Real closed fields with non-standard and standard analytic structure

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ABSTRACT

We consider the ordered field which is the completion of the Puiseux series field over $\mathbb{R}$ equipped with a ring of analytic functions on $[-1,1]^n$ which contains the standard subanalytic functions as well as functions given by $t$-adically convergent power series, thus combining the analytic structures of Denef and van den Dries [Ann. of Math. 128 (1988) 79–138] and Lipshitz and Robinson [Bull. London Math. Soc. 38 (2006) 897–906]. We prove quantifier elimination and o-minimality in the corresponding language. We extend these constructions and results to rank $n$ ordered fields $\mathbb{R}_n$ (the maximal completions of iterated Puiseux series fields). We generalize the example of Hrushovski and Peterzil [J. Symbolic Logic 72 (2007) 119–122] of a sentence which is not true in any o-minimal expansion of $\mathbb{R}$ (shown in [Bull. London Math. Soc. 38 (2006) 897–906] to be true in an o-minimal expansion of the Puiseux series field) to a tower of examples of sentences $\sigma_n$, true in $\mathbb{R}_n$, but not true in any o-minimal expansion of any of the fields $\mathbb{R}, \mathbb{R}_1, \ldots, \mathbb{R}_{n-1}$.

1. Introduction

In [13], it has been shown that the ordered field $K_1$ of Puiseux series in the variable $t$ over $\mathbb{R}$, equipped with a class of $t$-adically overconvergent functions such as $\sum_n (n+1)!/(tx)^n$, has quantifier elimination and is o-minimal in the language of ordered fields enriched with function symbols for these functions on $[-1,1]^n$. This study was motivated (indirectly) by the observation of Hrushovski and Peterzil [10], that there are sentences true in this structure that are not satisfiable in any o-minimal expansion of $\mathbb{R}$. This in turn was prompted by a question of van den Dries; see [10] for details.

In [5], it was observed that if $K$ is a maximally complete, non-archimedean real closed field with divisible value group, and if $f$ is an element of $R[[\xi]]$ with radius of convergence greater than 1, then $f$ extends naturally to an ‘analytic’ function $I^n \to K$, where $I = \{x \in K : -1 \leq x \leq 1\}$. Hence if $A$ is the ring of real power series with radius of convergence greater than 1, then $K$ has $A$-analytic structure; that is, this extension preserves all the algebraic properties of the ring $A$. In particular, the real quantifier elimination of [4] works in this context so $K$ has quantifier elimination, is o-minimal in the analytic field language, and is even elementarily equivalent to $\mathbb{R}$ with the subanalytic structure. See [6, 7] for extensions.

In Sections 2 and 3 below, we extend the results of [13] by proving quantifier elimination and o-minimality for $K_1$ in a larger language that contains the overconvergent functions together with the usual analytic functions on $[-1,1]^n$. In Section 4, we extend these results to a larger class of non-archimedean real-closed fields, including fields $\mathbb{R}_m$, of rank $m = 1, 2, 3, \ldots$, and
in Section 5 we show that the idea of the example of [10] can be iterated so that for each $m$ there is a sentence that is true in $\mathbb{R}_m$ but is not satisfiable in any o-minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$. In a subsequent paper, we will give a more comprehensive treatment of both Henselian fields with analytic structure and real closed fields with analytic structure; see [2].

2. Notation and quantifier elimination

In this section, we establish the notation and prove quantifier elimination (Theorem 2.14) for the field of Puiseux series over $\mathbb{R}$ in a language that contains function symbols for all the standard analytic functions on $[-1,1]^n$ and all the $t$-adically overconvergent functions on this set.

**Definition 2.1.** We have

$$K_1 := \bigcup_n \mathbb{R}(t^{1/n}),$$

the field of Puiseux series over $\mathbb{R}$,

$$K := \hat{K}_1,$$

the $t$-adic completion of $K_1$;

There, $K$ is a real closed, non-archimedean normed field. We shall use $\| \cdot \|$ to denote the (non-archimedean) $t$-adic norm on $K$, and $<$ to denote the order on $K$ that comes from the real closedness of $K$. We will use $| \cdot |$ to denote the corresponding absolute value, $|x| = \sqrt{x^2}$:

- $K^o := \{ x \in K : \|x\| \leq 1 \}$, the finite elements of $K$;
- $K^{\infty} := \{ x \in K : \|x\| < 1 \}$, the infinitesimal elements of $K$;
- $K_{alg} := K[\sqrt{-1}]$, the algebraic closure of $K$;
- $A_{n,\alpha} := \{ f \in \mathbb{R}[[\xi_1, \ldots, \xi_n]] : \text{radius of convergence of } f > \alpha \}, 0 < \alpha \in \mathbb{R}$;
- $\mathcal{R}_{n,\alpha} := A_{n,\alpha} \hat{\otimes} K = (A_{n,\alpha} \hat{\otimes} K)^\wedge$, where $\hat{\otimes}$ stands for the $t$-adic completion;
- $\mathcal{R}_{n,\alpha}^o := A_{n,\alpha} \hat{\otimes} K^o = (A_{n,\alpha} \hat{\otimes} K^o)^\wedge$;
- $\mathcal{R}_n := \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha}$;
- $\mathcal{R} := \bigcup_n \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha}$;
- $I := [-1,1] = \{ x \in K : |x| \leq 1 \}$.

**Remark 2.2.** (i) The field $K$ has $\mathcal{R}$-analytic structure; in [5] it is explained how the functions of $A_{n,\alpha}$ are defined on $I^n$. For example, if $f \in A_{1,\alpha}$, $\alpha > 1$, $a \in \mathbb{R} \cap [-1,1]$, $\beta \in K^{\infty}$, then $f(a + \beta) := \sum_n f^{(n)}(a) \beta^n / n!$. The extension to functions in $\mathcal{R}_{n,\alpha}$ is clear from the completeness of $K$. We define these functions to be zero outside $[-1,1]^n$. This extension naturally also works for maximally complete fields and fields of LE-series; see also [6, 7].

(ii) The ring $\mathcal{R}$ contains all the ‘standard’ real analytic functions on $[-1,1]^n$ and all the $t$-adically overconvergent functions in the sense of [13].

(iii) The elements of $A_{n,\alpha}$ in fact define complex analytic functions on the complex polydisc $\{ x \in \mathbb{C} : |x_i| \leq \alpha \text{ for } i = 1, \ldots, n \}$, and hence the elements of $\mathcal{R}_{n,\alpha}$ define ‘$K_{alg}$-analytic’ functions on the corresponding $K_{alg}$-polydisc $\{ x \in K_{alg} : |x_i| \leq \alpha \text{ for } i = 1, \ldots, n \}$.

(iv) We could as well work with $K_1$ instead of $K$. Then we must replace $\mathcal{R}_{n,\alpha}$ by

$$\mathcal{R}_{n,\alpha}' := \bigcup_m A_{n,\alpha} \hat{\otimes} \mathbb{R}(t^{1/m}) = \bigcup_m \left( (A_{n,\alpha} \hat{\otimes} \mathbb{R}(t^{1/m}))^\wedge \right).$$
(v) If $\beta \leq \alpha$ and $m \leq n$, then $R_{\alpha,m} \subset R_{\beta,n}$.

**Lemma 2.3.** Every non-zero $f \in R_{n,\alpha}$ has a unique representation

$$f = \sum_{i \in J \subset \mathbb{N}} f_i t^{\gamma_i},$$

where the $f_i \in A_{n,\alpha}$, $f_i \neq 0$, $\gamma_i \in \mathbb{Q}$, the $\gamma_i$ are increasing, $0 \in J$, and, either $J$ is finite of the form $\{0, \ldots, n\}$, or $\gamma_i \rightarrow \infty$ as $i \rightarrow \infty$ and $J = \mathbb{N}$. The function $f$ is a unit in $R_{n,\alpha}$ exactly when $f_0$ is a unit in $A_{n,\alpha}$.

**Proof.** Observe that if $a \in K$, $a \neq 0$, then $a$ has a unique representation $a = \sum_{i \in J \subset \mathbb{N}} a_i t^{\gamma_i}$, where the $a_i \in \mathbb{R}$, $a_i \neq 0$, $\gamma_i \in \mathbb{Q}$ and, either $J$ is finite, or $\gamma_i \rightarrow \infty$. If $f_0$ is a unit in $A_{n,\alpha}$, then $f \cdot f_0^{-1} \cdot t^{-\gamma_0} = 1 + \sum_{i=1}^{\infty} f_i \cdot f_0^{-1} \cdot t^{\gamma_i - \gamma_0}$ and $\gamma_i - \gamma_0 > 0$. \qed

**Definition 2.4.** (i) In the notation of the previous lemma, $f_0$ is called the top slice of $f$.

(ii) We call $f$ regular in $\xi_n$ of degree $s$ at $a \in [I \cap \mathbb{R}]^n$ if, in the classical sense, $f_0$ is regular in $\xi_n$ of degree $s$ at $a$.

(iii) We shall abuse the notation and use $\| \cdot \|$ to denote the $t$-adic norm on $K$ and the corresponding gauss-norm on $R_{n,\alpha}$; so with $f$ as in the above lemma, $\|f\| = \|t^{\gamma_0}\|$.

The standard Weierstrass preparation and division theorems for $A_{n,\alpha}$ extend to corresponding theorems for $R_{n,\alpha}$.

**Theorem 2.5** (Weierstrass preparation and division). If $f \in R_{n,\alpha}$ with $\|f\| = 1$ is regular in $\xi_n$ of degree $s$ at $0$, then there is a $\delta \in \mathbb{R}$, $\delta > 0$, such that there are unique $A_1, \ldots, A_s, U$ satisfying

$$f = [\xi_n^s + A_1(\xi')\xi_n^{s-1} + \ldots + A_s(\xi')]U(\xi)$$

and

$$A_1, \ldots, A_s, U \in R_{n-1,\delta} \quad \text{and} \quad U \in R_{n,\delta} \text{ a unit}.$$  

Then automatically

$$\|A_1\|, \ldots, \|A_s\|, \|U\| \leq 1, \quad \|A_1(0)\|, \ldots, \|A_s(0)\| < 1 \quad \text{and} \quad \|U(0)\| = 1.$$  

Furthermore, if $g \in R_{n,\alpha}$ then there are unique $Q \in R_{n,\delta}$ and $R_0(\xi'), \ldots, R_{s-1}(\xi') \in R_{n-1,\delta}$, satisfying

$$\|Q\|, \|R_i\| \leq \|g\|$$

and

$$g = Qf + R_0(\xi') + R_1(\xi')\xi_n + \ldots + R_{s-1}(\xi')\xi_n^{s-1}.$$  

**Proof.** We may assume that $f = \sum_{\gamma \in I} \sum_{\alpha \leq I} f_\alpha t^{\gamma}$, where $I \subset \mathbb{Q}^+$, $0 \in I$, $I = I + I$ and $I$ is well ordered. (We do not require that $f_\alpha \neq 0$ for all $\alpha \in I$, but we do require $f_0 \neq 0$.) We prove the preparation theorem. The proof of the division theorem is similar. We shall produce, inductively on $\gamma \in I$, monic polynomials $P_\gamma[\xi_n]$ with coefficients from $R_{n-1,\delta}$, and units $U_\gamma \in R_{n,\delta}$ such that, writing $\gamma'$ for the successor of $\gamma$ in $I$, we have

$$f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'}$$
and if $\gamma < \beta$, then

$$P_\gamma \equiv P_\beta \quad \text{and} \quad U_\gamma \equiv U_\beta \mod t^{\gamma}.$$  

Using [9, Theorem II.D.1 (p. 80)], or the proof on [16, pp. 142–144], we see that there is a $0 < \delta < \alpha$ such that for every $g \in A_{n,\delta}$ the Weierstrass data on dividing $g$ by $f_0$ are in $A_{n,\delta}$.

Now $P_0$ and $U_0$ are the classical Weierstrass data for $f_0$; that is, $f_0 = U_0P_0$, where $P_0 \in A_{n-1,\delta}[\xi_n]$ is monic of degree $s$ and $U \in A_{n,\delta}$ is a unit. Suppose that $P_\gamma$ and $U_\gamma$ have been found. Then

$$f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'},$$

so we have

$$U^{-1}_\gamma \cdot f \equiv P_\gamma + g_{\gamma'} t^{\gamma'} + o(t^{\gamma'}),$$

where $g_{\gamma'} \in A_{n,\delta}$ and we write $o(t^{\gamma'})$ to denote terms of order greater than $\gamma'$. By classical Weierstrass division we can write

$$g_{\gamma'} = P_0 \cdot Q_{\gamma'} + R_{\gamma'},$$

where $Q_{\gamma'} \in A_{n,\delta}$ and $R_{\gamma'} \in A_{n-1,\delta}[\xi_n]$ has degree less than $s$ in $\xi_n$. Let

$$P_{\gamma'} := P_\gamma + t^{\gamma'} R_{\gamma'}.$$  

Then

$$U^{-1}_\gamma \cdot f = P_\gamma + t^{\gamma'}(P_0 Q_{\gamma'} + R_{\gamma'}) + o(t^{\gamma'})$$

$$= (P_{\gamma'} + t^{\gamma'} P_{\gamma'} Q_{\gamma'}) + t^{\gamma'}(P_0 - P_{\gamma'})Q_{\gamma'} + o(t^{\gamma'})$$

$$= P_{\gamma'}(1 + t^{\gamma'} Q_{\gamma'}) + o(t^{\gamma'}),$$

since $P_0 - P_{\gamma'} = o(1)$; that is, it has positive order. Take $U_{\gamma'} := U_\gamma(1 + t^{\gamma'} Q_{\gamma'})$. The uniqueness of the $A_i$ and $U$ follows from the same induction. 

**Remark 2.6.** We remark, for use in a subsequent paper [2], that the argument of the previous proof works in the more general context that $I$ is a well-ordered subset of the value group $\Gamma$ of a suitably complete field, for example, a maximally complete field, cf. [14].

From the above proof or by direct calculation we have the following corollary.

**Corollary 2.7.** If $g \in \mathcal{R}_1$, $\beta \in [-1, 1]$ and $g(\beta) = 0$, then $\xi_1 - \beta$ divides $g$ in $\mathcal{R}_1$.

**Remark 2.8.** Let $f(\xi, \eta)$ be in $\mathcal{R}_{m+n,\alpha}$. Then there are unique $\bar{f}_\mu$ in $\mathcal{R}_{m,\alpha}$ such that

$$f(\xi, \eta) = \sum \bar{f}_\mu(\xi)\eta^\mu.$$  

The following lemma is used to prove Theorem 2.10.

**Lemma 2.9.** Let $f(\xi, \eta) = \sum \bar{f}_\mu(\xi)\eta^\mu \in \mathcal{R}_{m+n,\alpha}$. Then the $\bar{f}_\mu \in \mathcal{R}_{m,\alpha}$ and there exist an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$, and $g_\mu \in \mathcal{R}_{m+n,\beta}^\circ$ for $|\mu| < d$, such that

$$f = \sum_{|\mu| < d} \bar{f}_\mu(\xi)g_\mu(\xi, \eta),$$

in $\mathcal{R}_{m+n,\beta}$. 
Proof. We may assume that \( \|f\| = 1 \), and choose a \( \nu_0 \) such that \( \|\mathcal{F}_{\nu_0}\| = 1 \). Making an \( \mathbb{R} \)-linear change of variables, and shrinking \( \alpha \) if necessary, we may assume that \( \mathcal{F}_{\nu_0} \) is regular at \( 0 \) in \( \xi_m \) of degree \( s \), say. Write \( \xi' \) for \( (\xi_1, \ldots, \xi_{m-1}) \). By Weierstrass division (Theorem 2.5) there is a \( \beta > 0 \) and there are
\[
Q(\xi, \eta) \in \mathcal{R}_{m+n, \beta} \quad \text{and} \quad R(\xi, \eta) = R_0(\xi', \eta) + \ldots + R_{s-1}(\xi', \eta)\xi_m^{s-1} \in \mathcal{R}_{m+n-1, \beta}[\xi_m]
\]
such that
\[
f(\xi, \eta) = \mathcal{F}_{\nu_0}(\xi, \eta)Q(\xi, \eta) + R(\xi, \eta).
\]
By induction on \( m \), we may write
\[
R_0 = \sum_{|\mu| < d} \mathcal{R}_{\nu_0, \mu}(\xi')g_\mu(\xi', \eta)
\]
for some \( d \in \mathbb{N} \), some \( \beta > 0 \) and \( g_\mu(\xi', \eta) \in \mathcal{R}_{m+n-1, \beta}^\circ \). Writing \( R = \sum_\nu \mathcal{R}_\nu(\xi)\eta^\nu \), observe that each \( \mathcal{R}_\nu \) is an \( \mathcal{R}_{m, \beta}^\circ \)-linear combination of the \( \mathcal{F}_\nu \), since, taking the coefficient of \( \eta^\nu \) on both sides of the equation \( f(\xi, \eta) = \mathcal{F}_{\nu_0}(\xi, \eta)Q(\xi, \eta) + R(\xi, \eta) \), we have
\[
\mathcal{F}_\nu = \mathcal{F}_{\nu_0}\mathcal{R}_\nu + \mathcal{R}_\nu.
\]
Consider
\[
f - \mathcal{F}_{\nu_0}Q - \sum_{|\mu| < d} \mathcal{R}_\mu(\xi)g_\mu(\xi', \eta) =: S_1\xi_m + S_2\xi_m^2 + \ldots + S_{s-1}\xi_m^{s-1}
\]
\[
= \xi_m[S_1 + S_2\xi_m + \ldots + S_{s-1}\xi_m^{s-2}]
\]
\[
=: \xi_m^s S,
\]
say, where the \( S_i \in \mathcal{R}_{m+n-1, \beta}^\circ \). Again, observe that each \( \mathcal{S}_\nu \) is an \( \mathcal{R}_{m, \beta}^\circ \)-linear combination of the \( \mathcal{F}_\nu \). Complete the proof by induction on \( s \), working with \( S \) instead of \( R \).

Theorem 2.10 (Strong Noetherian property). Let \( f(\xi, \eta) = \sum_\mu \mathcal{F}_\mu(\xi)\eta^\mu \in \mathcal{R}_{m+n, \alpha} \). Then the \( \mathcal{F}_\mu \in \mathcal{R}_{m, \alpha} \) and there exist an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \), and units \( U_\mu(\xi, \eta) \in \mathcal{R}_{m+n, \beta}^\circ \) for \( |\mu| < d \) such that
\[
f = \sum_{\mu \in J} \mathcal{F}_\mu(\xi)\eta^\mu U_\mu(\xi, \eta)
\]
in \( \mathcal{R}_{m+n, \beta} \), where \( J \) is a subset of \( \{0, 1, \ldots, d\}^n \).

Proof. It is sufficient to show that there are an integer \( d \), a set \( J \subset \{0, 1, \ldots, d\}^n \), and \( g_\mu \in \mathcal{R}_{m+n, \beta}^\circ \) such that
\[
f = \sum_{\mu \in J} \mathcal{F}_\mu(\xi)\eta^\mu g_\mu(\xi, \eta),
\]
(2.1)
since then we may assume, rearranging the sum if necessary, that each \( g_\mu \) is of the form \( 1 + h_\mu \), where \( h_\mu \in (\eta)\mathcal{R}_{m+n, \beta}^\circ \). Shrinking \( \beta \) if necessary will guarantee that the \( g_\mu \) are units. However, then it is, in fact, sufficient to prove (2.1) for \( f \) replaced by
\[
f_I := \sum_{\mu \in I_i} \mathcal{F}_\mu(\xi)\eta^\mu
\]
for each \( I_i \) in a finite partition \( \{I_i\} \) of \( \mathbb{N}^n \).

By Lemma 2.9 there are an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \), and \( g_\mu \in \mathcal{R}_{m+n, \beta}^\circ \) for \( |\mu| \leq d \) such that
\[
f = \sum_{|\mu| \leq d} \mathcal{F}_\mu(\xi)g_\mu(\xi, \eta).
\]
Rearranging, we may assume for $\nu, \mu \in \{1, \ldots, d\}^n$ that $\bar{g}_{\mu, \nu}$ is equal to 1 if $\mu = \nu$ and that it is equal to 0 otherwise.

Focus on $f_{I_1}(\xi, \eta)$, defined as above by

$$f_{I_1}(\xi, \eta) = \sum_{\mu \in I_1} \bar{f}_\mu(\xi)\eta^\mu$$

with

$$I_1 := \{0, \ldots, d\}^n \cup \{\mu : \mu_i \geq d \text{ for all } i\}$$

and note that

$$f_{I_1}(\xi, \eta) = \sum_{|\nu| \leq d} \bar{f}_\nu(\xi)g_{\mu, I_1}(\xi, \eta)$$

with $g_{\mu, I_1}(\xi, \eta) \in \mathcal{R}_{m+n, \beta}$ defined by the corresponding sum

$$g_{\mu, I_1}(\xi, \eta) = \sum_{\nu \in I_1} \bar{f}_{\mu, \nu}(\xi)\eta^\nu.$$  

It is now clear that $g_{\mu, I_1}$ is of the form $\eta^\mu(1 + h_\mu)$, where $h_\mu \in (\eta)\mathcal{R}_{m+n, \beta}$.

One now proceeds by noting that $f - f_{I_1}$ is a finite sum of terms of the form $f_{I_j}$ for $j > 1$ and $\{I_j\}_j$ a finite partition of $\mathbb{N}^n$ and where each $f_{I_j}$ for $j > 1$ is of the form $\eta^\nu q(\xi, \eta')$, where $\eta'$ is $(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n)$ and $q$ is in $\mathcal{R}_{m+n-1, \beta}$.

These terms can be handled by induction on $n$.

**Definition 2.11.** For $\gamma \in K^{\text{alg}}_\beta$ let $\gamma^\circ$ denote the closest element of $\mathbb{C}$, that is, the unique element $\gamma^\circ$ of $\mathbb{C}$ such that $|\gamma - \gamma^\circ| \in K^{\text{alg}}_\beta$.

**Lemma 2.12.** Let $f \in \mathcal{R}_1$. If $f(\gamma) = 0$, then $f_0(\gamma^\circ) = 0$ (if $f_0$ is the top slice of $f$). Conversely, if $\beta \in \mathbb{R}$ (or $\mathbb{C}$) and $f_0(\beta) = 0$, then there is a $\gamma \in K^{\text{alg}}_\beta$ with $\gamma^\circ = \beta$ and $f(\gamma) = 0$. Indeed, $f_0$ has a zero of order $n$ at $\beta \in \mathbb{C}$ if, and only if, $f$ has $n$ zeros $\gamma$ (counting multiplicity) with $\gamma^\circ = \beta$.

**Proof.** Use Weierstrass preparation and [1, Proposition 3.4.1.1].

**Corollary 2.13.** A non-zero $f \in \mathcal{R}_{1, \alpha}$ has only finitely many zeros in the set $\{x \in K^{\text{alg}}_\beta : |x| \leq \alpha\}$. Indeed, there exist a polynomial $P(x) \in K[x]$ and a unit $U(x) \in \mathcal{R}_{1, \alpha}$ such that $f(x) = P(x) \cdot U(x)$.

**Proof.** Observe that $f_0$ has only finitely many zeros in $\{x \in \mathbb{C} : |x| \leq \alpha\}$, and that non-real zeros occur in complex conjugate pairs, and that $f$ is a unit exactly when $f_0$ has no zeros in this set (that is, when $f_0$ is a unit in $A_{1, \alpha}$), and use Lemma 2.12.

**Theorem 2.14** (Quantifier elimination theorem). Denote by $\mathcal{L}$ the language $\{+, \cdot, ^{-1}, 0, 1, <, \mathbb{R}\}$, where the functions in $\mathcal{R}_n$ are interpreted to be zero outside $\mathbb{R}^n$. Then $K$ admits quantifier elimination in $\mathcal{L}$.

**Proof.** This is a small modification of the real quantifier elimination of [4] as in [5], using the Weierstrass preparation theorem and the strong Noetherian property above. Crucial is that,
in Theorems 2.5 and 2.10, as in [4], β and δ are positive real numbers so one can use the compactness of $[-1,1]^n$ in $\mathbb{R}^n$.

3. The o-minimality

In this section, we prove the o-minimality of $K$ in the language $\mathcal{L}$. Let $\alpha > 1$. As we remarked above, each $f \in A_{n,\alpha}$ defines a function from the polydisc $(I_{C,\alpha})^n \to \mathbb{C}$, where $I_{C,\alpha} := \{x \in \mathbb{C} : |x| \leq \alpha\}$, and hence each $f \in \mathcal{R}_{n,\alpha}$ defines a function from $(I_{K_{\text{alg},\alpha}})^n \to K_{\text{alg}}$, where $I_{K_{\text{alg},\alpha}} := \{x \in K_{\text{alg}} : |x| \leq \alpha\}$. In general $A_{n,\alpha}$ is not closed under composition. However, if $F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha}$, $G_j(\xi) \in \mathcal{R}_{n,\beta}$ for $j = 1, \ldots, m$ and $|G_j(x)| \leq \alpha$ for all $x \in (I_{K_{\text{alg},\beta}})^n$, then $F(G_1(\xi), \ldots, G_m(\xi)) \in \mathcal{R}_{m,\beta}$. This is clear if $F \in A_{n,\alpha}$ and the $G_j \in A_{n,\beta}$. The general case follows easily.

For $c, r \in K$, $r > 0$, we denote the ‘closed interval’ with center $c$ and radius $r$ by

$$I(c, r) := \{x \in K : |x - c| \leq r\}$$

and for $c, \delta, \varepsilon \in K$, $0 < \delta < \varepsilon$, we denote the ‘closed annulus’ with center $c$, inner radius $\delta$ and outer radius $\varepsilon$ by

$$A(c, \delta, \varepsilon) := \{x \in K : \delta \leq |x - c| \leq \varepsilon\}.$$  

On occasion we will consider $I(c, r)$ as a disc in $K_{\text{alg}}$ and $A(c, \delta, \varepsilon)$ as an annulus in $K_{\text{alg}}$, replacing $K$ by $K_{\text{alg}}$ in the definitions. No confusion should result. Note that these discs and annuli are defined in terms of the real-closed order on $K$, not the non-archimedean absolute value $\| \cdot \|$, and hence are not discs or annuli in the sense of [1], [13] or [8], which we will refer to as affinoid discs and affinoid annuli. For $I = I(c, r)$, $A = A(c, \delta, \varepsilon)$ as above, we define the rings of analytic function on $I$ and $A$ as follows:

$$\mathcal{O}_I := \left\{f \left(\frac{x - c}{r}\right) : f \in \mathcal{R}_I \right\}$$

and

$$\mathcal{O}_A := \left\{g \left(\frac{\delta}{x - c}\right) + h \left(\frac{x - c}{\varepsilon}\right) : g, h \in \mathcal{R}_I, \ g(0) = 0 \right\}.$$  

The elements of $\mathcal{O}_I$ and $\mathcal{O}_A$ are analytic functions on, respectively, the corresponding $K_{\text{alg}}$-disc and the $K_{\text{alg}}$ annulus as well.

**Remark 3.1.** (i) Elements of $\mathcal{O}_A$ are multiplied using the relation $(\delta/(x - c))(\delta/(x - c)/\varepsilon) = \delta/\varepsilon$ and the fact that $|\delta/\varepsilon| < 1$. Indeed, let $g(\xi_1) = \sum_{i} a_i \xi_1^i$, $h(\xi_2) = \sum_{i} b_j \xi_2^i$. Then, using the relation $\xi_1 \xi_2 = \delta/\varepsilon$, we have

$$g \cdot h = \sum_{j<i} a_i b_j \left(\frac{\delta}{\varepsilon}\right)^j \xi_1^i + \sum_{i<j} a_i b_j \left(\frac{\delta}{\varepsilon}\right)^{j-i} \xi_2^{j-i} = f_1(\xi_1) + f_2(\xi_2).$$

If $g, h \in A_{1,\alpha}$ and $\delta/\varepsilon \in \mathbb{R}$, then $f_1, f_2 \in A_{1,\alpha}$, and this extends easily to the case $g, h \in \mathcal{R}_{1,\alpha}$ and $\delta/\varepsilon \in K^\circ$. Lemma 3.6 will show that, in fact, the only case of an annulus that we must consider is when $\delta/\varepsilon \in K^{\circ \circ}$.

(ii) We define the gauss-norm on $\mathcal{O}_I$ by

$$\|f \left(\frac{x - c}{r}\right)\| = \|f(\xi)\|,$$

and on $\mathcal{O}_A$ by

$$\|g \left(\frac{\delta}{x - c}\right) + h \left(\frac{x - c}{\varepsilon}\right)\| := \max\{\|g(\xi_1)\|, \|h(\xi_2)\|\}.$$
It is clear that the gauss-norm equals the supremum norm.

(iii) If \( f \in \mathcal{O}_I \), then \( \|(x - c)/r\|_f = \|f\|. \) If \( f \in \mathcal{O}_A \), then \( \|(x - c)/\varepsilon\|_f \leq \|f\| \) and if \( \delta/\varepsilon \in K^{\infty} \) (that is, is infinitesimal), then

\[
\left\| \left( \frac{x - c}{\varepsilon} \right) g \left( \frac{\delta}{x - c} \right) \right\| = \left\| \frac{\delta}{\varepsilon} \right\| \cdot \left\| g \left( \frac{\delta}{x - c} \right) \right\| < \left\| g \left( \frac{\delta}{x - c} \right) \right\|.
\]

and

\[
\left\| \left( \frac{x - c}{\varepsilon} \right) h \left( \frac{x - c}{\varepsilon} \right) \right\| = \left\| h \left( \frac{x - c}{\varepsilon} \right) \right\|.
\]

(iv) If \( \|f\| < 1 \), then \( 1 - f \) is a unit in \( \mathcal{O}_A \). (In fact, it is a strong unit — a unit \( u \) satisfying \( \|1 - u\| < 1 \).)

**Definition 3.2.** (i) We say that \( f \in \mathcal{O}_{I(0,1)} = \mathcal{R}_1 \) has a zero close to \( a \in I(0,1) \) if \( f \), as a \( K_{\text{alg}} \)-function defined on the \( K_{\text{alg}} \)-disc \( \{x : |x| \leq \alpha\} \) for some \( \alpha > 1 \), has a \( K_{\text{alg}} \)-zero \( b \) with \( a - b \) infinitesimal in \( K_{\text{alg}} \). We say \( f \) has a zero close to \( I(0,1) \) if it has a zero close to \( a \) for some \( a \in I(0,1) \). For an arbitrary interval \( I = I(c,r) \) we say that \( f = F((x - c)/r) \in \mathcal{O}_I \) has a zero close to \( a \in I \) if \( F \) has a zero close to \( (a - c)/r \in I(0,1) \), and that \( f \) has a zero close to \( I(c,r) \) if \( F \) has a zero close to \( I(0,1) \).

(ii) For \( a, b \in K^{\infty}, a, b > 0 \), we write \( a \sim b \) if \( a/b, b/a \in K^{\infty} \) and we write \( a \ll b \) if \( a/b \in K^{\infty} \).

(iii) Let \( X \) be an interval or an annulus, and let \( f \) be defined on a superset of \( X \). We shall write \( f \in \mathcal{O}_X \) to mean that there is a function \( g \in \mathcal{O}_X \) such that

\[
f|_X = g.
\]

**Lemma 3.3.** If \( f \in \mathcal{O}_{I(c,r)} \) has no zero close to \( I(c,r) \), then there is a cover of \( I(c,r) \) by finitely many closed intervals \( I_j = I(c_j, r_j) \) such that \( 1/f \in \mathcal{O}_{I_j} \) for each \( j \).

**Proof.** It is sufficient to consider the case \( I(c,r) = I(0,1) \). Cover \( I(0,1) \) by finitely many intervals \( I(c_j, r_j), c_j, r_j \in \mathbb{R} \), such that \( f \) has no \( K_{\text{alg}} \)-zero in the \( K_{\text{alg}} \)-disc \( I(c_j, r_j) \). Finally use Corollary 2.13.

**Remark 3.4.** The function \( f(x) = 1 + x^2 \) has no zeros close to \( I(0,1) \). It is not a unit in \( \mathcal{O}_{I(0,1)} \), but it is a unit in both \( \mathcal{O}_{I(-1,2/1,2)} \) and \( \mathcal{O}_{I(1,2,1/2)} \). The function \( g(x) = x \) is not a unit in \( \mathcal{O}_{I(\delta, \varepsilon)} \) for any \( 0 < \delta \in K^{\infty} \) and \( 0 < \varepsilon \in K^{\infty} \). It is of course a unit in \( \mathcal{O}_{A(\delta, \varepsilon)} \). The function \( 1/f = 1/x \in A(0, \delta, 1) \) for all \( \delta > 0 \) but is not in \( \mathcal{O}_I \) for \( I = I(1 + \delta)/2, (1 - \delta)/2 \), for any \( \delta \in K^{\infty} \). Thus we see that if \( X_1 \subset X_2 \) are annuli or intervals, then it does not necessarily follow that \( \mathcal{O}_{X_2} \subset \mathcal{O}_{X_1} \). However, the following are clear. If \( I_1 \subset I_2 \) are intervals, then \( \mathcal{O}_{I_2} \subset \mathcal{O}_{I_1} \). If \( 0 < \delta \in K^{\infty}, 0 < r \in K^{\infty} \setminus K^{\infty}, r < 1 \), and \( A = A(0, \delta, 1), I = I((1 + r)/2, (1 - r)/2) \), then \( \mathcal{O}_A \subset \mathcal{O}_I \). If \( A_1 \subset A_2 \) are annuli that have the same center, then \( \mathcal{O}_{A_2} \subset \mathcal{O}_{A_1} \). If \( 0 < \delta \ll c \) and \( 0 < r < c/\alpha \) for some \( \alpha \in \mathbb{R}, \alpha > 1 \), and \( I = I(c,r) \subset A(0, \delta, 1) = A \), then \( \mathcal{O}_A \subset \mathcal{O}_I \).

Writing \( x = c - y, |y| \leq r \) we see that

\[
\frac{\delta}{x} = \frac{\delta}{c - y} = \frac{\delta/c}{1 - y/c} + \frac{\delta}{c} \sum \left( \frac{y}{c} \right)^k = \frac{\delta}{c} \sum \left( \frac{y}{c} \right)^k.
\]

Restating Corollary 2.13 we have the following corollary.

**Corollary 3.5.** If \( f \in \mathcal{O}_{I(c,r)} \), then there exist a polynomial \( P \in K[\xi] \) and a unit \( U \in \mathcal{O}_{I(c,r)} \) such that \( f(\xi) = P(\xi) \cdot U(\xi) \).
Lemma 3.6. If $\varepsilon < N \delta$ for some $N \in \mathbb{N}$, then there is a covering of $A(c, \delta, \varepsilon)$ by finitely many intervals $I_j$ such that for every $f \in \mathcal{O}_{A(c, \delta, \xi)}$ and each $j$, we have $f \in \mathcal{O}_{I_j}$.

Proof. Use Lemma 3.3 or reduce directly to the case $\varepsilon = 1, 0 < \delta = r \in \mathbb{R}$ and the two intervals $[-1, -r] = I \left(\frac{-1 + r}{2}, \frac{1 - r}{2}\right)$ and $[r, 1] = I \left(\frac{1 + r}{2}, \frac{1 - r}{2}\right)$. □

The following lemma is key for proving o-minimality.

Lemma 3.7. Let $f \in A(c, \delta, \varepsilon)$. There are finitely many intervals and annuli $X_j$ that cover $A(c, \delta, \varepsilon)$, polynomials $P_j$ and units $U_j \in \mathcal{O}_{X_j}$ such that for each $j$ we have $f|_{X_j} = (P_j \cdot U_j)|_{X_j}$.

Proof. By the previous lemma, we may assume that $c = 0, \varepsilon = 1$ and $\delta \in K^{\circ\circ}$ (that is, $\delta$ is infinitesimal, say $\delta = t^\gamma$ for some $\gamma > 0$). Let

$$f(x) = g \left(\frac{\delta}{x}\right) + h(x) \quad \text{with} \quad g(\xi), h(\xi) \in \mathcal{R}_1, \ g(0) = 0,$$

and

$$g(\xi) = \sum_{i \in I \subseteq \mathbb{N}} t^{\alpha_i} \xi^{n_i} g_i(\xi)$$

with $n_i > 0, g_i(0) \neq 0, g_i \in A_{1, \alpha}$ for some $\alpha > 1$. Observe that

$$xg \left(\frac{\delta}{x}\right) = \sum_{i \in I} (t^{\alpha_i} \delta) \left(\frac{\delta}{x}\right)^{n_i - 1} g_i \left(\frac{\delta}{x}\right).$$

Hence (see Remark 3.1) for suitable $n \in \mathbb{N}$, absorbing the constant terms into $h$, we have

$$x^n f(x) = \overline{g} \left(\frac{\delta}{x}\right) + \overline{h}(x)$$

where $\overline{g}(0) = 0$ and $\| \overline{g} \| < \| \overline{h} \|$. (For use in Section 4, below, note that this argument does not use the restriction that $K$ is complete or of rank 1.) Multiplying by a constant, we may assume that $\| \overline{h} \| = 1$. Let

$$\overline{g}(\xi) = \sum_{i \in I} t^{\beta_i} \xi^{m_i} \overline{g}_i(\xi) \quad \text{with} \quad m_i > 0 \quad \text{and} \quad \beta_i > 0 \quad \text{for each} \ i,$$

and

$$\overline{h}(\xi) = \xi^{k_0} \overline{h}_0(\xi) + \sum_{i \in J \subseteq \mathbb{N}, 0} t^{\gamma_i} \overline{h}_i(\xi) \quad \text{with} \quad \overline{h}_0(0) \neq 0 \quad \text{and} \quad \gamma_i > 0 \quad \text{increasing}.$$ 

Since $\| \overline{g} \| = \| t^{\beta_0} \| < 1$ there is a $\delta'$ with $\delta \leq \delta' \in K^{\circ\circ}$ (that is, $\| \delta' \| < 1$) such that

$$\| t^{\gamma_i} \| < \| (\delta')^{k_0} \| \quad \text{and} \quad \| \overline{g} \| < \| (\delta')^{k_0} \|.$$ 

Splitting off some intervals of the form

$$I \left(\frac{-1 - r}{2}, \frac{1 + r}{2}\right) = [-1, -r] \quad \text{or} \quad I \left(\frac{1 + r}{2}, \frac{1 - r}{2}\right) = [r, 1] \quad \text{for} \ r > 0, \ r \in \mathbb{R},$$

we have
(on which the result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume that \( h_0 \) is a unit in \( A_{1,\alpha} \), for some \( \alpha > 1 \). Therefore,

\[
x^n f(x) = g \left( \frac{\delta}{x} \right) + h(x)
\]

\[
= h_0(x) x^{k_0} \left[ 1 + \sum_{i=1}^{\infty} \left( \frac{\delta}{P(x)} \right)^{k_i} t^{\alpha_i} \frac{h_1(x)}{h_0(x)} + \frac{1}{h_0(x)} \left( \frac{1}{\delta'} \right)^{k_0} \left( \frac{\delta'}{x} \right)^{\delta'} g \left( \frac{\delta'}{\delta} \right) \right].
\]

By our choice of \( \delta' \) and Remark 3.1 the quantity in square brackets is a (strong) unit. Hence we have taken care of an annulus of the form \( A(0, \delta', 1) \) for some \( \delta' \) with \( |\delta| \leq |\delta'| \) and \( \|\delta'\| < 1 \).

Observe that on the much bigger affinoid \( \mathcal{O} \) (on which the result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume \( \delta = \delta'/x \) interchanges the sets

\[
\{ x : \|x\| = \|\delta\| \text{ and } \delta \leq |x| \} \quad \text{and} \quad \{ y : \|y\| = 1 \text{ and } |y| \leq 1 \}.
\]

Hence, as above, there is a \( \delta'' \in K^{\infty} \) with \( \|\delta\| \leq \|\delta''\| \) and a covering of the annulus \( \delta \leq |x| \leq \delta'' \) by finitely many intervals and annuli with the required property.

It remains to treat the annulus \( \delta'' \leq |x| \leq \delta' \). Using the terminology of [11] and [13], observe that on the much bigger affinoid annulus \( \|\delta''\| \leq \|x\| \leq \|\delta'\| \) the function \( f \) is strictly convergent, indeed even overconvergent. Hence, as in [13, Lemma 3.6], on this affinoid annulus we can write

\[
f = \frac{P(x)}{x^t} \cdot U(x),
\]

where \( P(x) \) is a polynomial and \( U(x) \) is a strong unit (that is, \( \|U(x) - 1\| < 1 \)).

**Corollary 3.8.** If \( X \) is an interval or an annulus and \( f \in \mathcal{O}_X \), then the set \( \{ x \in X : f(x) \geq 0 \} \) is semialgebraic (that is, a finite union of (closed) intervals).

**Proof.** This is an immediate corollary of Corollary 3.5 and Lemma 3.7, since units do not change sign on intervals and since an annulus has two intervals as connected components.

**Definition 3.9.** For \( c = (c_1, \ldots, c_n) \), \( r = (r_1, \ldots, r_n) \) we define the poly-interval

\[
I(c, r) := \{ x \in K^n : |x_i - c_i| \leq r_i, \ i = 1, \ldots, n \}.
\]

This also defines the corresponding polydisk in \( (K_{\text{alg}})^n \). The ring of analytic functions on this poly-interval (or polydisc) is

\[
\mathcal{O}_I(c, r) := \left\{ f \left( \frac{x_1 - c_1}{r_1}, \ldots, \frac{x_n - c_n}{r_n} \right) : f \in \mathcal{R}_n \right\}.
\]

**Lemma 3.10.** Let \( \alpha, \beta > 1 \), \( F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha} \) and \( G_j(\xi_1, \ldots, \xi_n) \in \mathcal{R}_{n,\beta} \) with \( \|G_j\| \leq 1 \) for \( j = 1, \ldots, m \). Let \( X = \{ x \in [-1, 1]^n : |G_j(x)| \leq 1 \text{ for } j = 1, \ldots, m \} \). There are (finitely many) \( c_i = (c_{i1}, \ldots, c_{im}) \in \mathbb{R}^n \), \( \varepsilon_i \in \mathbb{R}, \varepsilon_i > 0 \) with \( \varepsilon_i < |c_{ij}| \) if \( c_{ij} \neq 0 \) such that the (poly) intervals \( I_j = \{ x \in K : |x_j - c_{ij}| < \varepsilon_i \text{ for } j = 1, \ldots, n \} \) cover \( X \), \( |G_j(x)| < \alpha \) for all \( x \in I_j, \ j = 1, \ldots, m \), and there are \( H_i \in \mathcal{O}_I \) such that

\[
F(G_1, \ldots, G_m)|_{I_j} = H_i|_{I_j}.
\]

**Proof.** Use the compactness of \([-1, 1]^n \cap \mathbb{R}^n\) and the following facts. If \( \|G_j\| = 1 \), then \( |G_j(x) - G_j(y)(x)| \) is infinitesimal for all \( x \in K^o \), where \( G_jy \) is the top slice of \( G_j \). If \( \|G_j\| < 1 \), then \( |G_j(x)| < K^\infty \) for all \( x \in [-1, 1]^n \).
Corollary 3.11. (i) Let $I$ be an interval, $F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_m$ and $G_j \in \mathcal{O}_I$ for $j = 1, \ldots, m$. Then there are finitely many intervals $I_i$ covering $I$ and functions $H_i \in \mathcal{O}_{I_i}$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}.$$ 

(ii) Let $A$ be an annulus, $F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_m$ and $G_j \in \mathcal{O}_A$ for $j = 1, \ldots, m$. Then there are finitely many $X_i$, each an interval or an annulus, covering $A$ and $H_i \in \mathcal{O}_{X_i}$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{X_i} = H_i|_{X_i}.$$ 

Proof. Part (i) reduces to Lemma 3.10 once we see that if $\|G_j\| > 1$, then we can use Corollary 2.13 to restrict to the subintervals of $I$ around the zeros of $G_j$ on which $|G_j| \leq C$ for some $C \in \mathbb{R}$, $C > 1$. On the rest of $I$, $F(G_1, \ldots, G_m)$ is zero.

For (ii), we may assume that $A = A(0, \delta, 1)$ with $\delta$ infinitesimal and that $G_j(x) = G_{j_1}(\delta/x) + G_{j_2}(x)$. As in (i), we may reduce to the case that $\|G_j\| \leq 1$, using Lemma 3.7 instead of Corollary 2.13, and using Lemma 3.6 and Remark 3.4. Apply Lemma 3.10 to the functions $F$ and $G'_j(\xi_1, \xi_2) = G_{j_1}(\xi_1) + G_{j_2}(\xi_2)$. The case $c = (0, 0)$ gives us the annulus $|\delta/x| \leq \varepsilon$, $|x| \leq \varepsilon$ that is $\varepsilon/\varepsilon \leq |x| \leq \varepsilon$. The case $c = (0, c_2)$ with $c_2 \neq 0$ gives us $|\delta/x| \leq \varepsilon$ and $|x - c_2| \leq \varepsilon$ with $\varepsilon < |c_2|$. This is equivalent to $|x - c_2| \leq \varepsilon$ since $\varepsilon, c_1 \in \mathbb{R}$, and hence equivalent to

$$-\varepsilon \leq x - c_2 \leq \varepsilon \quad \text{or} \quad c_2 - \varepsilon \leq x \leq c_2 + \varepsilon$$

which is an interval bounded away from 0. The case $c = (c_1, 0)$ gives us $|\delta/x - c_1| \leq \varepsilon$, $|x| \leq \varepsilon$ (since $\varepsilon < |c_1|$) which is equivalent to $|\delta/x - c_1| \leq \varepsilon$ or $c_1 - \varepsilon \leq \delta/x \leq c_1 + \varepsilon$ or (considering the case $c_1 > 0$; the case $c_1 < 0$ is similar) $\delta/(c_1 + \varepsilon) \leq x \leq \delta/(c_1 - \varepsilon)$ which is part of an annulus that can be reduced to intervals using Lemma 3.6. The case $c = (c_1, c_2)$ with both $c_1, c_2 \neq 0$ is vacuous since either $x$ or $\delta/x$ is infinitesimal on $A$ and $\varepsilon < |c_1|, |c_2|$.

Lemma 3.12. Let $X$ be an interval or an annulus and let $f, g \in \mathcal{O}_X$. There are finitely many subintervals and subannuli $X_i \subset X$, $i = 1, \ldots, \ell$, such that

$$\{x \in X : |f(x)| \leq |g(x)|\} \subset \bigcup_{i=1}^{\ell} X_i \subset X,$$

and, except at finitely many points,

$$\frac{f}{g}|_{X_i} \in \mathcal{O}_X.$$

Proof. We consider the case that $X$ is an interval, $I$, and may take $I = I(0, 1) = [-1, 1]$. We may assume by Corollary 3.5 that $f$ and $g$ have no common zero. If $g$ has no zeros close to $I(0, 1)$, then we are done by Lemma 3.3. Let $\alpha_1, \ldots, \alpha_n$ be the distinct elements of $[-1, 1] \cap \mathbb{R}$ such that $g$ has at least one zero close to $(\text{that is, within an infinitesimal of})$ $\alpha$. Breaking into subintervals and making changes of variables we may assume that $n = 1$ and that $\alpha_1 = 0$. Again making a change of variables (over $K$) we may assume that $g$ has at least one zero with zero ‘real’ part, that is, of the form $\alpha = \sqrt{-1}a$ for some $a \in K$. Let $N$ denote the number of zeros of $f \cdot g$ close to 0 in $I(0, 1)$. Let $\delta = 3 \cdot \max \{|x| : x \in K_{\text{alg}} \text{ close to } 0 \text{ and } f(x) \cdot g(x) = 0\}$. If $\delta = 0$, then $\alpha = \alpha = 0$ and there is no other zero of $f \cdot g$ close to zero in $I(0, 1)$. Then there is a $\delta' > 0$ such that for $|x| < \delta'$ we have $|g(x)| < |f(x)|$. Then the interval $I(0, \delta')$ drops away, and on the annulus $A(0, \delta', 1)$ the function $g$ is a unit. If $\delta > 0$, then we consider the interval $I(0, \delta)$ and the annulus $A(0, \delta, 1)$ separately, and proceed by induction on $N$. Therefore, suppose that $\delta > 0$ and $f \cdot g$ has $N$ zeros close to 0 in $I(0, 1)$. Let $\alpha$ be as above. If $\alpha \sim \delta$, then the zero
\[ a = \sqrt{-1} \alpha \text{ is not close to } I(0, \delta) \text{ and by restricting to } I(0, \delta) \text{ we have reduced } N. \text{ If } |\alpha| < \delta, \text{ then this zero is close to } 0 \text{ in } I(0, \delta), \text{ but the largest zeros (those of size } \delta/3 \text{) are not close to } 0 \text{ in } I(0, \delta), \text{ and hence restricting to } I(0, \delta) \text{ again reduces } N. \]

It remains to consider the case of the annulus \( A(0, \delta, 1) = \{ x : \delta \leq |x| \leq 1 \} \) where all the zeros of \( g \) are within \( \delta/3 \) of 0. By Lemma 3.7, we may assume that \( g(x) = P(x) \cdot U(x) \), where \( U \) is a unit and all the zeros of \( P \) are within \( \delta/3 \) of 0. Let \( \alpha_i, i = 1, \ldots, \ell, \) be these zeros. For \( |x| \geq \delta \) we may write

\[
\frac{1}{x - \alpha_i} = \frac{1}{x} \cdot \frac{1}{1 - \alpha_i/x} = \frac{1}{x} \sum_{j=0}^{\infty} \left( \frac{\alpha_i}{x} \right)^j = \frac{1}{x} \sum_{j=0}^{\infty} \left( \frac{\alpha_i}{\delta} \right)^j \left( \frac{\delta}{x} \right)^j \quad \text{and} \quad |\frac{\alpha_i}{\delta}| \leq \frac{1}{3}. \]

Hence \( f/g \in O_{A(0, \delta, 1)} \). This completes the case that \( X \) is an interval.

The case that \( X \) is an annulus is similar — one can cover \( X \) with finitely many subannuli \( X_i \) and subintervals \( Y_j \) such that for each \( i, g|_{X_i} \) is a unit in \( O_{X_i} \), and for each \( j, f|_{Y_j}, g|_{Y_j} \in O_{Y_j}. \)

From Corollary 3.11 and Lemma 3.12, we now have the following proposition, by induction on terms.

**Proposition 3.13.** Let \( f_1, \ldots, f_\ell \) be \( \mathcal{L} \)-terms in one variable, \( x \). There is a covering of \([-1, 1]\) by finitely many intervals and annuli \( X_i \) such that except for finitely many values of \( x \), we have for each \( i \) and \( j \) that \( f_i|_{X_i} \in O_{X_i} \) (that is, \( f_i|_{X_i} \) agrees with an element of \( O_{X_i} \) except at finitely many points of \( X_i \)).

This, together with Corollary 3.8, gives the following theorem.

**Theorem 3.14.** The field \( K \) is o-minimal in \( \mathcal{L} \).

### 4. Further extensions

In this section, we give extensions of the results of Sections 2 and 3 and the results of [13].

Let \( G \) be an (additive) ordered abelian group. Let \( t \) be a symbol. Then \( t^G \) is a (multiplicative) ordered abelian group. Following the notation of [5, 6] (but not [7] or [12]) we define \( \mathbb{R}((t^G)) \) to be the maximally complete valued field with additive value group \( G \) (or multiplicative value group \( t^G \)) and residue field \( \mathbb{R} \). Therefore,

\[
\mathbb{R}((t^G)) := \left\{ \sum_{g \in I} a_g t^g : a_g \in \mathbb{R} \text{ and } I \subset G \text{ well ordered} \right\}.
\]

We shall be a bit sloppy about mixing the additive and multiplicative valuations. \( I \subset G \) is well ordered exactly when \( t^I \subset t^G \) is reverse well ordered. The field \( K \) of Puiseux series, or its completion, is a proper subfield of \( \mathbb{R}_1 := \mathbb{R}((t^G)) \). Considering \( G = \mathbb{Q}^m \) with the lexicographic ordering, we define

\[
\mathbb{R}_m := \mathbb{R}((t^{G_m})).
\]

It is clear that if \( G_1 \subset G_2 \) as ordered groups, then \( \mathbb{R}((t^{G_1})) \subset \mathbb{R}((t^{G_2})) \) as valued fields. Also, \( \mathbb{R}((t^G)) \) is Henselian and, if \( G \) is divisible, then \( \mathbb{R}((t^G)) \) is real closed. We shall continue to use ‘\(<\)’ for the corresponding order on \( \mathbb{R}((t^G)) \).

In analogy with Section 2, we define the following.
Definition 4.1. We have
\[
\mathcal{R}_{n,\alpha}(G) := A_{n,\alpha} \hat{\otimes}_{\mathbb{R}} \mathbb{R}((t^G))
\]
\[
= \left\{ \sum_{g \in I} f_g t^g : f_g \in A_{n,\alpha} \text{ and } I \subset G \text{ well ordered} \right\}
\]
\[
\mathcal{R}_n(G) := \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha}(G)
\]
\[
\mathcal{R}(G) := \bigcup_n \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha}(G).
\]

As in Section 2, the elements of \(\mathcal{R}_n(G)\) define functions from \(I^n \subset \mathbb{R}((t^G))^n\) to \(\mathbb{R}((t^G))\). Indeed this interpretation is a ring endomorphism. In other words, the field \(\mathbb{R}((t^G))\) has analytic \(\mathcal{R}(G)\)-structure. (See [3] and especially [2] for more about fields with analytic structure.) The elements of \(\mathcal{R}_n(G)\) are interpreted as zero on \(\mathbb{R}((t^G)) \setminus I^n\).

Theorem 4.2. The Weierstrass preparation theorem (Theorem 2.5) and the strong Noetherian property (Theorem 2.10) hold with \(\mathcal{R}\) replaced by \(\mathcal{R}(G)\).

Proof. Only minor modifications to the proofs of Theorems 2.5 and 2.10 are needed.

The arguments of Sections 2 and 3 prove the following theorem.

Theorem 4.3. If \(G\) is divisible, then \(\mathbb{R}((t^G))\) admits quantifier elimination and is \(\alpha\)-minimal in \(\mathcal{L}_G\).

Corollary 4.4. If \(G_1 \subset G_2\) are divisible, then \(\mathbb{R}((t^{G_1})) \prec \mathbb{R}((t^{G_2}))\) in \(\mathcal{L}_{G_1}\).

We shall show in Section 5 that, though for \(m < n\) we have \(\mathbb{R}_m \prec \mathbb{R}_n\) in \(\mathcal{L}_{Q^m}\), there is a sentence of \(\mathcal{L}_{Q^n}\) that is true in \(\mathbb{R}_n\) but is not true in any \(\alpha\)-minimal expansion of \(\mathbb{R}_m\).

The results of [13] also extend to this more general setting.

Definition 4.5. We define the ring of *strictly convergent power series over \(\mathbb{R}((t^G))\) as
\[
\mathbb{R}((t^G))^*\langle \xi \rangle := \left\{ \sum_{g \in I} a_g(\xi)t^g : a_g(\xi) \in \mathbb{R}[\xi] \text{ and } I \text{ well ordered} \right\},
\]
and the subring of *overconvergent power series over \(\mathbb{R}((t^G))\) as
\[
\mathbb{R}((t^G))^*\langle \langle \xi \rangle \rangle := \{ f : f(\gamma\xi) \in \mathbb{R}((t^G))^*\langle \xi \rangle \text{ for some } \gamma \in \mathbb{R}((t^G)), \|\gamma\| > 1 \},
\]
\[
\mathcal{R}(G)_{\text{over}} := \bigcup_n \mathbb{R}((t^G))^*\langle \langle \xi_1, \ldots, \xi_n \rangle \rangle,
\]
and the corresponding overconvergent language as
\[
\mathcal{L}_{G,\text{over}} := \langle +, -, 1, 0, <, \mathcal{R}(G)_{\text{over}} \rangle.
\]
As in [13] we have the following theorem.

**Theorem 4.6.** If $G$ is divisible, then $\mathbb{R}((t^G))$ admits quantifier elimination and is o-minimal in $\mathcal{L}_{G, \text{over}}$.

Of course the o-minimality follows immediately from Theorem 4.3.

## 5. Extensions of the example of Hrushovski and Peterzil

In this section, we show that with minor modifications, the idea of the example of [10] can be iterated to give a nested family of examples. This relates to a question of Hrushovski and Peterzil as to whether there exists a small class of o-minimal structures such that any sentence, true in some o-minimal structure, can be satisfied in an expansion of a model in the class. Combining this with expansions with the exponential function, one can perhaps elaborate the tower of examples further.

Consider the functional equation

$$F(\beta z) = \alpha z F(z) + 1,$$  \hspace{1cm} (*)

and suppose that $F$ is a ‘complex analytic’ solution for $|z| \leq 1$. By this we mean that, writing $z = x + \sqrt{-1}y$, $F(z) = f(x, y) + \sqrt{-1}g(x, y)$, $F(z)$ is differentiable as a function of $z$. This is a definable condition on the two ‘real’ functions, $f$, $g$ of the two ‘real’ variables $x$, $y$. Then

$$F(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where

$$a_k = \frac{\alpha^k}{\beta^{k(k+1)/2}}$$

($\alpha$, and $\beta$ are parameters).

By this we mean that for each $n \in \mathbb{N}$ there is a constant $A_n$ such that

$$\left| F(z) - \sum_{k=0}^{n} a_k z^k \right| \leq A_n |z^{n+1}|$$  \hspace{1cm} (**)

is true for all $z$ with $|z| \leq 1$. Indeed, by [15, Theorem 2.50], one can take $A_n = C \cdot 2^{n+1}$, for $C$ a constant independent of $n$.

Consider the following statement: $F(z)$ is a complex analytic function (in the above sense) on $|z| \leq 1$ that satisfies (**); the number $\beta > 0$ is within the radius of convergence of the function $f(z) = \sum_{n=1}^{\infty} (n-1)!z^n$ and $\alpha > 0$.

This statement is not satisfiable by any functions in any o-minimal expansion of the field of Puiseux series $K_1$, or the maximally complete field $\mathbb{R}_1 = \mathbb{R}((t^Q))$, because, if it were, we would have $\|\beta\| = \|t^\gamma\|$ and $\|\alpha\| = \|t^\delta\|$, for some $\gamma, \delta \in Q$, $\gamma, \delta > 0$, and for suitable choice of $n$ the condition (**) would be violated. On the other hand, if we choose $\alpha, \beta \in \mathbb{R}_2$ with $\text{ord}(\alpha) = (1, 0)$ and $\text{ord}(\beta) = (0, 1)$ then $\sum a_k z^k \in \mathbb{R}_2((z))$ satisfies the statement on $\mathbb{R}_2$.

This process can clearly be iterated to give, in the notation of Section 4, the following proposition.

**Proposition 5.1.** For each $m$ there is a sentence of $\mathcal{L}_{Q^m}$ that is true in $\mathbb{R}_m$ but is not satisfiable in any o-minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$.
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