. Symb. Logic **7** (2001), 262–269.
[2] *Denef, J.*, p-adic semi-algebraic sets and cell decomposition, J. reine angew. Math. **369** (1986), 154–166.
[3] *Denef, J.*, The rationality of the Poincaré series associated to the p-adic points on a variety, Invent. Math. **77** (1984), 1–23.
[4] *van den Dries, L.*, Tame topology and o-minimal structures, LMS Lect. Note Ser. **248** CUP (1998).
[5] *Macintyre, A.*, On definable subsets of p-adic fields, J. Symb. Logic **41** (1976), 605–610.
[6] *Scowcroft, P., van den Dries, L.*, On the structure of semialgebraic sets over p-adic fields, J. Symb. Logic **53** (1988), no. 4, 1138–1164.

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