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Rectilinearization of semi-algebraic p-adic sets and Denef's rationality of Poincaré series

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Abstract

In [R. Cluckers, Classification of semi-algebraic sets up to semi-algebraic bijection, J. Reine Angew. Math. 540 (2001) 105–114], it is shown that a p-adic semi-algebraic set can be partitioned in such a way that each part is semi-algebraically isomorphic to a Cartesian product $\prod_{i=1}^{l} R^{(k)}$ where the sets $R^{(k)}$ are very basic subsets of $\mathbb{Q}_p$. It is suggested in [R. Cluckers, Classification of semi-algebraic sets up to semi-algebraic bijection, J. Reine Angew. Math. 540 (2001) 105–114] that this result can be adapted to become useful to p-adic integration theory, by controlling the Jacobians of the occurring isomorphisms. In this paper we show that the isomorphisms can be chosen in such a way that the valuations of their Jacobians equal the valuations of products of coordinate functions, hence obtaining a kind of explicit p-adic resolution of singularities for semi-algebraic p-adic functions. We do this by restricting the used isomorphisms to a few specific types of functions, and by controlling the order in which they appear. This leads to an alternative proof of the rationality of the Poincaré series associated to the p-adic points on a variety, as proven by Denef in [J. Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, Invent. Math. 77 (1984) 1–23].

Keywords: p-Adic integration; p-Adic cell decomposition; Rectilinearization; Rationality of Serre Poincaré series; Igusa local zeta function

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1. Introduction

We give an alternative, selfcontained proof of the main theorem of [3] by Denef on the rationality of the Poincaré series associated to the p-adic points on a variety. As one of the two proofs given in [3], it uses p-adic cell decomposition (a selfcontained proof of which is given in [4]), but it uses almost no p-adic integration theory. Instead, our approach uses the rectilinearization result proven in this paper. This result refines a similar rectilinearization result of [1] which says that any semi-algebraic p-adic set can be partitioned into finitely many parts each of which is semi-algebraically isomorphic to a Cartesian power of basic subsets $\mathbb{Z}_p^{(k)}$ of $\mathbb{Q}_p$, where $\mathbb{Z}_p^{(k)}$ is the set of p-adic integers (of any nonnegative order) having coefficients $1, 0, \ldots, 0$ in p-adic expansion, with $k-1$ zeros, see (1). The rectilinearization result of this paper improves this result of [1] by controlling the order of the Jacobian of the occurring isomorphisms. Namely, the order of the Jacobian equals the order of a monomial with integer powers. Since by [1] also the norm of the pullback of given semi-algebraic functions is the norm of a monomial, we can pullback p-adic semi-algebraic integrals to simpler p-adic integrals, consisting of an integrand which is (the norm of) a monomial integrated over a Cartesian power of some $\mathbb{Z}_p^{(k)}$. Since the measure of $\mathbb{Z}_p^{(k)}$ is easily calculable, since integrating a monomial over a Cartesian product of $\mathbb{Z}_p^{(k)}$ is a product of integrals of monomials in one variable over the factors $\mathbb{Z}_p^{(k)}$, the pulled back integral is trivial what concerns integrability as well as its calculation. Since the Denef–Serre Poincaré series as well as the other Poincaré series of [3] can be written as a semi-algebraic p-adic integral in the form of (7), cf. [3], the rationality of these series follows by calculating the pullback of this integral along a rectilinearization.

1.1. Notation and terminology

Let p denote a fixed prime number, $\mathbb{Q}_p$ the field of p-adic numbers and K a fixed finite field extension of $\mathbb{Q}_p$. For $x \in K$ let $v(x) \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of x. Let $R = \{x \in K \mid v(x) \geqslant 0\}$ be the valuation ring, and let q_K denote the cardinality of the residue field of K. Write $\mathbb{N}$ for the nonnegative integers and $\mathbb{N}_0$ for $\mathbb{N} \setminus \{0\}$. Put $K^\times = K \setminus \{0\}$ and for $n \in \mathbb{N}_0$ let P_n be the set $\{x \in K^\times \mid \exists y \in K\colon\ y^n = x\}$. We call a subset of K^n *semi-algebraic* if it is a Boolean combination (i.e. obtained by taking finite unions, complements and finite intersections) of sets of the form $\{x \in K^m \mid f(x) \in P_n\}$, with $f(x) \in K[X_1, \ldots, X_m]$. The collection of semi-algebraic sets is closed under taking images under projections $K^m \to K^{m-1}$; further we have that sets of the form $\{x \in K^m \mid v(f(x)) \leqslant v(g(x))\}$ with $f(x), g(x) \in K[X_1, \ldots, X_m]$ are semi-algebraic (see [4] and [5]). A function $f: A \to B$ is semi-algebraic if its graph is a semi-algebraic set; if further f is a bijection, we call f an *isomorphism*. By a *finite partition* of a semi-algebraic set we mean a partition into finitely many semi-algebraic sets.

Let π be a fixed element of R with $v(\pi) = 1$, thus π is a uniformizing parameter for R. For a semi-algebraic set $X \subset K$ and $k > 0$ we write

$$X^{(k)} = \bigl\{x \in X \bigm| x \neq 0 \text{ and } v\bigl(\pi^{-v(x)}x - 1\bigr) \geqslant k\bigr\}, \tag{1}$$

which is semi-algebraic (see [4, Lemma 2.1]); $X^{(k)}$ consists of those points $x \in X$ which have a p-adic expansion $x = \sum_{i=s}^{\infty} a_i \pi^i$ with $a_s = 1$ and $a_i = 0$ for $i = s+1, \ldots, s+k-1$.

We recall a form of Hensel's lemma and a corollary.

Lemma 1 *(Hensel). Let $f(t)$ be a polynomial over R in one variable t, and let $\alpha \in R$, $e \in \mathbb{N}$. Suppose that $v(f(\alpha)) > 2e$ and $v(f'(\alpha)) \leqslant e$, where f' denotes the derivative of f. Then there exists a unique $\bar{\alpha} \in R$ such that $f(\bar{\alpha}) = 0$ and $v(\bar{\alpha} - \alpha) > e$.*

Corollary 2. *(See [1].) Let $n > 1$ be a natural number. For each $k > v(n)$, and $k' = k + v(n)$ the function $K^{(k)} \to P_n^{(k')}$: $x \mapsto x^n$ is an isomorphism.*

The following definitions are central.

Definition 3. We say that a semi-algebraic function $f : A \subseteq K^n \to K$ satisfies condition (2) (with constants μ_i, β) if we have constants $\mu_i \in \mathbb{Z}$, $\beta \in K$ such that each $x = (x_i) \in A$ satisfies $x_i \neq 0$ if $\mu_i < 0$ and

$$v\big(f(x)\big) = v\left(\beta \prod_i x_i^{\mu_i}\right). \tag{2}$$

We say that a semi-algebraic function g from an open set $B \subseteq K^n \to K^n$ satisfies condition (3) (with constants μ_i, β) if g is C^1 on B and there exist constants $\mu_i \in \mathbb{Z}$, $\beta \in K$ such that each $x = (x_i) \in B$ satisfies $x_i \neq 0$ if $\mu_i < 0$ and

$$v\big(\mathrm{Jac}\big(g(x)\big)\big) = v\left(\beta \prod_i x_i^{\mu_i}\right). \tag{3}$$

Definition 4. Let $f : X \to Y$ be a semi-algebraic isomorphism. Say that the isomorphism f is of type f_0 when f equals an isomorphism of the following kind:

$$f_0 : X \subset K^n \to Y \subset K^{n+1}: \quad x \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

for some $n \geqslant 0$.

Say that the isomorphism f is of type f_1, respectively of type f_2, t_c or t, when $X \subset K^n$, $Y \subset K^n$ for some $n \geqslant 0$, and f equals an isomorphism of the following kind:

$$\begin{aligned}
f_1 : X \to Y: &\quad x \mapsto \big(\alpha_1 x_1^{a_1}, \ldots, \alpha_n x_n^{a_n}\big),\\
f_2 : X \to Y: &\quad x \mapsto \left(x_1, \ldots, x_{i-1}, x_i \prod_{j \neq i} x_j^{b_j}, x_{i+1}, \ldots, x_n\right),\\
t_c : X \to Y: &\quad x \mapsto \big(x_1, \ldots, x_{i-1}, x_i + c(x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_n\big), \quad \text{or}\\
t : X \to Y: &\quad x \mapsto (x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n),
\end{aligned}$$

with $a_i, b_j \in \mathbb{Z}$, $a_i < 0$ implies $x_i \neq 0$ for $x \in X$, $b_j < 0$ implies $x_j \neq 0$ for $x \in X$, $\alpha_i, c_i \in K$, $x = (x_1, \ldots, x_n)$, c a semi-algebraic function which is moreover C^1 in the case that X is open in K^n, and such that for each j and each $x \in X$, $c_j = 0$ or $v(c_j) < v(x_j)$ holds.

Remark 5. If a function $g = (g_1, \ldots, g_m) : X \subset K^n \to Y \subset K^m$ is a composition of functions of type f_0, f_1, f_2, and t, then the component functions g_i all satisfy condition (2).

2. Rectilinearization with good Jacobians

Using cell decomposition in the form of Denef [3] or [4], the following can be easily derived, cf. [1, Lemma 4], or [2].

Lemma 6. *Let $X \subset K^m$ be semi-algebraic and $b_j : K^m \to K$ semi-algebraic functions for $j = 1, \dots, r$. Then there exists a finite partition of X s.t. each part A has the form*

$$A = \{x \in K^m \mid \hat{x} \in D,\ v(a_1(\hat{x})) \,\square_1\, v(x_m - c(\hat{x})) \,\square_2\, v(a_2(\hat{x})),\ x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have

$$v(b_j(x)) = \frac{1}{n} v\big((x_m - c(\hat{x}))^{\mu_j} d_j(\hat{x})\big),$$

with $\hat{x} = (x_1, \dots, x_{m-1})$, $D \subset K^{m-1}$ semi-algebraic, the set A projecting surjectively onto D, $n > 0$, $\mu_j \in \mathbb{Z}$, $\lambda \in K$, $c, d_j : K^{m-1} \to K$ and $a_i : K^{m-1} \to K^\times$ semi-algebraic functions, and each $\square_i$ either $\leqslant$ or no condition. If $\lambda = 0$, we use the conventions $\mu_j = 0$ and $0^0 = 1$, and thus $v(b_j(x)) = \frac{1}{n} v(d_j(\hat{x}))$.

Moreover, D has the structure of a K-analytic manifold and the functions c, a_i and d_j are K-analytic on D.

We can now state our main result. If $l = 0$, then $\prod_{i=1}^{l} R^{(k)}$ denotes the set $\{0\}$.

Theorem 7 *(Rectilinearization with good Jacobians). Let $X \subset K^m$ be a semi-algebraic set and $b_j : X \to K$ semi-algebraic functions for $j = 1, \dots, r$. Then there exists a finite partition of X such that for each part A we have constants $l \in \mathbb{N}$, $k \in \mathbb{N}_0$, and an isomorphism*

$$f : \prod_{i=1}^{l} R^{(k)} \to A,$$

such that the functions $b_j \circ f$ satisfy condition (2). *Moreover, for each part A with isomorphism f with $l = m$ we can ensure that f satisfies condition* (3).

The first author proved a similar result in [1, Proposition 3], but without condition (3) for the isomorphisms f. Theorem 7 can be understood as an embedded resolution of singularities for the p-adic semi-algebraic functions b_j. We will prove Theorem 7 using the following stronger result.

Theorem 8 *(Rectilinearization with basic functions). With the notation and in the conclusion of Theorem 7, we can moreover ensure that for each part A we have an isomorphism of the form*

$$f : \prod_{i=1}^{l} R^{(k)} \to A: \quad x \mapsto (T_c \circ g)(x),$$

with g a composition of bijective maps of the types f_0, f_1, f_2, t, and T_c a composition of functions of type t_c.

Proof of Theorem 8. Recall that $X \subset K^m$. We work by induction on m.

First we note a generality. Say that a semi-algebraic function $h : B \subset K^\ell \to K$ satisfies condition (4) when there are $n > 0$, $\beta \in K$ and $\mu_i \in \mathbb{Z}$ such that

$$v\big(h(x)\big) = \frac{1}{n} v\left(\beta \prod_i x_i^{\mu_i}\right) \quad \text{for all } x \text{ in } B, \tag{4}$$

and such that $x_i \neq 0$ for all $x \in B$ when $\mu_i < 0$. When moreover $B = \prod_{i=1}^{\ell} R^{(k)}$ for some k, then condition (4) for h implies that h satisfies condition (2). Indeed, since h takes values in K and K has $\mathbb{Z}$ as value group, and since $v(x_i)$ runs over $\mathbb{N}$, μ_i and $v(\beta)$ must be divisible by n.

Let $X \subset K$ and $b_j : X \to K$ be semi-algebraic functions, $j = 1, \ldots, r$. By Lemma 6 there is a partition such that each part A is of the form

$$A = \big\{x \in K \bigm| v(a_1) \,\square_1\, v(x - c) \,\square_2\, v(a_2),\ x - c \in \lambda P_n\big\},$$

and such that for each $x \in A$ we have $v(b_j(x)) = \frac{1}{n} v(\beta_j (x - c)^{\mu_j})$, with $c, \lambda, \beta_j \in K$, $a_i \in K^\times$, $n > 0$, $\mu_j \in \mathbb{Z}$, and each $\square_i$ either $\leqslant$ or no condition. If A is a singleton (that is, $\lambda = 0$), then the desired isomorphism $\{0\} = \prod_{i=1}^{0} R^{(1)} \to A$ is easily constructed. Now assume that $\lambda \neq 0$. Since we build the isomorphisms f in the reverse order (that is, beginning with an isomorphism onto A), we first use a translation of type t_c,

$$\bar{t}_c : A_{c=0} \to A: \quad x \mapsto x + c$$

with

$$A_{c=0} := \big\{x \in K \bigm| v(a_1) \,\square_1\, v(x) \,\square_2\, v(a_2),\ x \in \lambda P_n\big\}.$$

Note that $v(b_j \circ \bar{t}_c(x)) = \frac{1}{n} v(\beta_j x^{\mu_j})$ for each $x \in A_{c=0}$, hence the functions $b_j \circ \bar{t}_c$ satisfy condition (4).

We will now reduce to the case where $\square_1$ is $\leqslant$, thus to sets of the form B:

$$B = \big\{x \in K \bigm| v(a_1) \leqslant v(x) \,\square_2\, v(a_2),\ x \in \lambda P_n\big\}, \tag{5}$$

with $\square_2$ either $\leqslant$ or no condition. If $\square_1$ is no condition and $\square_2$ is $\leqslant$, we can apply the isomorphism

$$f_1^{(1)} : \left\{x \in K \Bigm| v\left(\frac{1}{a_2}\right) \leqslant v(x),\ x \in \frac{1}{\lambda} P_n\right\} \to A_{c=0}: \quad x \mapsto \frac{1}{x},$$

and replace μ_j by $-\mu_j$. When both $\square_1$ and $\square_2$ are no condition, the set $A_{c=0}$ can be partitioned in parts

$$A_{c=0}^1 = \big\{x \in A_{c=0} \bigm| 0 \leqslant v(x)\big\},$$
$$A_{c=0}^2 = \big\{x \in A_{c=0} \bigm| v(x) \leqslant -1\big\}.$$

The set $A^1_{c=0}$ now has the desired form. For the set $A^2_{c=0}$ we can proceed as in the previous case and use an isomorphism similar to $f_1^{(1)}$.

We have thus reduced to the case that we have a partition of X in parts A such that for each part there exists an isomorphism f_B from a set B as in (5) onto A, with f_B a composition of an isomorphism of type t_c with an isomorphism of type f_1, and such that clearly condition (4) holds for the functions $b_j \circ f_B$.

Case 1: $\square_2$ *is* $\leqslant$ *in (5).* By Hensel's lemma we can partition B into finitely many parts of the form $y + \pi^s R$ for some fixed $s > v(a_2)$ and with $v(a_1) \leqslant v(y) \leqslant v(a_2)$ for each y (see Example 9). For each such part there is a finite partition $y + \pi^s R = \bigcup_{\gamma\in\Gamma} B_\gamma \cup \{y\}$, with $B_\gamma = y + \pi^s \gamma R^{(1)}$ and $v(\gamma) = 0$ for each γ. Now the functions

$$f_\gamma : R^{(1)} \to B_\gamma: \quad x \mapsto y + \pi^s \gamma x$$

are isomorphisms between $R^{(1)}$ and the sets B_γ. Note that f_γ equals $\bar{t} \circ f_1^{(2)}$ with

$$f_1^{(2)} : x \mapsto \pi^s \gamma x \quad \text{and} \quad \bar{t} : x \mapsto x + y,$$

where $v(\bar{t}(x)) = v(y)$ since $v(y) < s$ and $x \in \pi^s \gamma R^{(1)}$.

Put $f := f_B \circ f_\gamma$. Then f is as desired and the functions $b_j \circ f$ clearly satisfy condition (4) and thus also (2) by the generality in the beginning of the proof. This proves Case 1.

Case 2: $\square_2$ *is no condition in (5).* The map

$$f_1^{(3)} : R \cap \lambda' P_n \to B: \quad x \mapsto a_1 x,$$

with $\lambda' = \lambda / a_1$ is an isomorphism. Choose $k > v(n)$ and put $k' = k + v(n)$. Let $R \cap \lambda' P_n = \bigcup_\gamma C_\gamma$ be a finite partition, with $C_\gamma = \gamma(R \cap P_n^{(k')})$ and $0 \leqslant v(\gamma) < n$. Now we have that the map $f_1^{(\gamma)} : R^{(k)} \to C_\gamma$: $x \mapsto \gamma x^n$ is an isomorphism by Corollary 2. Let g_γ be the semi-algebraic function $f_1^{(3)} \circ f_1^{(\gamma)}$, which is an isomorphism from $R^{(k)}$ onto a semi-algebraic set $B_\gamma \subset B$. The sets B_γ form a finite partition of B. This induces a finite partition of X in parts A_γ and isomorphisms $f = f_B \circ g_\gamma : R^{(k)} \to A_\gamma$.

Clearly the $b_j \circ f$ satisfy condition (2), since they satisfy condition (4). Since g_γ is a composition of isomorphisms of type f_1, f is as desired. This proves Case 2 and thus the case $m = 1$ is proved.

Now let $m > 1$. We will show that we can reduce to the case described in Eq. (6) below. The theorem then follows by Lemma 11. Using Lemma 6 and its notation, we find a finite partition of X such that each part A has the form

$$A = \bigl\{x \in K^m \bigm| \hat{x} \in D,\ v\bigl(a_1(\hat{x})\bigr) \,\square_1\, v\bigl(x_m - c(\hat{x})\bigr) \,\square_2\, v\bigl(a_2(\hat{x})\bigr),\ x_m - c(\hat{x}) \in \lambda P_n\bigr\},$$

and such that for each $x \in A$ we have $v(b_j(x)) = \frac{1}{n} v((x_m - c(\hat{x}))^{\mu_{mj}} d_j(\hat{x}))$, with $\mu_{mj} \in \mathbb{Z}$.

First we use a translation $t_c^{(m)}: A_{c=0} \to A$: $x \mapsto (x_1, \ldots, x_m + c(\hat{x}))$ to get rid of $c(\hat{x})$, as in the case $m = 1$. If we apply the induction hypotheses to the set $D \subset K^{m-1}$ and the functions a_1, a_2, d_j, we get a partition of D in parts $\tilde{D}$ and functions

$$\tilde{f}: \prod_{i=1}^{l} R^{(k)} \to \tilde{D}: \quad (x_1, \ldots, x_l) \mapsto (\tilde{T}_c \circ \tilde{g})(x_1, \ldots, x_l),$$

with $\tilde{T}_c$ a composition of functions of type t_c and $\tilde{g}$ a composition of functions of types f_0, f_1, f_2, t. This induces a finite partition of $A_{c=0}$, such that for each part $\tilde{A}_{c=0}$ there is an isomorphism of the form

$$\tilde{f}': B \to \tilde{A}_{c=0}: \quad x \mapsto \big(\tilde{f}(x_1, \ldots, x_l), x_{l+1}\big).$$

Here B is a set of the form

$$B = \left\{ x \in \prod_{i=1}^{l} R^{(k)} \times \lambda P_n \;\middle|\; v\left(\alpha_1 \prod_{i=1}^{l} x_i^{\eta_i}\right) \square_1 \, v(x_{l+1}) \, \square_2 \, v\left(\alpha_2 \prod_{i=1}^{l} x_i^{\varepsilon_i}\right)\right\}.$$

If we compose the functions $\tilde{f}'$ with the translation $t_c^{(m)}$, we obtain isomorphisms of the form

$$f = t_c^{(m)} \circ \tilde{f}': B \to A_B: \quad (x_1, \ldots, x_l, x_{l+1}) \mapsto \big((\tilde{T}_c \circ \tilde{g})(x_1, \ldots, x_l), x_{l+1} + c(x_1, \ldots, x_l)\big)$$

between sets B and sets $A_B \subset X$. The sets A_B form a finite partition of X. Clearly the $b_j \circ f$ satisfy condition (4). We will show that we only need functions of type f_0, f_1, f_2 and t to rectilinearize B. Thus the final isomorphisms $\prod R^{(k)} \to A$ will have the form $x \mapsto (T_c \circ g)(x)$, where every translation contained in g is of type t, and T_c is a composition of functions of type t_c.

If $\lambda = 0$, then $B = \prod_{i=1}^{l} R^{(k)} \times \{0\}$. In this case our isomorphism has the form

$$\prod_{i=1}^{l} R^{(k)} \to A: \quad x \mapsto (T_c \circ g \circ f_0)(x).$$

Recall that by Lemma 6, $\alpha_1 \neq 0 \neq \alpha_2$. From now on suppose that $\lambda \neq 0$.

Analogously as for $m = 1$ we may suppose that $\square_2$ is either $\leqslant$ or no condition and $\square_1$ is the symbol $\leqslant$ (possibly after partitioning or applying $x \mapsto (x_1, \ldots, x_l, 1/x_{l+1})$).

Choose $\bar{k} > v(n)$ and put $k' = \bar{k} + v(n)$. We may suppose that $k' > k$, so we have a finite partition $B = \bigcup_\gamma B_\gamma$ with $\gamma = (\gamma_1, \ldots, \gamma_{l+1}) \in K^{l+1}$, $0 \leqslant v(\gamma_i) < n$ and

$$B_\gamma = \big\{ x \in B \;\big|\; x_i \in \gamma_i P_n^{(k')}, \text{ for } i = 1, \ldots, l+1 \big\}.$$

Now we have isomorphisms

$$f_\gamma: C_\gamma \to B_\gamma: \quad x \mapsto \big(\gamma_1 x_1^n, \ldots, \gamma_{l+1} x_{l+1}^n\big),$$

with

$$C_\gamma = \left\{ x \in \left(\prod_{i=1}^{l} R^{(\bar{k})} \right) \times K^{(\bar{k})} \;\middle|\; v\left(\alpha_1' \prod_{i=1}^{l} x_i^{\eta_i} \right) \leqslant v(x_{l+1}) \,\square_2\, v\left(\alpha_2' \prod_{i=1}^{l} x_i^{\varepsilon_i} \right) \right\},$$

for appropriate choices of $\alpha_i' \in K$. Put $\bar{f} = f \circ f_\gamma$. Clearly, the $b_j \circ \bar{f}$ satisfy condition (4). Put $\nu_i = \varepsilon_i - \eta_i$, $\beta = \alpha_2'/\alpha_1'$. Then the following is an isomorphism

$$D_\gamma \to C_\gamma\colon \quad x \mapsto \left(x_1, \ldots, x_l, \alpha_1' x_{l+1} \prod_{i=1}^{l} x_i^{\eta_i} \right),$$

with $D_\gamma = \{x \in \prod_{i=1}^{l+1} R^{(\bar{k})} \mid v(x_{l+1}) \,\square_2\, v(\beta \prod_{i=1}^{l} x_i^{\nu_i})\}$.

The case that $\square_2$ is no condition is now trivial. Summarizing, it follows that we can reduce to the case of an isomorphism

$$f : E = \left\{ x \in \prod_{i=1}^{l+1} R^{(\bar{k})} \;\middle|\; v(x_{l+1}) \leqslant v\left(\beta \prod_{i=1}^{l} x_i^{\nu_i} \right) \right\} \to X \tag{6}$$

with $\beta \neq 0$, $\bar{k} > 0$, and $\nu_i \in \mathbb{Z}$, such that each $b_j \circ f$ satisfies condition (2), and such that the projection of E to the first l coordinates is surjective, onto $\prod_{i=1}^{l} R^{(\bar{k})}$ (this surjectivity comes from the original application of Lemma 6 in the beginning of the case $m > 1$). It follows by this surjectivity that $v(\beta) \geqslant 0$ and that $\nu_i \geqslant 0$ for each i, in (6).

Use Lemma 11 to obtain a partition of E in parts E_i and isomorphisms $\phi_i : \prod R^{(k)} \to E_i$. The ϕ_i are composed of functions of types f_0, f_1, f_2, t and the components of ϕ_i all satisfy condition (2). Therefore each $b_j \circ f \circ \phi_i$ will satisfy condition (2). This finishes the proof of Theorem 8. □

Example 9. $X = \{x \mid x \in P_n,\ v(x) = 0\}$ is a finite union of sets of the form $y + \pi^s R$ for some fixed $s > 0$ and with $v(y) = 0$ for each y. Indeed, for s, take an odd number such that $s \geqslant 2v(n)$. For the numbers y, take the (different) elements of the set X mod π^s. Then obviously $X \subset \bigcup_y (y + \pi^s R)$, and all the sets in this union are disjoint. Now fix y and suppose $x \in y + \pi^s R$. Since $v(x) = v(y)$, we only have to prove that $x \in P_n$. We know that $y \in P_n$, so there exists $b \in K^\times$ such that $y = b^n$. Put $f(X) := X^n - x$, then by our choice of s, $f(b) \equiv 0 \bmod \pi^s$ and $f'(b) \not\equiv 0 \bmod \pi^{\frac{s+1}{2}}$. So by Hensel's lemma, there exists $b' \equiv b \bmod \pi^{\frac{s+1}{2}}$ such that $x = (b')^n$.

[Likewise, the set $X = \{x \mid x \in \lambda P_n,\ v(x) = a\}$, is a finite union of sets of the form $y + \pi^s R$ for some fixed $s > 0$ and with $v(y) = a$ for each y.]

It is then an exercise, cf. the proof of Theorem 8, to find a partition and isomorphisms as in Theorem 8 for the set X.

Remark 10. In Theorem 8 it is also possible to change the order such that all functions of type f_0 are applied first among the functions of type f_0, f_1, f_2, t. To do this, we need to slightly modify the functions used in the proof of Theorem 8. We then get isomorphisms of the form

$$f : \prod_{i=1}^{l} R^{(k)} \to A\colon \quad x \mapsto (T_c \circ g \circ F_0)(x),$$

with T_c a composition of functions of type t_c, F_0 a composition of functions of type f_0 and g a composition of functions of type f_1, f_2, t.

Lemma 11. *Let E be a set of the form*

$$E := \left\{ x \in \prod_{i=1}^{m} R^{(k)} \;\middle|\; v(x_m) \leqslant v\left(\beta \prod_{i=1}^{m-1} x_i^{\nu_i}\right) \right\},$$

where $\beta \neq 0$, $k \in \mathbb{N}_0$, $\nu_i \in \mathbb{N}$, with $v(\beta) \geqslant 0$. Then there exists a finite partition of E such that for each part A there is an isomorphism of the form

$$\prod_{i=1}^{l} R^{(k)} \to A: \quad x \mapsto (g \circ F_0)(x)$$

with $l \leqslant m$, g a composition of functions of types f_1, f_2, t, and F_0 a composition of functions of type f_0.

Proof. We work by induction on m.

Case $m = 1$. We can partition $E = \{x \in R^{(k)} \mid v(x) \leqslant v(\beta)\}$ as

$$E = \bigcup_{s} \left\{ x \in R^{(k)} \;\middle|\; v(x) = s \right\},$$

with $s \in \{0, \ldots, v(\beta)\}$. Now proceed as for Case 1 for $m = 1$ in the proof of Theorem 8.

Case $m > 1$. Note that there are no conditions on x_i if $\nu_i = 0$. Hence, we may suppose that $\nu_1 > 0$. We first prove the proposition when $\nu_1 = 1$. We can partition E into parts E_1 and E_2, with

$$E_1 = \left\{ x \in E \;\middle|\; v(x_m) \leqslant v\left(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}\right) \right\},$$

$$E_2 = \left\{ x \in E \;\middle|\; v\left(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}\right) < v(x_m) \right\}$$

$$= \left\{ x \in \prod_{i=1}^{m} R^{(k)} \;\middle|\; v\left(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}\right) < v(x_m) \leqslant v\left(\beta x_1 \prod_{i=2}^{m-1} x_i^{\nu_i}\right) \right\}.$$

Since $v(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}) \leqslant v(x_1 \beta \prod_{i=2}^{m-1} x_i^{\nu_i})$ for $x \in E_1$, it follows that

$$E_1 = R^{(k)} \times \left\{ (x_2, \ldots, x_m) \in \prod_{i=2}^{m} R^{(k)} \;\middle|\; v(x_m) \leqslant v\left(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}\right) \right\},$$

and the lemma follows for E_1 by the induction hypothesis.

For E_2, let D_{m-1} be the set

$$D_{m-1} = \left\{ (x_2, \dots, x_m) \in \prod_{i=2}^{m} R^{(k)} \;\middle|\; v\left(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}\right) < v(x_m) \right\}.$$

We may suppose that $\beta \in K^{(k)}$. Then, the map

$$R^{(k)} \times D_{m-1} \to E_2\colon \quad x \mapsto \left(\frac{x_1 x_m}{\beta \prod_{i=2}^{m-1} x_i^{\nu_i}}, x_2, \dots, x_m \right)$$

is an isomorphism which is a composition of isomorphisms of type f_1 and f_2. Also

$$\prod_{i=1}^{m} R^{(k)} \to R^{(k)} \times D_{m-1}\colon \quad x \mapsto \left(x_1, \dots, x_{m-1}, \pi\beta x_m \prod_{i=2}^{m-1} x_i^{\nu_i} \right)$$

is an isomorphism which is a composition of isomorphisms of type f_1 and f_2. This proves the lemma when $\nu_1 = 1$.

Suppose now that $\nu_1 > 1$. We prove that we can reduce to the case $\nu_1 = 1$ by partitioning and applying appropriate power maps. Choose $\tilde{k} > v(\nu_1)$ and put $\tilde{k}' = \tilde{k} + v(\nu_1)$. We may suppose that $\tilde{k} \geqslant k$, so we have a finite partition $E = \bigcup_\alpha E_\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_m) \in K^m$, $v(\alpha_1) = 0$, $0 \leqslant v(\alpha_i) < \nu_1$ for $i = 2, \dots, m$ and

$$E_\alpha = \left\{ x \in E \;\middle|\; x_1 \in \alpha_1 R^{(\tilde{k})},\ x_i \in \alpha_i P_{\nu_1}^{(\tilde{k}')} \text{ for } i = 2, \dots, m \right\}.$$

By Corollary 2 we have isomorphisms

$$f_\alpha : C_\alpha \to E_\alpha\colon \quad x \mapsto \left(\alpha_1 x_1, \alpha_2 x_2^{\nu_1}, \dots, \alpha_m x_m^{\nu_1} \right),$$

with $C_\alpha = \{x \in \prod_{i=1}^m R^{(\tilde{k})} \mid v(x_m) \leqslant v(\beta' x_1 \prod_{i=2}^{m-1} x_i^{\nu_i})\}$, which are isomorphisms of type f_1. Here $\beta' \in K^\times$ depends only on α. This reduces the problem to the case with $\nu_1 = 1$ and thus the lemma is proved. □

To derive Theorem 7 from Theorem 8, we use three simple lemmas.

Lemma 12. *Isomorphisms of type f_1, f_2, t and of type t_c, from an open in K^n onto a subset of K^n, satisfy condition* (3).

Proof. With obvious notation,

$$\operatorname{Jac} f_1(x) = \det \begin{pmatrix} \alpha_1 a_1 x_1^{a_1 - 1} & & 0 \\ & \ddots & \\ 0 & & \alpha_n a_n x_n^{a_n - 1} \end{pmatrix} = \prod_{i=1}^{n} \alpha_i a_i x_i^{a_i - 1},$$

$$\operatorname{Jac} f_2(x) = \det \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & * & * & \prod_{i \neq j} x_j^{b_j} & * & * & * \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} = \prod_{i \neq j} x_j^{b_j},$$

and similarly for t_c and t. □

Lemma 13. *Let $T_c : X \subset K^m \to Y \subset K^m$ be a composition of isomorphisms of type t_c, with X open in K^m. Then T_c satisfies condition* (3).

Proof. Automatically T_c has the form

$$T_c(x) = \big(x_1 + c_1, x_2 + c_2(x_1), \dots, x_m + c_m(x_1, \dots, x_{m-1})\big)$$

for some C^1 semi-algebraic functions c_i. Now one can calculate $\operatorname{Jac} T_c$. □

Lemma 14 *(Compositions). Let $g : X \subset K^m \to K^m$ and $T : Y \subset K^m \to K^m$ be isomorphisms with $g(X) \subset Y$. Write $g = (g_1, \dots, g_m)$. If T and g satisfy condition* (3) *and the functions g_i satisfy condition* (2)*, then the composition $T \circ g$ satisfies condition* (3).

Proof. By condition (3) for T, there exist constants β, μ_i such that $v(\operatorname{Jac} T(x)) = v(\beta \prod_{i=1}^m x_i^{\mu_i})$. By the chain rule,

$$\begin{aligned} v\big(\operatorname{Jac}(T \circ g)(x)\big) &= v\big(\operatorname{Jac} T|_{g(x)} \cdot \operatorname{Jac} g(x)\big) \\ &= v\left(\beta \prod_{i=1}^m g_i(x)^{\mu_i} \operatorname{Jac} g(x)\right). \end{aligned}$$

The lemma now follows from condition (3) for g and condition (2) for the g_i. □

Proof of Theorem 7. By Theorem 8, there exists a finite partition of X in parts A such that for each part, the isomorphism f has the form

$$f : \prod_{i=1}^{l} R^{(k)} \to A: \quad x \mapsto (T_c \circ g)(x),$$

with g a composition of isomorphisms of type f_1, f_2, f_0, t and T_c a composition of isomorphisms of type t_c. We still have to prove that f satisfies condition (3) when $l = m$.

By Lemma 12, g is composed of functions which satisfy condition (3). Also, the component functions of g satisfy condition (2), by Remark 5. Hence, by Lemma 14, g satisfies condition (3). On the other hand, T_c satisfies condition (3) by Lemma 13. By Lemma 14, $T_c \circ g$ satisfies condition (3). □

3. Application: Rationality of p-adic integrals

We give an alternative, simple and selfcontained proof of the following rationality result by combining Theorem 7 with the basic fact that $\sum_{\ell=0}^{\infty} T^{\ell}$ equals $\frac{1}{1-T}$ for small T. Let $|dx|$ be the Haar measure on K^m such that the measure of R^m equals 1.

Corollary 15 *(Rationality). (See [3].) Let $S \subset K^m$ be a semi-algebraic set and $f, g : S \to K$ semi-algebraic functions. If the following integral exists for $s \in \mathbb{R}$, $s \gg 0$ (that is, if the integrand is absolutely integrable for s sufficiently big), then*

$$I(s) := \int_S \left|f(x)\right|^s \cdot \left|g(x)\right| |dx| \tag{7}$$

is rational in q_K^{-s} and the denominator of $I(s)$ is a product of factors of the form $(1 - q_K^{-sa-b})$ with $a, b \in \mathbb{Z}$, and $(a, b) \neq (0, 0)$.

Proof. By Theorem 7 and the change of variables formula for p-adic integrals, we may suppose that $S = \prod_{i=1}^m R^{(k)}$ and that $f(x) = \beta \prod_i x_i^{\mu_i}$ and $g(x) = \gamma \prod_i x_i^{\nu_i}$, for all x in S, with $\mu_i, \nu_i \in \mathbb{Z}$ and $\beta, \gamma \in K$. We may suppose that $\beta \neq 0 \neq \gamma$. Since the integral has become a product integral, we may suppose that $m = 1$. In this case, we can write

$$\begin{aligned} I(s) &= \sum_{\ell \in \mathbb{N}} \int_{v(x_1)=\ell,\, x_1 \in R^{(k)}} \left|\beta x_1^{\mu_1}\right|^s \left|\gamma x_1^{\nu_1}\right| |dx_1| \\ &= c q_K^{-v(\beta)s} \sum_{\ell \in \mathbb{N}} q_K^{-(\mu_1 s + \nu_1 + 1)\ell} \end{aligned}$$

with $c = q_K^{-k-v(\gamma)}$. By this calculation, the integrand is absolutely integrable for $s \gg 0$ if and only if $\mu_1 \geqslant 0$ and $\mu_1 > 0$ when $\nu_1 < 0$. In this case $I(s)$ equals

$$\frac{c q_K^{-v(\beta)s}}{1 - q_K^{-(\mu_1 s + \nu_1 + 1)}},$$

which is as desired. □

Corollary 15 corresponds to the main rationality results of [3], namely Theorems 1.1, 3.2, and 7.4 of [3], of which the rationality of the Denef–Serre Poincaré series (namely Theorem 1.1 of [3]) is a special case.

Remark 16. The advantage of our proof of Corollary 15 is that the rationality can be directly derived from Theorem 7; the disadvantage is that the integrals cannot depend on parameters for our proof. Other estimates, for example on the multiplicity of the poles of $I(s)$ as in [3], can be controlled with our method.

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