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Real closed fields with non-standard and standard analytic structure

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Abstract

We consider the ordered field which is the completion of the Puiseux series field over $\mathbb{R}$ equipped with a ring of analytic functions on $[-1,1]^n$ which contains the standard subanalytic functions as well as functions given by t-adically convergent power series, thus combining the analytic structures of Denef and van den Dries [*Ann. of Math.* 128 (1988) 79–138] and Lipshitz and Robinson [*Bull. London Math. Soc.* 38 (2006) 897–906]. We prove quantifier elimination and o-minimality in the corresponding language. We extend these constructions and results to rank n ordered fields $\mathbb{R}_n$ (the maximal completions of iterated Puiseux series fields). We generalize the example of Hrushovski and Peterzil [*J. Symbolic Logic* 72 (2007) 119–122] of a sentence which is not true in any o-minimal expansion of $\mathbb{R}$ (shown in [*Bull. London Math. Soc.* 38 (2006) 897–906] to be true in an o-minimal expansion of the Puiseux series field) to a tower of examples of sentences σ_n, true in $\mathbb{R}_n$, but not true in any o-minimal expansion of any of the fields $\mathbb{R}, \mathbb{R}_1, \ldots, \mathbb{R}_{n-1}$.

1. *Introduction*

In [**13**], it has been shown that the ordered field K_1 of Puiseux series in the variable t over $\mathbb{R}$, equipped with a class of t-adically overconvergent functions such as $\sum_n (n+1)!(tx)^n$, has quantifier elimination and is o-minimal in the language of ordered fields enriched with function symbols for these functions on $[-1,1]^n$. This study was motivated (indirectly) by the observation of Hrushovski and Peterzil [**10**], that there are sentences true in this structure that are not satisfiable in any o-minimal expansion of $\mathbb{R}$. This in turn was prompted by a question of van den Dries; see [**10**] for details.

In [**5**], it was observed that if K is a maximally complete, non-archimedean real closed field with divisible value group, and if f is an element of $\mathbb{R}[[\xi]]$ with radius of convergence greater than 1, then f extends naturally to an 'analytic' function $I^n \to K$, where $I = \{x \in K : -1 \leqslant x \leqslant 1\}$. Hence if $\mathcal{A}$ is the ring of real power series with radius of convergence greater than 1, then K has $\mathcal{A}$-analytic structure; that is, this extension preserves all the algebraic properties of the ring $\mathcal{A}$. In particular, the real quantifier elimination of [**4**] works in this context so K has quantifier elimination, is o-minimal in the analytic field language, and is even elementarily equivalent to $\mathbb{R}$ with the subanalytic structure. See [**6**, **7**] for extensions.

In Sections 2 and 3 below, we extend the results of [**13**] by proving quantifier elimination and o-minimality for K_1 in a larger language that contains the overconvergent functions together with the usual analytic functions on $[-1,1]^n$. In Section 4, we extend these results to a larger class of non-archimedean real-closed fields, including fields $\mathbb{R}_m$, of rank $m = 1, 2, 3, \ldots$, and

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in Section 5 we show that the idea of the example of [10] can be iterated so that for each m there is a sentence that is true in $\mathbb{R}_m$ but is not satisfiable in any o-minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$. In a subsequent paper, we will give a more comprehensive treatment of both Henselian fields with analytic structure and real closed fields with analytic structure; see [2].

2. *Notation and quantifier elimination*

In this section, we establish the notation and prove quantifier elimination (Theorem 2.14) for the field of Puiseux series over $\mathbb{R}$ in a language that contains function symbols for all the standard analytic functions on $[-1,1]^n$ and all the t-adically overconvergent functions on this set.

Definition 2.1. We have

$$K_1 := \bigcup_n \mathbb{R}((t^{1/n})), \text{ the field of Puiseux series over } \mathbb{R},$$
$$K := \widehat{K}_1, \text{ the } t\text{-adic completion of } K_1;$$

There, K is a real closed, non-archimedean normed field. We shall use $\|\cdot\|$ to denote the (non-archimedean) t-adic norm on K, and $<$ to denote the order on K that comes from the real closedness of K. We will use $|\cdot|$ to denote the corresponding absolute value, $|x| = \sqrt{x^2}$:

$$K^\circ := \{x \in K : \|x\| \leqslant 1\}, \text{ the finite elements of } K;$$
$$K^{\circ\circ} := \{x \in K : \|x\| < 1\}, \text{ the infinitesimal elements of } K;$$
$$K_{\text{alg}} := K[\sqrt{-1}], \text{ the algebraic closure of } K;$$
$$A_{n,\alpha} := \{f \in \mathbb{R}[[\xi_1, \ldots, \xi_n]] : \text{ radius of convergence of } f > \alpha\}, \ 0 < \alpha \in \mathbb{R};$$
$$\mathcal{R}_{n,\alpha} := A_{n,\alpha} \widehat{\bigotimes}_{\mathbb{R}} K = (A_{n,\alpha} \bigotimes_{\mathbb{R}} K)\widehat{}, \text{ where `}\widehat{}\text{' stands for the } t\text{-adic completion;}$$
$$\mathcal{R}^\circ{}_{n,\alpha} := A_{n,\alpha} \widehat{\bigotimes}_{\mathbb{R}} K^\circ = (A_{n,\alpha} \bigotimes_{\mathbb{R}} K^\circ)\widehat{};$$
$$\mathcal{R}_n := \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha};$$
$$\mathcal{R} := \bigcup_n \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha};$$
$$I := [-1,1] = \{x \in K : |x| \leqslant 1\}.$$

Remark 2.2. (i) The field K has $\mathcal{R}$-analytic structure; in [5] it is explained how the functions of $A_{n,\alpha}$ are defined on I^n. For example, if $f \in A_{1,\alpha}$, $\alpha > 1$, $a \in \mathbb{R} \cap [-1,1]$, $\beta \in K^{\circ\circ}$, then $f(a+\beta) := \sum_n f^{(n)}(a)\beta^n/n!$. The extension to functions in $\mathcal{R}_{n,\alpha}$ is clear from the completeness of K. We define these functions to be zero outside $[-1,1]^n$. This extension naturally also works for maximally complete fields and fields of LE-series; see also [6, 7].

(ii) The ring $\mathcal{R}$ contains all the 'standard' real analytic functions on $[-1,1]^n$ and all the t-adically overconvergent functions in the sense of [13].

(iii) The elements of $A_{n,\alpha}$ in fact define complex analytic functions on the complex polydisc $\{x \in \mathbb{C} : |x_i| \leqslant \alpha \text{ for } i = 1, \ldots, n\}$, and hence the elements of $\mathcal{R}_{n,\alpha}$ define 'K_{alg}-analytic' functions on the corresponding K_{alg}-polydisc $\{x \in K_{\text{alg}} : |x_i| \leqslant \alpha \text{ for } i = 1, \ldots, n\}$.

(iv) We could as well work with K_1 instead of K. Then we must replace $\mathcal{R}_{n,\alpha}$ by

$$\mathcal{R}'{}_{n,\alpha} := \bigcup_m A_{n,\alpha} \widehat{\bigotimes}_{\mathbb{R}} \mathbb{R}((t^{1/m})) = \bigcup_m \Big((A_{n,\alpha} \bigotimes_{\mathbb{R}} \mathbb{R}((t^{1/m})))\widehat{} \Big).$$

(v) If $\beta \leqslant \alpha$ and $m \leqslant n$, then $\mathcal{R}_{\alpha,m} \subset \mathcal{R}_{\beta,n}$.

LEMMA 2.3. *Every non-zero $f \in \mathcal{R}_{n,\alpha}$ has a unique representation*

$$f = \sum_{i \in J \subset \mathbb{N}} f_i t^{\gamma_i},$$

where the $f_i \in A_{n,\alpha}$, $f_i \neq 0$, $\gamma_i \in \mathbb{Q}$, the γ_i are increasing, $0 \in J$, and, either J is finite of the form $\{0, \dots, n\}$, or $\gamma_i \to \infty$ as $i \to \infty$ and $J = \mathbb{N}$. The function f is a unit in $\mathcal{R}_{n,\alpha}$ exactly when f_0 is a unit in $A_{n,\alpha}$.

Proof. Observe that if $a \in K$, $a \neq 0$, then a has a unique representation $a = \sum_{i \in J \subset \mathbb{N}} a_i t^{\gamma_i}$, where the $a_i \in \mathbb{R}$, $a_i \neq 0$, $\gamma_i \in \mathbb{Q}$ and, either J is finite, or $\gamma_i \to \infty$. If f_0 is a unit in $A_{n,\alpha}$, then $f \cdot f_0^{-1} \cdot t^{-\gamma_0} = 1 + \sum_{i=1}^{\infty} f_i \cdot f_0^{-1} \cdot t^{\gamma_i - \gamma_0}$ and $\gamma_i - \gamma_0 > 0$. □

DEFINITION 2.4. (i) In the notation of the previous lemma, f_0 is called the *top slice* of f.
(ii) We call f *regular in ξ_n of degree s at $a \in [I \cap \mathbb{R}]^n$* if, in the classical sense, f_0 is regular in ξ_n of degree s at a.
(iii) We shall abuse the notation and use $\|\cdot\|$ to denote the t-adic norm on K and the corresponding gauss-norm on $\mathcal{R}_{n,\alpha}$; so with f as in the above lemma, $\|f\| = \|t^{\gamma_0}\|$.

The standard Weierstrass preparation and division theorems for $A_{n,\alpha}$ extend to corresponding theorems for $\mathcal{R}_{n,\alpha}$.

THEOREM 2.5 (Weierstrass preparation and division). *If $f \in \mathcal{R}_{n,\alpha}$ with $\|f\| = 1$ is regular in ξ_n of degree s at 0, then there is a $\delta \in \mathbb{R}$, $\delta > 0$, such that there are unique $A_1, \dots, A_s, U$ satisfying*

$$f = [\xi_n^s + A_1(\xi')\xi_n^{s-1} + \ldots + A_s(\xi')]U(\xi)$$

and

$$A_1, \dots, A_s, \in \mathcal{R}_{n-1,\delta} \quad \text{and} \quad U \in \mathcal{R}_{n,\delta} \text{ a unit.}$$

Then automatically

$$\|A_1\|, \dots, \|A_s\|, \|U\| \leqslant 1, \quad \|A_1(0)\|, \dots, \|A_s(0)\| < 1 \quad \text{and} \quad \|U(0)\| = 1.$$

Furthermore, if $g \in \mathcal{R}_{n,\alpha}$ then there are unique $Q \in \mathcal{R}_{n,\delta}$ and $R_0(\xi'), \dots, R_{s-1}(\xi') \in \mathcal{R}_{n-1,\delta}$, satisfying

$$\|Q\|, \|R_i\| \leqslant \|g\|$$

and

$$g = Qf + R_0(\xi') + R_1(\xi')\xi_n + \ldots + R_{s-1}(\xi')\xi_n^{s-1}.$$

Proof. We may assume that $f = \sum_{\gamma \in I} f_\gamma t^\gamma$, where $I \subset \mathbb{Q}^+$, $0 \in I$, $I = I + I$ and I is well ordered. (We do not require that $f_\alpha \neq 0$ for all $\alpha \in I$, but we do require $f_0 \neq 0$.) We prove the preparation theorem. The proof of the division theorem is similar. We shall produce, inductively on $\gamma \in I$, monic polynomials $P_\gamma[\xi_n]$ with coefficients from $\mathcal{R}_{n-1,\delta}$, and units $U_\gamma \in \mathcal{R}_{n,\delta}$ such that, writing γ' for the successor of γ in I, we have

$$f \equiv P_\gamma \cdot U_\gamma \bmod t^{\gamma'}$$

and if $\gamma < \beta$, then

$$P_\gamma \equiv P_\beta \quad \text{and} \quad U_\gamma \equiv U_\beta \bmod t^{\gamma'}.$$

Using [**9**, Theorem II.D.1 (p. 80)], or the proof on [**16**, pp. 142–144], we see that there is a $0 < \delta \leqslant \alpha$ such that for every $g \in A_{n,\delta}$ the Weierstrass data on dividing g by f_0 are in $A_{n,\delta}$.

Now P_0 and U_0 are the classical Weierstrass data for f_0; that is, $f_0 = U_0 P_0$, where $P_0 \in A_{n-1,\delta}[\xi_n]$ is monic of degree s, and $U \in A_{n,\delta}$ is a unit. Suppose that P_γ and U_γ have been found. Then

$$f \equiv P_\gamma \cdot U_\gamma \bmod t^{\gamma'}$$

so we have

$$U_\gamma{}^{-1} \cdot f \equiv P_\gamma + g_{\gamma'} t^{\gamma'} + \mathrm{o}(t^{\gamma'}),$$

where $g_{\gamma'} \in A_{n,\delta}$ and we write $\mathrm{o}(t^{\gamma'})$ to denote terms of order greater than γ'. By classical Weierstrass division we can write

$$g_{\gamma'} = P_0 \cdot Q_{\gamma'} + R_{\gamma'},$$

where $Q_{\gamma'} \in A_{n,\delta}$ and $R_{\gamma'} \in A_{n-1,\delta}[\xi_n]$ has degree less than s in ξ_n. Let

$$P_{\gamma'} := P_\gamma + t^{\gamma'} R_{\gamma'}.$$

Then

$$\begin{aligned} U_\gamma{}^{-1} \cdot f &= P_\gamma + t^{\gamma'}(P_0 Q_{\gamma'} + R_{\gamma'}) + \mathrm{o}(t^{\gamma'}) \\ &= (P_{\gamma'} + t^{\gamma'} P_{\gamma'} Q_{\gamma'}) + t^{\gamma'}(P_0 - P_{\gamma'}) Q_{\gamma'} + \mathrm{o}(t^{\gamma'}) \\ &= P_{\gamma'}(1 + t^{\gamma'} Q_{\gamma'}) + \mathrm{o}(t^{\gamma'}), \end{aligned}$$

since $P_0 - P_{\gamma'} = \mathrm{o}(1)$; that is, it has positive order. Take $U_{\gamma'} := U_\gamma(1 + t^{\gamma'} Q_{\gamma'})$. The uniqueness of the A_i and U follows from the same induction. □

Remark 2.6. We remark, for use in a subsequent paper [**2**], that the argument of the previous proof works in the more general context that I is a well-ordered subset of the value group Γ of a suitably complete field, for example, a maximally complete field, cf. [**14**].

From the above proof or by direct calculation we have the following corollary.

Corollary 2.7. *If $g \in \mathcal{R}_1$, $\beta \in [-1,1]$ and $g(\beta) = 0$, then $\xi_1 - \beta$ divides g in $\mathcal{R}_1$.*

Remark 2.8. Let $f(\xi,\eta)$ be in $\mathcal{R}_{m+n,\alpha}$. Then there are unique $\overline{f}_\mu$ in $\mathcal{R}_{m,\alpha}$ such that

$$f(\xi,\eta) = \sum_\mu \overline{f}_\mu(\xi)\eta^\mu.$$

The following lemma is used to prove Theorem 2.10.

Lemma 2.9. *Let $f(\xi,\eta) = \sum_\mu \overline{f}_\mu(\xi)\eta^\mu \in \mathcal{R}_{m+n,\alpha}$. Then the $\overline{f}_\mu \in \mathcal{R}_{m,\alpha}$ and there exist an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$, and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ for $|\mu| < d$, such that*

$$f = \sum_{|\mu|<d} \overline{f}_\mu(\xi) g_\mu(\xi,\eta),$$

in $\mathcal{R}_{m+n,\beta}$.

Proof. We may assume that $\|f\| = 1$, and choose a ν_0 such that $\|\overline{f}_{\nu_0}\| = 1$. Making an $\mathbb{R}$-linear change of variables, and shrinking α if necessary, we may assume that $\overline{f}_{\nu_0}$ is regular at 0 in ξ_m of degree s, say. Write ξ' for $(\xi_1, \dots, \xi_{m-1})$. By Weierstrass division (Theorem 2.5) there is a $\beta > 0$ and there are

$$Q(\xi,\eta) \in \mathcal{R}_{m+n,\beta} \quad \text{and} \quad R(\xi,\eta) = R_0(\xi',\eta) + \dots + R_{s-1}(\xi',\eta)\xi_m^{s-1} \in \mathcal{R}_{m+n-1,\beta}[\xi_m]$$

such that

$$f(\xi,\eta) = \overline{f}_{\nu_0}(\xi)Q(\xi,\eta) + R(\xi,\eta).$$

By induction on m, we may write

$$R_0 = \sum_{|\mu|<d} \overline{R}_{0\mu}(\xi')g_\mu(\xi',\eta)$$

for some $d \in \mathbb{N}$, some $\beta > 0$ and $g_\mu(\xi',\eta) \in \mathcal{R}^\circ_{m+n-1,\beta}$. Writing $R = \sum_\nu \overline{R}_\nu(\xi)\eta^\nu$, observe that each $\overline{R}_\nu$ is an $\mathcal{R}^\circ_{m,\beta}$-linear combination of the $\overline{f}_\nu$, since, taking the coefficient of η^ν on both sides of the equation $f(\xi,\eta) = \overline{f}_{\nu_0}(\xi)Q(\xi,\eta) + R(\xi,\eta)$, we have

$$\overline{f}_\nu = \overline{f}_{\nu_0}\overline{Q}_\nu + \overline{R}_\nu.$$

Consider

$$\begin{aligned} f - \overline{f}_{\nu_0}Q - \sum_{|\mu|<d} \overline{R}_\mu(\xi)g_\mu(\xi',\eta) &=: \ S_1\xi_m + S_2{\xi_m}^2 + \dots + S_{s-1}\xi_m^{s-1} \\ &= \xi_m[S_1 + S_2\xi_m + \dots + S_{s-1}\xi_m^{s-2}] \\ &=: \ \xi_m \cdot S, \end{aligned}$$

say, where the $S_i \in \mathcal{R}^\circ_{m+n-1,\beta}$. Again, observe that each $\overline{S}_\nu$ is an $\mathcal{R}^\circ_{m,\beta}$-linear combination of the $\overline{f}_{\nu'}$. Complete the proof by induction on s, working with S instead of R. □

THEOREM 2.10 (Strong Noetherian property). *Let $f(\xi,\eta) = \sum_\mu \overline{f}_\mu(\xi)\eta^\mu \in \mathcal{R}_{m+n,\alpha}$. Then the $\overline{f}_\mu \in \mathcal{R}_{m,\alpha}$ and there exist an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$, and units $U_\mu(\xi,\eta) \in \mathcal{R}^\circ_{m+n,\beta}$ for $|\mu| < d$ such that*

$$f = \sum_{\mu \in J} \overline{f}_\mu(\xi)\eta^\mu U_\mu(\xi,\eta)$$

in $\mathcal{R}_{m+n,\beta}$, where J is a subset of $\{0,1,\dots,d\}^n$.

Proof. It is sufficient to show that there are an integer d, a set $J \subset \{0,1,\dots,d\}^n$, and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ such that

$$f = \sum_{\mu \in J} \overline{f}_\mu(\xi)\eta^\mu g_\mu(\xi,\eta), \tag{2.1}$$

since then we may assume, rearranging the sum if necessary, that each g_μ is of the form $1 + h_\mu$, where $h_\mu \in (\eta)\mathcal{R}^\circ_{m+n,\beta}$. Shrinking β if necessary will guarantee that the g_μ are units. However, then it is, in fact, sufficient to prove (2.1) for f replaced by

$$f_{I_i} := \sum_{\mu \in I_i} \overline{f}_\mu(\xi)\eta^\mu$$

for each I_i in a finite partition $\{I_i\}$ of $\mathbb{N}^n$.

By Lemma 2.9 there are an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$, and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ for $|\mu| \leqslant d$ such that

$$f = \sum_{|\mu| \leqslant d} \overline{f}_\mu(\xi)g_\mu(\xi,\eta).$$

Rearranging, we may assume for $\nu, \mu \in \{1, \ldots, d\}^n$ that $(\bar{g}_\mu)_\nu$ is equal to 1 if $\mu = \nu$ and that it is equal to 0 otherwise.

Focus on $f_{I_1}(\xi, \eta)$, defined as above by

$$f_{I_1}(\xi, \eta) = \sum_{\mu \in I_1} \overline{f}_\mu(\xi)\eta^\mu$$

with

$$I_1 := \{0, \ldots, d\}^n \cup \{\mu : \mu_i \geqslant d \text{ for all } i\}$$

and note that

$$f_{I_1}(\xi, \eta) = \sum_{|\mu| \leqslant d} \overline{f}_\mu(\xi) g_{\mu, I_1}(\xi, \eta) \tag{2.2}$$

with $g_{\mu,I_1}(\xi, \eta) \in \mathcal{R}^\circ_{m+n,\beta}$ defined by the corresponding sum

$$g_{\mu,I_1}(\xi, \eta) = \sum_{\nu \in I_1} \overline{g}_{\mu,\nu}(\xi)\eta^\nu.$$

It is now clear that g_{μ,I_1} is of the form $\eta^\mu(1 + h_\mu)$, where $h_\mu \in (\eta)\mathcal{R}^\circ_{m+n,\beta}$.

One now proceeds by noting that $f - f_{I_1}$ is a finite sum of terms of the form f_{I_j} for $j > 1$ and $\{I_j\}_j$ a finite partition of $\mathbb{N}^n$ and where each f_{I_j} for $j > 1$ is of the form $\eta_i^\ell q(\xi, \eta')$, where η' is $(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n)$ and q is in $\mathcal{R}^\circ_{m+n-1,\beta}$. These terms can be handled by induction on n. □

Definition 2.11. For $\gamma \in K^\circ_{\mathrm{alg}}$ let γ° denote the closest element of $\mathbb{C}$, that is, the unique element γ° of $\mathbb{C}$ such that $|\gamma - \gamma^\circ| \in K^{\circ\circ}_{\mathrm{alg}}$.

Lemma 2.12. *Let $f \in \mathcal{R}_1$. If $f(\gamma) = 0$, then $f_0(\gamma^\circ) = 0$ (f_0 is the top slice of f). Conversely, if $\beta \in \mathbb{R}$ (or $\mathbb{C}$) and $f_0(\beta) = 0$, then there is a $\gamma \in K_{\mathrm{alg}}$ with $\gamma^\circ = \beta$ and $f(\gamma) = 0$. Indeed, f_0 has a zero of order n at $\beta \in \mathbb{C}$ if, and only if, f has n zeros γ (counting multiplicity)* with *$\gamma^\circ = \beta$.*

Proof. Use Weierstrass preparation and [**1**, Proposition 3.4.1.1]. □

Corollary 2.13. *A non-zero $f \in \mathcal{R}_{1,\alpha}$ has only finitely many zeros in the set $\{x \in K_{\mathrm{alg}} : |x| \leqslant \alpha\}$. Indeed, there exist a polynomial $P(x) \in K[x]$ and a unit $U(x) \in \mathcal{R}_{1,\alpha}$ such that $f(x) = P(x) \cdot U(x)$.*

Proof. Observe that f_0 has only finitely many zeros in $\{x \in \mathbb{C} : |x| \leqslant \alpha\}$, and that non-real zeros occur in complex conjugate pairs, and that f is a unit exactly when f_0 has no zeros in this set (that is, when f_0 is a unit in $A_{1,\alpha}$), and use Lemma 2.12. □

Theorem 2.14 (Quantifier elimination theorem). *Denote by $\mathcal{L}$ the language $\langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R}\rangle$, where the functions in $\mathcal{R}_n$ are interpreted to be zero outside I^n. Then K admits quantifier elimination in $\mathcal{L}$.*

Proof. This is a small modification of the real quantifier elimination of [**4**] as in [**5**], using the Weierstass preparation theorem and the strong Noetherian property above. Crucial is that, in Theorems 2.5 and 2.10, as in [4], β and δ are positive real numbers so one can use the compactness of $[-1,1]^n$ in $\mathbb{R}^n$. □

3. *The o-minimality*

In this section, we prove the o-minimality of K in the language $\mathcal{L}$. Let $\alpha > 1$. As we remarked above, each $f \in A_{n,\alpha}$ defines a function from the polydisc $(I_{\mathbb{C},\alpha})^n \to \mathbb{C}$, where $I_{\mathbb{C},\alpha} := \{x \in \mathbb{C} : |x| \leqslant \alpha\}$, and hence each $f \in \mathcal{R}_{n,\alpha}$ defines a function from $(I_{K_{\rm alg},\alpha})^n \to K_{\rm alg}$, where $I_{K_{\rm alg},\alpha} := \{x \in K_{\rm alg} : |x| \leqslant \alpha\}$. In general $A_{n,\alpha}$ is not closed under composition. However, if $F(\eta_1,\dots,\eta_m) \in \mathcal{R}_{m,\alpha}$, $G_j(\xi) \in \mathcal{R}_{n,\beta}$ for $j = 1,\dots,m$ and $|G_j(x)| \leqslant \alpha$ for all $x \in (I_{K_{\rm alg},\beta})^n$, then $F(G_1(\xi),\dots,G_m(\xi)) \in \mathcal{R}_{n,\beta}$. This is clear if $F \in A_{n,\alpha}$ and the $G_j \in A_{n,\beta}$. The general case follows easily.

For $c, r \in K$, $r > 0$, we denote the 'closed interval' with center c and radius r by

$$I(c,r) := \{x \in K : |x - c| \leqslant r\}$$

and for $c, \delta, \varepsilon \in K$, $0 < \delta < \varepsilon$, we denote the 'closed annulus' with center c, inner radius δ and outer radius ε by

$$A(c,\delta,\varepsilon) := \{x \in K : \delta \leqslant |x - c| \leqslant \varepsilon\}.$$

On occasion we will consider $I(c,r)$ as a disc in $K_{\rm alg}$ and $A(c,\delta,\varepsilon)$ as an annulus in $K_{\rm alg}$, replacing K by $K_{\rm alg}$ in the definitions. No confusion should result. Note that these discs and annuli are defined in terms of the real-closed order on K, not the non-archimedean absolute value $\|\cdot\|$, and hence are not discs or annuli in the sense of [1],[13] or [8], which we will refer to as *affinoid discs* and *affinoid annuli*. For $I = I(c,r)$, $A = A(c,\delta,\varepsilon)$ as above, we define the rings of analytic function on I and A as follows:

$$\mathcal{O}_I := \left\{ f\left(\frac{x-c}{r}\right) : f \in \mathcal{R}_1 \right\}$$
$$\mathcal{O}_A := \left\{ g\left(\frac{\delta}{x-c}\right) + h\left(\frac{x-c}{\varepsilon}\right) : g, h \in \mathcal{R}_1,\ g(0) = 0 \right\}.$$

The elements of $\mathcal{O}_I$ and $\mathcal{O}_A$ are analytic functions on, respectively, the corresponding $K_{\rm alg}$-disc and the $K_{\rm alg}$ annulus as well.

REMARK 3.1. (i) Elements of $\mathcal{O}_A$ are multiplied using the relation $(\delta/(x-c))((x-c)/\varepsilon) = \delta/\varepsilon$ and the fact that $|\delta/\varepsilon| < 1$. Indeed, let $g(\xi_1) = \sum_i a_i \xi_1^i$, $h(\xi_2) = \sum_j b_j \xi_2^j$. Then, using the relation $\xi_1\xi_2 = \delta/\varepsilon$, we have

$$\begin{aligned} g \cdot h &= \sum_{j<i} a_i b_j \left(\frac{\delta}{\varepsilon}\right)^j \xi_1^i + \sum_{i \leqslant j} a_i b_j \left(\frac{\delta}{\varepsilon}\right)^{j-i} \xi_2^{j-i} \\ &= f_1(\xi_1) + f_2(\xi_2). \end{aligned}$$

If $g, h \in A_{1,\alpha}$ and $\delta/\varepsilon \in \mathbb{R}$, then $f_1, f_2 \in A_{1,\alpha}$, and this extends easily to the case $g, h \in \mathcal{R}_{1,\alpha}$ and $\delta/\varepsilon \in K^\circ$. Lemma 3.6 will show that, in fact, the only case of an annulus that we must consider is when $\delta/\varepsilon \in K^{\circ\circ}$.

(ii) We define the gauss-norm on $\mathcal{O}_I$ by

$$\left\| f\left(\frac{x-c}{r}\right) \right\| := \|f(\xi)\|,$$

and on $\mathcal{O}_A$ by

$$\left\| g\left(\frac{\delta}{x-c}\right) + h\left(\frac{x-c}{r}\right) \right\| := \max\{\|g(\xi_1)\|, \|h(\xi_2)\|\}.$$

It is clear that the gauss-norm equals the supremum norm.

(iii) If $f \in \mathcal{O}_I$, then $\|((x-c)/r)f\| = \|f\|$. If $f \in \mathcal{O}_A$, then $\|((x-c)/\varepsilon)f\| \leqslant \|f\|$ and if $\delta/\varepsilon \in K^{\circ\circ}$ (that is, is infinitesimal), then

$$\left\|\left(\frac{x-c}{\varepsilon}\right) g\left(\frac{\delta}{x-c}\right)\right\| = \left\|\frac{\delta}{\varepsilon}\right\| \cdot \left\|g\left(\frac{\delta}{x-c}\right)\right\| < \left\|g\left(\frac{\delta}{x-c}\right)\right\|$$

and

$$\left\|\left(\frac{x-c}{\varepsilon}\right) h\left(\frac{x-c}{\varepsilon}\right)\right\| = \left\|h\left(\frac{x-c}{\varepsilon}\right)\right\|.$$

(iv) If $\|f\| < 1$, then $1-f$ is a unit in $\mathcal{O}_A$. (In fact, it is a *strong unit* — a unit u satisfying $\|1-u\| < 1$.)

DEFINITION 3.2. (i) We say that $f \in \mathcal{O}_{I(0,1)} = \mathcal{R}_1$ has a zero close to $a \in I(0,1)$ if f, as a K_{alg}-function defined on the K_{alg}-disc $\{|x| : |x| \leqslant \alpha\}$ for some $\alpha > 1$, has a K_{alg}-zero b with $a-b$ infinitesimal in K_{alg}. We say f has a zero close to $I(0,1)$ if it has a zero close to a for some $a \in I(0,1)$. For an arbitrary interval $I = I(c,r)$ we say that $f = F((x-c)/r) \in \mathcal{O}_I$ has a zero close to $a \in I$ if F has a zero close to $(a-c)/r \in I(0,1)$, and that f has a zero close to $I(c,r)$ if F has a zero close to $I(0,1)$.

(ii) For $a, b \in K^\circ$, $a, b > 0$, we write $a \sim b$ if $a/b, b/a \in K^\circ$ and we write $a << b$ if $a/b \in K^{\circ\circ}$.

(iii) Let X be an interval or an annulus, and let f be defined on a superset of X. We shall write $f \in \mathcal{O}_X$ to mean that there is a function $g \in \mathcal{O}_X$ such that

$$f|_X = g.$$

LEMMA 3.3. *If $f \in \mathcal{O}_{I(c,r)}$ has no zero close to $I(c,r)$, then there is a cover of $I(c,r)$ by finitely many closed intervals $I_j = I(c_j, r_j)$ such that $1/f \in \mathcal{O}_{I_j}$ for each j.*

Proof. It is sufficient to consider the case $I(c,r) = I(0,1)$. Cover $I(0,1)$ by finitely many intervals $I(c_j, r_j)$, $c_j, r_j \in \mathbb{R}$, such that f has no K_{alg}-zero in the K_{alg}-disc $I(c_j, r_j)$. Finally use Corollary 2.13. □

REMARK 3.4. The function $f(x) = 1 + x^2$ has no zeros close to $I(0,1)$. It is not a unit in $\mathcal{O}_{I(0,1)}$, but it is a unit in both $\mathcal{O}_{I(-1/2,1/2)}$ and $\mathcal{O}_{I(1/2,1/2)}$. The function $g(x) = x$ is not a unit in $\mathcal{O}_{I(\delta,\varepsilon)}$ for any $0 < \delta \in K^{\circ\circ}$ and $0 < \varepsilon \in K^\circ \setminus K^{\circ\circ}$. It is of course a unit in $\mathcal{O}_{A(0,\delta,\varepsilon)}$. The function $1/g = 1/x \in A(0,\delta,1)$ for all $\delta > 0$ but is not in $\mathcal{O}_I$ for $I = I((1+\delta)/2, (1-\delta)/2)$, for any $\delta \in K^{\circ\circ}$. Thus we see that if $X_1 \subset X_2$ are annuli or intervals, then it does not necessarily follow that $\mathcal{O}_{X_2} \subset \mathcal{O}_{X_1}$. However, the following are clear. If $I_1 \subset I_2$ are intervals, then $\mathcal{O}_{I_2} \subset \mathcal{O}_{I_1}$. If $0 < \delta \in K^{\circ\circ}$, $0 < r \in K^\circ \setminus K^{\circ\circ}$, $r < 1$, and $A = A(0,\delta,1)$, $I = I((1+r)/2, (1-r)/2)$, then $\mathcal{O}_A \subset \mathcal{O}_I$. If $A_1 \subset A_2$ are annuli that have the same center, then $\mathcal{O}_{A_2} \subset \mathcal{O}_{A_1}$. If $0 < \delta << c$ and $0 < r < c/\alpha$ for some $\alpha \in \mathbb{R}$, $\alpha > 1$, and $I = I(c,r) \subset A(0,\delta,1) = A$, then $\mathcal{O}_A \subset \mathcal{O}_I$. Writing $x = c - y$, $|y| \leqslant r$ we see that

$$\frac{\delta}{x} = \frac{\delta}{(c-y)} = \frac{\delta/c}{1-y/c} + \frac{\delta}{c}\sum\left(\frac{y}{c}\right)^k = \frac{\delta}{c}\sum\left(\frac{r}{c}\frac{y}{r}\right)^k.$$

Restating Corollary 2.13 we have the following corollary.

COROLLARY 3.5. *If $f \in \mathcal{O}_{I(c,r)}$, then there exist a polynomial $P \in K[\xi]$ and a unit $U \in \mathcal{O}_{I(c,r)}$ such that $f(\xi) = P(\xi) \cdot U(\xi)$.*

LEMMA 3.6. *If $\varepsilon < N\delta$ for some $N \in \mathbb{N}$, then there is a covering of $A(c,\delta,\varepsilon)$ by finitely many intervals I_j such that for every $f \in \mathcal{O}_{A(c,\delta,\xi)}$ and each j, we have $f \in \mathcal{O}_{I_j}$.*

Proof. Use Lemma 3.3 or reduce directly to the case $\varepsilon = 1$, $0 < \delta = r \in \mathbb{R}$ and the two intervals

$$[-1,-r] = I\left(-\frac{1+r}{2}, \frac{1-r}{2}\right) \quad \text{and} \quad [r,1] = I\left(\frac{1+r}{2}, \frac{1-r}{2}\right). \qquad \square$$

The following lemma is key for proving o-minimality.

LEMMA 3.7. *Let $f \in A(c,\delta,\varepsilon)$. There are finitely many intervals and annuli X_j that cover $A(c,\delta,\varepsilon)$, polynomials P_j and units $U_j \in \mathcal{O}_{X_j}$ such that for each j we have $f|_{X_j} = (P_j \cdot U_j)|_{X_j}$.*

Proof. By the previous lemma, we may assume that $c = 0$, $\varepsilon = 1$ and $\delta \in K^{\circ\circ}$ (that is, δ is infinitesimal, say $\delta = t^\gamma$ for some $\gamma > 0$). Let

$$f(x) = g\left(\frac{\delta}{x}\right) + h(x) \quad \text{with } g(\xi), h(\xi) \in \mathcal{R}_1,\ g(0) = 0,$$

and

$$g(\xi) = \sum_{i \in I \subset \mathbb{N}} t^{\alpha_i} \xi^{n_i} g_i(\xi)$$

with $n_i > 0$, $g_i(0) \neq 0$, $g_i \in A_{1,\alpha}$ for some $\alpha > 1$. Observe that

$$xg\left(\frac{\delta}{x}\right) = \sum_{i \in I} (t^{\alpha_i}\delta) \left(\frac{\delta}{x}\right)^{n_i - 1} g_i\left(\frac{\delta}{x}\right).$$

Hence (see Remark 3.1) for suitable $n \in \mathbb{N}$, absorbing the constant terms into h, we have

$$x^n f(x) = \overline{g}\left(\frac{\delta}{x}\right) + \overline{h}(x)$$

where $\overline{g}(0) = 0$ and $\|\overline{g}\| < \|\overline{h}\|$. (For use in Section 4, below, note that this argument does not use the restriction that K is complete or of rank 1.) Multiplying by a constant, we may assume that $\|\overline{h}\| = 1$. Let

$$\overline{g}(\xi) = \sum_{i \in \overline{I}} t^{\beta_i} \xi^{m_i} \overline{g}_i(\xi) \quad \text{with } m_i > 0 \text{ and } \beta_i > 0 \text{ for each } i,$$

and

$$\overline{h}(\xi) = \xi^{k_0} \overline{h}_0(\xi) + \sum_{i \in J \subset \mathbb{N} \setminus 0} t^{\gamma_i} \overline{h}_i(\xi) \quad \text{with } \overline{h}_0(0) \neq 0 \text{ and the } \gamma_i > 0 \text{ increasing.}$$

Since $\|\overline{g}\| = \|t^{\beta_0}\| < 1$ there is a δ' with $\delta \leqslant \delta' \in K^{\circ\circ}$ (that is, $\|\delta'\| < 1$) such that

$$\|t^{\gamma_1}\| < \|(\delta')^{k_0}\| \quad \text{and} \quad \|\overline{g}\| < \|(\delta')^{k_0}\|.$$

Splitting off some intervals of the form

$$I\left(\frac{-1-r}{2}, \frac{-1+r}{2}\right) = [-1,-r] \quad \text{or} \quad I\left(\frac{1+r}{2}, \frac{1-r}{2}\right) = [r,1] \quad \text{for } r > 0,\ r \in \mathbb{R},$$

(on which the result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume that $\overline{h}_0$, is a unit in $A_{1,\alpha}$, for some $\alpha > 1$. Therefore,

$$\begin{aligned} x^n f(x) &= \overline{g}\left(\frac{\delta}{x}\right) + \overline{h}(x) \\ &= \overline{h}_0(x)x^{k_0}\left[1 + \sum_{i=1}^{\infty}\left(\frac{\delta'}{Px}\right)^{k_0} \frac{t^{\gamma_i}}{(\delta')^{k_0}}\frac{\overline{h}_i(x)}{\overline{h}_0(x)} + \frac{1}{\overline{h}_0(x)}\left(\frac{1}{\delta'}\right)^{k_0}\left(\frac{\delta'}{x}\right)^{k_0} g\left(\frac{\delta}{\delta'}\frac{\delta'}{x}\right)\right]. \end{aligned}$$

By our choice of δ' and Remark 3.1 the quantity in square brackets is a (strong) unit. Hence we have taken care of an annulus of the form $A(0, \delta', 1)$ for some δ' with $|\delta| \leqslant |\delta'|$ and $\|\delta'\| < 1$.

Observe that the change of variables $y = \delta/x$ interchanges the sets

$$\{x : \|x\| = \|\delta\| \text{ and } \delta \leqslant |x|\} \quad \text{and} \quad \{y : \|y\| = 1 \text{ and } |y| \leqslant 1\}.$$

Hence, as above, there is a $\delta'' \in K^{\circ\circ}$ with $\|\delta\| < \|\delta''\|$ and a covering of the annulus $\delta \leqslant |x| \leqslant \delta''$ by finitely many intervals and annuli with the required property.

It remains to treat the annulus $\delta'' \leqslant |x| \leqslant \delta'$. Using the terminology of [**11**] and [**13**], observe that on the much bigger *affinoid* annulus $\|\delta''\| \leqslant \|x\| \leqslant \|\delta'\|$ the function f is strictly convergent, indeed even overconvergent. Hence, as in [**13**, Lemma 3.6], on this affinoid annulus we can write

$$f = \frac{P(x)}{x^{\ell}} \cdot U(x),$$

where $P(x)$ is a polynomial and $U(x)$ is a strong unit (that is, $\|U(x) - 1\| < 1$.) □

Corollary 3.8. *If X is an interval or an annulus and $f \in \mathcal{O}_X$, then the set $\{x \in X : f(x) \geqslant 0\}$ is semialgebraic (that is, a finite union of (closed) intervals).*

Proof. This is an immediate corollary of Corollary 3.5 and Lemma 3.7, since units do not change sign on intervals and since an annulus has two intervals as connected components. □

Definition 3.9. For $c = (c_1, \ldots, c_n)$, $r = (r_1, \ldots, r_n)$ we define the *poly-interval*

$$I(c, r) := \{x \in K^n : |x_i - c_i| \leqslant r_i,\ i = 1, \ldots, n\}.$$

This also defines the corresponding polydisc in $(K_{\text{alg}})^n$. The ring of analytic functions on this poly-interval (or polydisc) is

$$\mathcal{O}_{I(c,r)} := \left\{f\left(\frac{x_1 - c_1}{r_1}, \ldots, \frac{x_n - c_n}{r_n}\right) : f \in \mathcal{R}_n\right\}.$$

Lemma 3.10. *Let $\alpha, \beta > 1, F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha}$ and $G_j(\xi_1, \ldots, \xi_n) \in \mathcal{R}_{n,\beta}$ with $\|G_j\| \leqslant 1$ for $j = 1, \ldots, m$. Let $X = \{x \in [-1, 1]^n : |G_j(x)| \leqslant 1$ for $j = 1, \ldots, m\}$. There are (finitely many) $c_i = (c_{i1}, \ldots, c_{in}) \in \mathbb{R}^n$, $\varepsilon_i \in \mathbb{R}$, $\varepsilon_i > 0$ with $\varepsilon_i < |c_{ij}|$ if $c_{ij} \neq 0$ such that the (poly) intervals $I_i = \{x \in K : |x_j - c_{ij}| < \varepsilon_i$ for $j = 1, \ldots, n\}$ cover X, $|G_j(x)| < \alpha$ for all $x \in I_i$, $j = 1, \ldots, m$, and there are $H_i \in \mathcal{O}_{I_i}$ such that*

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}.$$

Proof. Use the compactness of $[-1, 1]^n \cap \mathbb{R}^n$ and the following facts. If $\|G_j\| = 1$, then $|G_j(x) - G_{j0}(x)|$ is infinitesimal for all $x \in K^\circ$, where G_{j0} is the top slice of G_j. If $\|G_j\| < 1$, then $|G_j(x)| \in K^{\circ\circ}$ for all $x \in [-1, 1]^n$. □

Corollary 3.11. (i) *Let I be an interval, $F(\eta_1,\dots,\eta_m)\in\mathcal{R}_m$ and $G_j\in\mathcal{O}_I$ for $j=1,\dots,m$. Then there are finitely many intervals I_i covering I and functions $H_i\in\mathcal{O}_{I_i}$ such that for each i*

$$F(G_1,\dots,G_m)|_{I_i}=H_i|_{I_i}.$$

(ii) *Let A be an annulus, $F(\eta_1,\dots,\eta_m)\in\mathcal{R}_m$ and $G_j\in\mathcal{O}_A$ for $j=1,\dots,m$. Then there are finitely many X_i, each an interval or an annulus, covering A and $H_i\in\mathcal{O}_{X_i}$ such that for each i*

$$F(G_1,\dots,G_m)|_{X_i}=H_i|_{X_i}.$$

Proof. Part (i) reduces to Lemma 3.10 once we see that if $\|G_j\|>1$, then we can use Corollary 2.13 to restrict to the subintervals of I around the zeros of G_j on which $|G_j|\leqslant C$ for some $C\in\mathbb{R}$, $C>1$. On the rest of I, $F(G_1,\dots,G_m)$ is zero.

For (ii), we may assume that $A=A(0,\delta,1)$ with δ infinitesimal and that $G_j(x)=G_{j_1}(\delta/x)+G_{j_2}(x)$. As in (i), we may reduce to the case that $\|G_j\|\leqslant 1$, using Lemma 3.7 instead of Corollary 2.13, and using Lemma 3.6 and Remark 3.4. Apply Lemma 3.10 to the functions F and $G'_j(\xi_1,\xi_2)=G_{j_1}(\xi_1)+G_{j_2}(\xi_2)$. The case $c=(0,0)$ gives us the annulus $|\delta/x|\leqslant\varepsilon$, $|x|\leqslant\varepsilon$ that is, $\delta/\varepsilon\leqslant|x|\leqslant\varepsilon$. The case $c=(0,c_2)$ with $c_2\neq 0$ gives us $\delta/|x|\leqslant\varepsilon$ and $|x-c_2|\leqslant\varepsilon$ with $\varepsilon<|c_2|$. This is equivalent to $|x-c_2|\leqslant\varepsilon$ since $\varepsilon,c_1\in\mathbb{R}$, and hence equivalent to

$$-\varepsilon\leqslant x-c_2\leqslant\varepsilon \quad\text{or}\quad c_2-\varepsilon\leqslant x\leqslant c_2+\varepsilon$$

which is an interval bounded away from 0. The case $c=(c_1,0)$ gives us $|\delta/x-c_1|\leqslant\varepsilon$, $|x|\leqslant\varepsilon$ (since $\varepsilon<|c_1|$) which is equivalent to $|\delta/x-c_1|\leqslant\varepsilon$ or $c_1-\varepsilon\leqslant\delta/x\leqslant c_1+\varepsilon$ or (considering the case $c_1>0$; the case $c_1<0$ is similar) $\delta/(c_1+\varepsilon)\leqslant x\leqslant\delta/(c_1-\varepsilon)$ which is part of an annulus that can be reduced to intervals using Lemma 3.6. The case $c=(c_1,c_2)$ with both $c_1,c_2\neq 0$ is vacuous since either x or δ/x is infinitesimal on A and $\varepsilon<|c_1|,|c_2|$. □

Lemma 3.12. *Let X be an interval or an annulus and let $f,g\in\mathcal{O}_X$. There are finitely many subintervals and subannuli $X_i\subset X$, $i=1,\dots,\ell$, such that*

$$\{x\in X:|f(x)|\leqslant|g(x)|\}\subset\bigcup X_i\subset X,$$

and, except at finitely many points,

$$\frac{f}{g}\Big|_{X_i}\in\mathcal{O}_{X_i}.$$

Proof. We consider the case that X is an interval, I, and may take $I=I(0,1)=[-1,1]$. We may assume by Corollary 3.5 that f and g have no common zero. If g has no zeros close to $I(0,1)$, then we are done by Lemma 3.3. Let $\alpha_1,\dots,\alpha_n$ be the distinct elements α of $[-1,1]\cap\mathbb{R}$ such that g has at least one zero close to (that is, within an infinitesimal of) α. Breaking into subintervals and making changes of variables we may assume that $n=1$ and that $\alpha_1=0$. Again making a change of variables (over K) we may assume that g has at least one zero with zero 'real' part, that is, of the form $a=\sqrt{-1}\alpha$ for some $\alpha\in K$. Let N denote the number of zeros of $f\cdot g$ close to 0 in $I(0,1)$. Let $\delta=3\cdot\max\{|x|:x\in K_{\text{alg}}$ close to 0 and $f(x)\cdot g(x)=0\}$. If $\delta=0$, then $a=\alpha=0$ and there is no other zero of $f\cdot g$ close to zero in $I(0,1)$. Then there is a $\delta'>0$ such that for $|x|<\delta'$ we have $|g(x)|<|f(x)|$. Then the interval $I(0,\delta')$ drops away, and on the annulus $A(0,\delta',1)$ the function g is a unit. If $\delta>0$, then we consider the interval $I(0,\delta)$ and the annulus $A(0,\delta,1)$ separately, and proceed by induction on N. Therefore, suppose that $\delta>0$ and $f\cdot g$ has N zeros close to 0 in $I(0,1)$. Let α be as above. If $\alpha\sim\delta$, then the zero

$a = \sqrt{-1}\alpha$ is not close to $I(0,\delta)$ and by restricting to $I(0,\delta)$ we have reduced N. If $|\alpha| << \delta$, then this zero is close to 0 in $I(0,\delta)$, but the largest zeros (those of size $\delta/3$) are not close to 0 in $I(0,\delta)$, and hence restricting to $I(0,\delta)$ again reduces N.

It remains to consider the case of the annulus $A(0,\delta,1) = \{x : \delta \leqslant |x| \leqslant 1\}$ where all the zeros of g are within $\delta/3$ of 0. By Lemma 3.7, we may assume that $g(x) = P(x) \cdot U(x)$, where U is a unit and all the zeros of P are within $\delta/3$ of 0. Let α_i, $i = 1, \ldots, \ell$, be these zeros. For $|x| \geqslant \delta$ we may write

$$\frac{1}{x-\alpha_i} = \frac{1}{x}\frac{1}{1-\alpha_i/x} = \frac{1}{x}\sum_{j=0}^{\infty}\left(\frac{\alpha_i}{x}\right)^j = \frac{1}{x}\sum_{j=0}^{\infty}\left(\frac{\alpha_i}{\delta}\right)^j\left(\frac{\delta}{x}\right)^j \quad \text{and} \quad \left|\frac{\alpha_i}{\delta}\right| \leqslant \frac{1}{3}.$$

Hence $f/g \in \mathcal{O}_{A(0,\delta,1)}$. This completes the case that X is an interval.

The case that X is an annulus is similar — one can cover X with finitely many subannuli X_i and subintervals Y_j such that for each i, $g|_{X_i}$ is a unit in $\mathcal{O}_{X_i}$ and for each j, $f|_{Y_j}, g|_{Y_j} \in \mathcal{O}_{Y_j}$. □

From Corollary 3.11 and Lemma 3.12, we now have the following proposition, by induction on terms.

Proposition 3.13. *Let $f_1, \ldots, f_\ell$ be $\mathcal{L}$-terms in one variable, x. There is a covering of $[-1,1]$ by finitely many intervals and annuli X_i such that except for finitely many values of x, we have for each i and j that $f_i|_{X_j} \in \mathcal{O}_{X_j}$ (that is, $f_i|_{X_j}$ agrees with an element of $\mathcal{O}_{X_j}$ except at finitely many points of X_j).*

This, together with Corollary 3.8, gives the following theorem.

Theorem 3.14. *The field K is o-minimal in $\mathcal{L}$.*

4. *Further extensions*

In this section, we give extensions of the results of Sections 2 and 3 and the results of [**13**].

Let G be an (additive) ordered abelian group. Let t be a symbol. Then t^G is a (multiplicative) ordered abelian group. Following the notation of [**5**, **6**] (but not [**7**] or [**12**]) we define $\mathbb{R}((t^G))$ to be the maximally complete valued field with additive value group G (or multiplicative value group t^G) and residue field $\mathbb{R}$. Therefore,

$$\mathbb{R}((t^G)) := \left\{\sum_{g\in I} a_g t^g : a_g \in \mathbb{R} \text{ and } I \subset G \text{ well ordered}\right\}.$$

We shall be a bit sloppy about mixing the additive and multiplicative valuations. $I \subset G$ is well ordered exactly when $t^I \subset t^G$ is reverse well ordered. The field K of Puiseux series, or its completion, is a proper subfield of $\mathbb{R}_1 := \mathbb{R}((t^{\mathbb{Q}}))$. Considering $G = \mathbb{Q}^m$ with the lexicographic ordering, we define

$$\mathbb{R}_m := \mathbb{R}((t^{\mathbb{Q}^m})).$$

It is clear that if $G_1 \subset G_2$ as ordered groups, then $\mathbb{R}((t^{G_1})) \subset \mathbb{R}((t^{G_2}))$ as valued fields. Also, $\mathbb{R}((t^G))$ is Henselian and, if G is divisible, then $\mathbb{R}((t^G))$ is real closed. We shall continue to use '$<$' for the corresponding order on $\mathbb{R}((t^G))$.

In analogy with Section 2, we define the following.

Definition 4.1. We have

$$\begin{aligned}\mathcal{R}_{n,\alpha}(G) &:= A_{n,\alpha}\widehat{\bigotimes}^{*}_{\mathbb{R}}\mathbb{R}((t^G))\\ &:= \left\{\sum_{g\in I} f_g t^g : f_g \in A_{n,\alpha} \text{ and } I \subset G \text{ well ordered}\right\}\\ \mathcal{R}_n(G) &:= \bigcup_{\alpha>1}\mathcal{R}_{n,\alpha}(G)\\ \mathcal{R}(G) &:= \bigcup_{n}\bigcup_{\alpha>1}\mathcal{R}_{n,\alpha}(G).\end{aligned}$$

As in Section 2, the elements of $\mathcal{R}_n(G)$ define functions from $I^n \subset \mathbb{R}((t^G))^n$ to $\mathbb{R}((t^G))$. Indeed this interpretation is a ring endomorphism. In other words, the field $\mathbb{R}((t^G))$ has analytic $\mathcal{R}(G)$-structure. (See [**3**] and especially [**2**] for more about fields with analytic structure.) The elements of $\mathcal{R}_n(G)$ are interpreted as zero on $\mathbb{R}((t^G))^n \setminus I^n$. Let

$$\mathcal{L}_G := \langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R}(G)\rangle,$$

and so $\mathbb{R}((t^G))$ is an $\mathcal{L}_G$-structure. Indeed, if $G_1 \subset G_2$, then $\mathbb{R}((t^{G_2}))$ is an $\mathcal{L}_{G_1}$-structure.

Theorem 4.2. *The Weierstrass preparation theorem* (*Theorem* 2.5) *and the strong Noetherian property* (*Theorem* 2.10) *hold with* $\mathcal{R}$ *replaced by* $\mathcal{R}(G)$.

Proof. Only minor modifications to the proofs of Theorems 2.5 and 2.10 are needed. □

The arguments of Sections 2 and 3 prove the following theorem.

Theorem 4.3. *If* G *is divisible, then* $\mathbb{R}((t^G))$ *admits quantifier elimination and is o-minimal in* $\mathcal{L}_G$.

Corollary 4.4. *If* $G_1 \subset G_2$ *are divisible, then* $\mathbb{R}((t^{G_1})) \prec \mathbb{R}((t^{G_2}))$ *in* $\mathcal{L}_{G_1}$.

We shall show in Section 5 that, though for $m < n$ we have $\mathbb{R}_m \prec \mathbb{R}_n$ in $\mathcal{L}_{\mathbb{Q}^m}$, there is a sentence of $\mathcal{L}_{\mathbb{Q}^n}$ that is true in $\mathbb{R}_n$ but is not true in any o-minimal expansion of $\mathbb{R}_m$.

The results of [**13**] also extend to this more general setting.

Definition 4.5. We define the ring of **strictly convergent power series over* $\mathbb{R}((t^G))$ as

$$\mathbb{R}((t^G))^*\langle\xi\rangle := \left\{\sum_{g\in I} a_g(\xi)t^g : a_g(\xi) \in \mathbb{R}[\xi] \text{ and } I \text{ well ordered}\right\},$$

and the subring of **overconvergent power series over* $\mathbb{R}((t^G))$ as

$$\mathbb{R}((t^G))^*\langle\langle\xi\rangle\rangle := \{f : f(\gamma\xi) \in \mathbb{R}((t^G))^*\langle\xi\rangle \text{ for some } \gamma \in \mathbb{R}((t^G)),\ \|\gamma\| > 1\},$$

$$\mathcal{R}(G)_{\text{over}} := \bigcup_n \mathbb{R}((t^G))^*\langle\langle\xi_1, \ldots, \xi_n\rangle\rangle,$$

and the corresponding *overconvergent language* as

$$\mathcal{L}_{G,\,\text{over}} := \langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R}(G)_{\text{over}}\rangle.$$

As in [13] we have the following theorem.

THEOREM 4.6. *If G is divisible, then* $\mathbb{R}((t^G))$ *admits quantifier elimination and is o-minimal in* $\mathcal{L}_{G,\,\mathrm{over}}$.

Of course the o-minimality follows immediately from Theorem 4.3.

5. *Extensions of the example of Hrushovski and Peterzil*

In this section, we show that with minor modifications, the idea of the example of [10] can be iterated to give a nested family of examples. This relates to a question of Hrushovski and Peterzil as to whether there exists a small class of o-minimal structures such that any sentence, true in some o-minimal structure, can be satisfied in an expansion of a model in the class. Combining this with expansions with the exponential function, one can perhaps elaborate the tower of examples further.

Consider the functional equation

$$F(\beta z) = \alpha z F(z) + 1, \qquad (*)$$

and suppose that F is a 'complex analytic' solution for $|z| \leqslant 1$. By this we mean that, writing $z = x + \sqrt{-1}y$, $F(z) = f(x, y) + \sqrt{-1}g(x, y)$, $F(z)$ is differentiable as a function of z. This is a definable condition on the two 'real' functions, f, g of the two 'real' variables x, y. Then

$$F(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where

$$a_k = \frac{\alpha^k}{\beta^{k(k+1)/2}}$$

(α, and β are parameters).

By this we mean that for each $n \in \mathbb{N}$ there is a constant A_n such that

$$\left| F(z) - \sum_{k=0}^{n} a_k z^k| \leqslant A_n |z^{n+1} \right| \qquad (**)$$

is true for all z with $|z| \leqslant 1$. Indeed, by [15, Theorem 2.50], one can take $A_n = C \cdot 2^{n+1}$, for C a constant independent of n.

Consider the following statement: $F(z)$ is a complex analytic function (in the above sense) on $|z| \leqslant 1$ that satisfies $(*)$; the number $\beta > 0$ is within the radius of convergence of the function $f(z) = \sum_{n=1}^{\infty} (n-1)! z^n$ and $\alpha > 0$.

This statement is not satisfiable by any functions in any o-minimal expansion of the field of Puiseux series K_1, or the maximally complete field $\mathbb{R}_1 = \mathbb{R}((t^{\mathbb{Q}}))$, because, if it were, we would have $\|\beta\| = \|t^\gamma\|$ and $\|\alpha\| = \|t^\delta\|$, for some $\gamma, \delta \in \mathbb{Q}$, $\gamma, \delta > 0$, and for suitable choice of n the condition $(**)$ would be violated. On the other hand, if we choose $\alpha, \beta \in \mathbb{R}_2$ with $\mathrm{ord}(\alpha) = (1, 0)$ and $\mathrm{ord}(\beta) = (0, 1)$ then $\Sigma a_k z^k \in \mathbb{R}_2\langle\langle z\rangle\rangle^*$ satisfies the statement on $\mathbb{R}_2$.

This process can clearly be iterated to give, in the notation of Section 4, the following proposition.

PROPOSITION 5.1. *For each* m *there is a sentence of* $\mathcal{L}_{\mathbb{Q}^m}$ *that is true in* $\mathbb{R}_m$ *but is not satisfiable in any o-minimal expansion of* $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$.

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