MOTIVIC FUNCTIONS, INTEGRABILITY, AND APPLICATIONS TO HARMONIC ANALYSIS ON p-ADIC GROUPS

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ABSTRACT. We provide a short and self-contained overview of the techniques based on motivic integration as they are applied in harmonic analysis on p-adic groups; our target audience is mainly representation theorists with no background in model theory (and model theorists with an interest in recent applications of motivic integration in representation theory, though we do not provide any representation theory background). We aim to give a fairly comprehensive survey of the results in harmonic analysis that were proved by such techniques in the last ten years, with emphasis on the most recent techniques and applications from [13], [8], and [32, Appendix B].

1. INTRODUCTION

The purpose of this short article is three-fold: to announce the new results related to analytic properties of motivic (exponential) functions, which we hope could be widely applicable; to summarize the recent applications of these results in harmonic analysis on p-adic groups; and to provide a short "user-guide" to these "motivic" techniques aimed at representation theorists.

Concretely, motivic (exponential) functions are complex-valued functions on p-adic manifolds defined uniformly in p by means of a rich yet sufficiently manageable language of logic. The word "exponential" refers to the situation when additive characters of the local field (leading to exponential sums) are included in the set of ingredients one is allowed to use when making such a function. The class of motivic exponential functions has two key features that make it particularly

useful. On the one hand, these functions can be described in a field-independent manner, and thus are an ideal tool for transferring results between different local fields. On the other hand, the class of motivic (exponential) functions is closed under integration with respect to parameters, and thanks to this property one can show that many functions naturally arising in harmonic analysis on p-adic groups belong to this class. We give a user-oriented short introduction to motivic (exponential) functions and motivic integration in §2 below.

The project of using motivic integration in harmonic analysis on p-adic groups was initiated by T. C. Hales in 1998 and goes under the emerging name of "motivic harmonic analysis." One of the goals of this article is to collect results about "definability" of various objects arising in representation theory of p-adic groups and the applications in harmonic analysis that follow from such definability, creating an easy reference and summarizing the state of the art of this project. We do that in §5. A reader who just wishes to see the kind of results in harmonic analysis one can prove using these techniques should look at §6 and §7, specifically Theorems 15, 16, 17, and 18, whose statements are independent of any logic terminology.

Since the main goal of the paper is to illustrate the use of motivic integration techniques and make them accessible to the large community working on p-adic harmonic analysis, in the following sections, we summarize most of the definitions we use that relate to motivic functions, motivic measures, etc. However, we do not include here any definitions related to harmonic analysis on reductive groups, instead referring the reader, for example, to the beautiful article [25]. We have also tried to include as few technical details as possible in this exposition, while still including precise statements of the results, so that they can be used as stated here.

2. What is a motivic (exponential) function?

Informally, motivic functions are built from definable functions in the Denef-Pas language and motivic *exponential* functions are built from motivic functions, additive characters on local fields, and definable functions serving as arguments of the characters. As such, they are given independently of the local field and can be interpreted in any local field and with any additive character. Let us recall the definition of the Denef-Pas language first, and proceed to motivic (exponential) functions afterwards, where we slightly simplify the terminology and notation of [13].

2.1. **Denef-Pas language.** The Denef-Pas language is a first order language of logic designed for working with valued fields. Formulas in this language will allow us to uniformly handle sets and functions for all local fields. We start by defining two sublanguages of the language of Denef-Pas: the language of rings and the Presburger language.

2.1.1. *The language of rings.* Apart from the symbols for variables $x_1, \ldots, x_n, \ldots$ and the usual logical symbols equality $=$, parentheses (and), the quantifiers $\exists$ and $\forall$, and the logical operations conjunction $\wedge$, negation $\neg$, and disjunction $\vee$, the language of rings consists of the following symbols:

- constants 0, 1;
- binary functions $\times$, $+$.

A (first-order) formula in the language of rings is any syntactically correct formula built out of these symbols. One usually omits the words 'first order.'

If a formula in the language of rings has n free (i.e., unquantified) variables then it defines a subset of R^n for any ring R. For example the formula "$\exists x_2\,(x_2 \times x_1 = 1)$" defines the set of units $R^\times$ in any ring R; note that quantifiers (by convention) always run over the ring in question. Note also that quantifier-free formulas in the language of rings define constructible sets (as they appeared in classical algebraic geometry).

2.1.2. *Presburger language.* A formula in Presburger language is built out of variables running over $\mathbb{Z}$, the logical symbols (as above) and symbols $+$, $\leq$, 0, 1, and for each $d = 2, 3, 4, \ldots$, a symbol $\equiv_d$ to denote the binary relation $x \equiv y \pmod{d}$. Note the absence of the symbol for multiplication.

Since multiplication is not allowed, sets and functions defined by formulas in the Presburger language are in fact very basic, cf. [5] or [30]. By [30], quantifiers are never needed to describe Presburger sets, so all definable sets are of simple form. In particular, the subsets of $\mathbb{Z}$ defined by Presburger formulas are finite unions of arithmetic progressions (in positive or negative direction) and points. This simple observation is at the source of the simple forms of the bounds in Theorems 11, 12, and 17, 18.

2.1.3. *Denef-Pas language.* The Denef-Pas language is a three-sorted language in the sense that its formulas speak about three different "sorts" of elements: those of the valued field, of the residue field, and of the value group (which will always be $\mathbb{Z}$ in our setting). Each variable in such a formula only runs over the elements of one of the sorts, i.e., there are three disjoint sets of symbols for the variables of the different sorts. For a formula to be syntactically correct, one has to pay attention to the sorts when composing functions and plugging them into relations.

In addition to the variables and the logical symbols, the formulas use the following symbols.

- In the valued field sort: the language of rings.
- In the residue field sort: the language of rings.
- In the $\mathbb{Z}$-sort: the Presburger language.
- The symbol $\mathrm{ord}(\cdot)$ for the valuation map from the non-zero elements of the valued field sort to the $\mathbb{Z}$-sort, and the symbol $\overline{\mathrm{ac}}(\cdot)$ for the so-called angular component, which is a multiplicative function from the valued field sort to the residue field sort (more about this function below).

A formula in this language can be interpreted in any discretely valued field F which comes with a uniformizing element ϖ, by letting the variables range over F, over its residue field k_F, and over $\mathbb{Z}$, respectively, depending on the sort they belong to; ord is the valuation map (defined on $F^\times$ and such that $\mathrm{ord}(\varpi) = 1$), and $\overline{\mathrm{ac}}$ is defined as follows. If x is a unit (that is, $\mathrm{ord}(x) = 0$), then $\overline{\mathrm{ac}}(x)$ is the residue of x modulo ϖ (thus, an element of the residue field); for all other non-zero x, one puts $\overline{\mathrm{ac}}(x) := \varpi^{-\mathrm{ord}(x)} x \bmod (\varpi)$. Thus, for $x \neq 0$, $\overline{\mathrm{ac}}(x)$ is the residue class of the first non-zero coefficient of the ϖ-adic expansion of x. Finally, we define $\overline{\mathrm{ac}}(0) = 0$.

Thus, a formula φ in this language with n free valued-field variables, m free residue-field variables, and r free $\mathbb{Z}$-variables gives naturally, for each discretely valued field F with a uniformizer ϖ, a subset $\varphi(F)$ of $F^n \times k_F^m \times \mathbb{Z}^r$: namely, $\varphi(F)$ is the set of all the tuples where the interpretation of φ in F is "true" (where a variable appearing inside a quantifier runs over either F, k_F, or $\mathbb{Z}$, respectively, depending on its sort).

2.1.4. *Adding constants to the language.* In many situations one needs to work with geometric objects defined over some fixed base number field, or over its ring of integers $\mathcal{O}$. In such situations, on top of the symbols for the constants that are already present in the above language (like 0 and 1), we will add to the Denef-Pas language all elements of $\mathcal{O}[[t]]$ as extra symbols for constants in the valued field sort. We denote the resulting language by $\mathcal{L}_{\mathcal{O}}$.

Given a complete discretely valued field F which is an algebra over $\mathcal{O}$ via a chosen ring homomorphism (mapping 1 to 1) $\iota : \mathcal{O} \to F$, with a chosen uniformizer ϖ of the valuation ring $\mathcal{O}_F$ of F, one can interpret the formulas in $\mathcal{L}_{\mathcal{O}}$ as described in the previous subsection, where the new constants from $\mathcal{O}$ are interpreted as elements of F by using ι, the constant symbol t is interpreted as the uniformizer ϖ, and thus, by the completeness of F, elements of $\mathcal{O}[[t]]$ can be naturally interpreted in F as well.

Definition 1. Let $\mathcal{C}_{\mathcal{O}}$ be the collection of all triples (F, ι, ϖ), where F is a non-Archimedean local field which allows at least one ring homomorphism from $\mathcal{O}$ to F, the map $\iota : \mathcal{O} \to F$ is such a ring homomorphism, and ϖ is a uniformizer for F. Let $\mathcal{A}_{\mathcal{O}}$ be the collection of triples (F, ι, ϖ) in $\mathcal{C}_{\mathcal{O}}$ with F of characteristic zero, and let $\mathcal{B}_{\mathcal{O}}$ be the collection of the triples (F, ι, ϖ) such that F has positive characteristic. Here by a non-Archimedean local field we mean a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

Given an integer M, let $\mathcal{C}_{\mathcal{O},M}$ be the collection of (F, ι, ϖ) in $\mathcal{C}_{\mathcal{O}}$ such that the residue field of F has characteristic larger than M, and similarly for $\mathcal{A}_{\mathcal{O},M}$ and $\mathcal{B}_{\mathcal{O},M}$.

Since our results and proofs are independent of the choices of the map ι and the uniformizer ϖ (in the sense that we ultimately prove all our statements for all fields $F \in \mathcal{C}_{\mathcal{O}}$ with sufficiently large residue characteristic, with any choices of ι and ϖ), we will often just write $F \in \mathcal{C}_{\mathcal{O}}$, instead of naming the whole triple. For any $F \in \mathcal{C}_{\mathcal{O}}$, write $\mathcal{O}_F$ for the valuation ring of F, k_F for its residue field, and q_F for the cardinality of k_F.

2.2. **Definable sets and motivic functions.** The $\mathcal{L}_{\mathcal{O}}$-formulas introduced in the previous section allow us to obtain a field-independent notion of subsets of $F^n \times k_F^m \times \mathbb{Z}^r$ for $F \in \mathcal{C}_{\mathcal{O}}$.

Definition 2. A collection $X = (X_F)_F$ of subsets $X_F \subset F^n \times k_F^m \times \mathbb{Z}^r$ for F in $\mathcal{C}_{\mathcal{O}}$ is called a *definable set* if there is an $\mathcal{L}_{\mathcal{O}}$-formula φ and an integer M such that $X_F = \varphi(F)$ for each F with residue characteristic at least M (cf. Remark 4), where $\varphi(F)$ is as explained at the end of §2.1.3.

By Definition 2, a "definable set" is actually a collection of sets indexed by $F \in \mathcal{C}_{\mathcal{O}}$; such practice is not uncommon in model theory and has its analogues in classical algebraic geometry. A particularly simple definable set is $(F^n \times k_F^m \times \mathbb{Z}^r)_F$ with $\mathcal{O} = \mathbb{Z}$, for which we introduce the simplified notation $\mathrm{VF}^n \times \mathrm{RF}^m \times \mathbb{Z}^r$, where VF stands for valued field and RF for residue field. We apply the typical set-theoretical notation to definable sets X, Y, e.g., $X \subset Y$ (if $X_F \subset Y_F$ for each $F \in \mathcal{C}_{\mathcal{O}}$), $X \times Y$, and so on.

Definition 3. For definable sets X and Y, a collection $f = (f_F)_F$ of functions $f_F : X_F \to Y_F$ for $F \in \mathcal{C}_{\mathcal{O}}$ is called a definable function and denoted by $f : X \to Y$ if the collection of graphs of the f_F is a definable set.

Remark 4. There is a subtle issue here caused by the fact that the same definable set can de defined by different formulas. Technically, it would be more elegant to think of a definable set as an equivalence class of what we have called definable sets in the Definition 2 above, where we call two such definable sets equivalent if they are the same for all F with sufficiently large residue characteristic. In order not to complicate notation, we will not emphasize this point, but because of this all the results presented in this article will only be valid for sufficiently big residue characteristic. In particular, for any of the uniform (where by "uniform" we mean, defined in a field-independent way as above) objects X we introduced in Definitions 2, 5, 6, we are actually only interested in $(X_F)_{F \in \mathcal{C}_{\mathcal{O},M}}$ for M sufficiently big.

There are precise quantifier elimination statements for the Denef-Pas language, which lead to the study of the geometry of definable sets and functions for F in $\mathcal{C}_{\mathcal{O},M}$, where often M is large and depends on the data defining the set. To give an overview of these results is beyond of the scope of this survey, but the reader may consult [6], [12] and others.

We now come to motivic functions, for which definable functions are the building blocks as mentioned above. We note that while definable functions, by definition, have to be $\mathrm{VF}^n \times \mathrm{RF}^m \times \mathbb{Z}^r$-valued for some m, n, r, the *motivic* functions are built from definable sets and functions, and can be thought of as complex-valued functions (though values outside of $\mathbb{Q}$ appear only in Definition 6). This does not require thinking of rational or complex numbers in any logic context; instead, these are just usual complex-valued functions that happen to be built from definable ingredients in the way prescribed by the following definitions.

Definition 5. Let $X = (X_F)_F$ be a definable set. A collection $f = (f_F)_F$ of functions $f_F : X_F \to \mathbb{C}$ is called *a motivic function* on X if and only if there exist integers N, N', and N'' such that, for all $F \in \mathcal{C}_{\mathcal{O}}$,

$$f_F(x) = \sum_{i=1}^{N} q_F^{\alpha_{iF}(x)} \big(\#(Y_{iF})_x\big) \left(\prod_{j=1}^{N'} \beta_{ijF}(x) \right) \left(\prod_{\ell=1}^{N''} \frac{1}{1 - q_F^{a_{i\ell}}} \right), \text{ for } x \in X_F,$$

for some

- non-zero integers $a_{i\ell}$,
- definable functions $\alpha_i : X \to \mathbb{Z}$ and $\beta_{ij} : X \to \mathbb{Z}$,
- definable sets $Y_i \subset X \times \mathrm{RF}^{r_i}$,

where, for $x \in X_F$, $(Y_{iF})_x$ is the finite set $\{y \in k_F^{r_i} \mid (x, y) \in Y_{iF}\}$.

2.3. Motivic exponential functions. For any local field F, let $\mathscr{D}_F$ be the set of the additive characters on F that are trivial on the maximal ideal $\mathfrak{m}_F$ of $\mathcal{O}_F$ and nontrivial on $\mathcal{O}_F$. For $\Lambda \in \mathscr{D}_F$, we will write $\bar{\Lambda}$ for the induced additive character on k_F. Note that additive characters of $\mathbb{Q}_p$, for example, are obtained from characters of the finite field $\mathbb{F}_p$, which are exponential functions. Thus, expressions involving additive characters of p-adic fields often give rise to exponential sums, and this explains the term "exponential" in the definition below.

Definition 6. Let $X = (X_F)_F$ be a definable set. A collection $f = (f_{F,\Lambda})_{F,\Lambda}$ of functions $f_{F,\Lambda} : X_F \to \mathbb{C}$ for $F \in \mathcal{C}_{\mathcal{O}}$ and $\Lambda \in \mathscr{D}_F$ is called *a motivic exponential function* on X if there exist integers $N > 0$ and $r_i \geq 0$, motivic functions $f_i = (f_{iF})$, definable sets $Y_i \subset X \times \mathrm{RF}^{r_i}$ and definable functions $g_i : Y_i \to \mathrm{VF}$ and $e_i : Y_i \to \mathrm{RF}$

for $i = 1, \ldots, N$, such that for all $F \in \mathcal{C}_{\mathcal{O}}$ and all $\Lambda \in \mathscr{D}_F$,

$$f_{F,\Lambda}(x) = \sum_{i=1}^{N} f_{iF}(x) \left(\sum_{y \in (Y_{iF})_x} \Lambda\big(g_i(x,y)\big) \cdot \bar{\Lambda}\big(e_i(x,y)\big) \right) \text{ for all } x \in X_F.$$

Compared to Definition 5, the counting operation $\#$ has been replaced by taking exponential sums, which makes the motivic exponential functions a richer class than the motivic functions. Indeed, note that the sum as above gives just $\#(Y_{iF})_x$ if $g_i = 0$ and $e_i = 0$. Motivic (exponential) functions form classes of functions that are stable under integrating some of the variables out (see Theorem 8 below). In fact, the search for this very property led R. Cluckers and F. Loeser to this definition for motivic exponential functions, introduced in [10].

2.4. **Families.** It is often convenient to talk about families of objects; in particular, we will talk about families of definable sets and of definable, motivic, or motivic exponential functions, even though formally these notions are just special cases of the above Definitions 2, 5 and 6.

Definition 7. Suppose that A is a definable set. A *family of definable sets* with parameter in A is a definable set $X \subset A \times Y$ with some definable set Y. We denote such a family by $(X_a)_{a \in A}$. For each $F \in \mathcal{C}_{\mathcal{O}}$ and $a \in A_F$, the sets $X_{F,a} := \{y \in Y_F \mid (a,y) \in X_F\}$ are called members of the family $X_F = (X_{F,a})_{a \in A_F}$.

Similarly, for $(X_a)_{a \in A}$ as above, an (exponential) motivic, resp. definable, function f on X can be considered as a *family $(f_a)_{a \in A}$ of (exponential) motivic*, resp., *definable functions* with parameter in A. Its members are functions $f_{F,a}$ or $f_{F,\Lambda,a}$ for $F \in \mathcal{C}_{\mathcal{O}}$, $\Lambda \in \mathscr{D}_F$ and $a \in A_F$.

Whenever we call a specific function f_F motivic, we implicitly think of F as varying and we mean that there exists an $M > 0$ and a motivic function g with $f_F = g_F$ for all $F \in C_{\mathcal{O},M}$. Likewise, we think of F as varying whenever a specific set X_F or function f_F is called definable.

2.5. **Measure and integration.** To integrate a motivic function f on a definable set X, we need a uniformly given family of measures on each X_F. For $X = \mathrm{VF}$, we put the Haar measure on $X_F = F$ so that $\mathcal{O}_F$ has measure 1; on k_F and on $\mathbb{Z}$, we use the counting measure. To obtain measures on arbitrary definable sets X, we use "definable volume forms" as follows.

Any F-analytic subvariety of F^n, say, everywhere of equal dimension, together with an F-analytic volume form, carries a natural measure associated to the volume form, cf. [4, §5.1] and [35, §2.2]. This carries over to the motivic setting, namely, there is a notion of a definable volume form on a definable set, introduced in [7, §8]. In a nutshell, a definable volume form ω on a definable set X is a volume form defined by means of definable functions on definable coordinate charts, see [8, §3.5.1] and references therein for details. Such a definable volume form specializes to a volume form ω_F on X_F for every F with sufficiently large residue characteristic; more precisely, a definable volume form specializes to a top degree differential form for every F, but its non-vanishing can be assured only when the residue characteristic of F is sufficiently large. By a "motivic measure," we mean a family of measures on each X_F that arises in this way from a definable volume form.

The classes of motivic (exponential) functions are natural to work with for the purposes of integration, as shown by the following theorem, which generalizes a

similar result of [10] to the class of functions summable in the classical sense of L^1-integrability.

Theorem 8. [13, Theorem 4.3.1] *Let f be a motivic function, resp. a motivic exponential function, on $X \times Y$ for some definable sets X and Y, with Y equipped with a motivic measure μ. Then there exist a motivic function, resp. a motivic exponential function, I on X and an integer $M > 0$ such that for each $F \in \mathcal{C}_{\mathcal{O},M}$, each $\Lambda \in \mathscr{D}_F$ and for each $x \in X_F$ one has*

$$I_F(x) = \int_{y \in Y_F} f_F(x, y)\, d\mu_F, \ \ resp. \ I_{F,\Lambda}(x) = \int_{y \in Y_F} f_{F,\Lambda}(x, y)\, d\mu_F,$$

whenever the function $Y_F \to \mathbb{C} : y \mapsto f_F(x, y)$, resp. $y \mapsto f_{F,\Lambda}(x, y)$ lies in $L^1(Y_F, \mu_F)$.

3. New transfer principles for motivic exponential functions

Theorem 8 leads to the transfer of integrability and transfer of boundedness principles, which we quote from [13]. (For simplicity, we quote the version without parameters; that is, below we present [13, Theorem 4.4.1] and [13, Theorem 4.4.2] in the case that the parameter space is a point.)

Theorem 9. [13, Theorem 4.4.1] *Let f be a motivic exponential function on VF^n. Then there exists $M > 0$, such that for the fields $F \in \mathcal{C}_{\mathcal{O},M}$, the truth of the statement that $f_{F,\Lambda}$ is (locally) integrable for all $\Lambda \in \mathscr{D}_F$ depends only on the isomorphism class of the residue field of F.*

Theorem 10. [13, Theorem 4.4.2] *Let f be a motivic exponential function on VF^n. Then, for some $M > 0$, for the fields $F \in \mathcal{C}_{\mathcal{O},M}$, the truth of the statement that $f_{F,\Lambda}$ is (locally) bounded for all $\Lambda \in \mathscr{D}_F$ depends only on the isomorphism class of the residue field of F.*

In the versions with parameters of these results, f is a family of motivic exponential functions with parameter in a definable set A. In that case, each of the two theorems states that there exists a motivic exponential function h on A such that the statement under consideration is true for $f_{F,\Lambda,a}$ if and only if $h_{F,\Lambda}(a) = 0$, that is, we relate the *locus of integrability*, resp. the *locus of boundedness* of the motivic (exponential) function f to the zero locus of another motivic (exponential) function h.

4. New "uniform boundedness" results for motivic functions

It turns out that the field-independent description of motivic functions leads not only to transfer principles, but also to uniform (in the residue characteristic of the field) bounds for those motivic functions whose specializations are known to be bounded for individual local fields (for now, for motivic functions without the exponential). We quote the following theorems from [32].

Theorem 11. [32, Theorem B.6] *Let f be a motivic function on $W \times \mathbb{Z}^n$, where W is a definable set. If for almost all fields $F \in \mathcal{A}_{\mathcal{O}}$ or almost all fields $F \in \mathcal{B}_{\mathcal{O}}$ there exists any (set-theoretical) function $\alpha^F : \mathbb{Z}^n \to \mathbb{R}$ (depending on F) such that*

$$|f_F(w, \lambda)|_{\mathbb{C}} \leq \alpha^F(\lambda) \ on \ W_F \times \mathbb{Z}^n,$$

then there exist integers a, b, and M such that for all $F \in \mathcal{C}_{\mathcal{O},M}$,

$$|f_F(w, \lambda)|_{\mathbb{C}} \leq q_F^{a+b\|\lambda\|} \ on \ W_F \times \mathbb{Z}^n,$$

where $\|\lambda\| = \sum_{i=1}^n |\lambda_i|$, and where $|\cdot|_{\mathbb{C}}$ is the norm on $\mathbb{C}$.

We observe that in the case with $n = 0$, the theorem yields that if a motivic function f on W is such that f_F is bounded on W_F, say, for each $F \in \mathcal{A}_{\mathcal{O},M}$, then the bound for $|f_F|_{\mathbb{C}}$ can be taken to be q_F^a for some $a \geq 0$, uniformly in F with large residue characteristic, and moreover, the same constant works for $F \in \mathcal{B}_{\mathcal{O},M}$ as well.

The following statement allows one to transfer bounds which are known for local fields of characteristic zero to local fields of positive characteristic, and vice versa.

Theorem 12. [32, Theorem B.7] *Let f be a motivic function on $W \times \mathbb{Z}^n$, where W is a definable set, and let a and b be integers. Then there exists M such that, for any $F \in \mathcal{C}_{\mathcal{O},M}$, whether the statement*

$$f_F(w, \lambda) \leq q_F^{a+b\|\lambda\|} \ for \ all \ (w, \lambda) \in W_F \times \mathbb{Z}^n \tag{1}$$

holds or not, only depends on the isomorphism class of the residue field of F.

4.1. **A few words about proofs.** The common strategy in the proofs of Theorems 8 – 12 (assuming the set-up of motivic integration developed in [7] and [10]) is the following. The first step is to reduce the problem from arbitrary motivic exponential functions f to motivic exponential functions g_i of a simpler form by writing f as a sum $\sum_i g_i$. For some of the results (e.g., to obtain a precise description of the locus of integrability or boundedness of f), one needs to prove that not too much cancellation can occur in this sum. The precise statements are [13, Propositions 3.2.5 and 4.5.8]; it is in fact one of the main difficulties in the proofs of Theorems 9 and 10.

In the next step, we eliminate all the valued-field variables, possibly at the cost of introducing more residue-field and $\mathbb{Z}$-valued variables, which uses the Cell Decomposition Theorem (cf., for example, [13, Theorems 3.3.2 and 4.5.3]). Once we have a motivic function that depends only on the residue-field and value-group variables, we again break it into a sum of simpler terms, and then get rid of the residue-field variables. Finally, the question is reduced to the study of functions $h : \mathbb{Z}^r \to \mathbb{R}$ (this study is done in [13, §2]), which are, up to a finite partition of $\mathbb{Z}^r$ into sets which are definable in the Presburger language (see Section 2.1.2), sums of products of linear functions in $x \in \mathbb{Z}^r$ and of powers of q_F, where the power also depends linearly on x. For most x, the behavior of such a sum h is governed by a dominant term, and for a single term, each of the theorems is easy to prove. To control the locus where no single term of h is dominant, we use the powerful result by Wilkie from [36] about "o-minimality of $\mathbb{R}_{\exp}$."

5. WHICH OBJECTS ARISING IN HARMONIC ANALYSIS ARE DEFINABLE?

In the remainder of the article, when we say "definable," we mean definable in the Denef-Pas language $\mathcal{L}_{\mathbb{Z}}$ (introduced in Section 2.1.4). If one wishes to think specifically of reductive groups defined over a given global field **F** with the ring of integers $\mathcal{O}$ and its completions, then one can use the language $\mathcal{L}_{\mathcal{O}}$ instead. We write $\mathcal{A}_M$, $\mathcal{B}_M$, and $\mathcal{C}_M$ for $\mathcal{A}_{\mathbb{Z},M}$, $\mathcal{B}_{\mathbb{Z},M}$ and $\mathcal{C}_{\mathbb{Z},M}$, respectively (see Definition 1).

5.1. **Connected reductive groups.** We recall that a split connected reductive group over a local field is determined by its root datum $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$, see, e.g., [33, §10]. By an *absolute* root datum of $\mathbf{G}$ we will mean the root datum of $\mathbf{G}$ over the separable closure of the given local field; it is a quadruple of this form.

By an unramified root datum we mean a root datum of an unramified reductive group over a local field F, i.e., a quintuple $\xi = (X^*, \Phi, X_*, \Phi^\vee, \theta)$, where θ is the action of the Frobenius element of F^{ur}/F on the first four components of ξ, compatible with $\Phi \subset X^*$, $\Phi^\vee \subset X_*$, and the pairing between X^* and X_*. We will usually denote a quadruple $(X^*, \Phi, X_*, \Phi^\vee)$ by Ψ, and a quintuple that includes Galois action by ξ.

Given an unramified root datum ξ, it is shown in [9, §4] that a connected reductive group $\mathbf{G}(F)$ with root datum ξ is "definable using parameters," for all local fields $F \in \mathcal{C}_M$, where M is determined by the root datum. This result is extended to all connected reductive groups (not necessarily unramified) in [8, §3.1], [32, B.4.3]. More precisely, by "definable using parameters," we mean that $\mathbf{G}(F)$ is a member of a family (in the sense of Definition 7) of definable connected reductive groups with parameter in a suitable definable set Z. This parameter encodes information about the extension over which $\mathbf{G}$ splits (see [8, §3.1] for details). In the remainder of this article, every definable or motivic object in $\mathbf{G}(F)$ is parametrized by Z, and all results are uniform in the family.

5.2. **Haar measure and Moy-Prasad filtrations.** If $\mathbf{G}$ is unramified, the canonical measure on $\mathbf{G}(F)$ defined by Gross [21] is motivic for $F \in \mathcal{C}_M$ (with M depending only on the absolute root datum of $\mathbf{G}$), see [9, §7.1]. For a root datum defining a ramified group $\mathbf{G}$, to the best of our knowledge, this is presently not known. However, in [32, Appendix B] we prove that there is a constant c that depends only on the absolute root datum, such that Gross' measure can be renormalized by a factor between q^{-c} and q^c (possibly depending on F) to give a motivic measure on $\mathbf{G}(F)$, where $q = p^r$ is the cardinality of the residue field of F, and p is assumed large enough so that the group is tamely ramified.

In fact, this question is closely related to whether Moy-Prasad filtration subgroups $\mathbf{G}(F)_{x,r}$ (defined in [28, §2]) are definable (here x is a point in the building of $\mathbf{G}(F)$, and $r \geq 0$). This question has been answered in a number of important special cases. Namely, if $\mathbf{G}$ is unramified over F, and x is a hyper-special point, then we prove in [8, §3] that $\mathbf{G}(F)_{x,r}$ is definable for all $r \geq 0$. The same proof (which relies on the split case and taking Galois-fixed points) works in the case when $\mathbf{G}$ is ramified over F, x is a special point, and $r > 0$, but fails for $r = 0$ since Galois descent no longer works. This is exactly the reason that the question whether the canonical measure is motivic is open in the ramified case. We also prove in [8, §3] that for x an optimal point in the alcove (in the sense of [28, §6.1]), $\mathbf{G}(F)_{x,r}$ is definable for $r > 0$, and for all $r \geq 0$ if $\mathbf{G}$ is unramified over F, and therefore the $\mathbf{G}(F)$-domain $\mathbf{G}(F)_r$ is definable. We also prove the corresponding results for the Lie algebra. We note that the first results on definability of $\mathbf{G}(F)_{x,0}$ in some special cases appeared in [18].

5.3. **Orbital integrals and their Fourier transforms.** There are two ways to think of an orbital integral: as a distribution on the space $C_c^\infty(\mathbf{G}(F))$ of locally constant compactly supported functions on $\mathbf{G}(F)$ (or, respectively, on the Lie algebra), or to fix a test function f on $\mathbf{G}(F)$ (respectively, on $\mathfrak{g}(F)$), and consider the

orbital integral $\mathcal{O}_\gamma(f)$ (respectively, $\mathcal{O}_X(f)$) as a function on $\mathbf{G}(F)$ (respectively, on the Lie algebra $\mathfrak{g}(F)$).

The earliest results concerning definability of orbital integrals use the second approach: in [14], C. Cunningham and T. C. Hales prove that when $\mathbf{G}$ is unramified and f is the characteristic function of $\mathfrak{g}(\mathcal{O}_F)$, $\mathcal{O}_X$ is a motivic function on the set of so-called *good* regular elements in $\mathfrak{g}(F)$. In fact, they prove a slightly more refined result, giving explicitly the residue field parameters that control the behavior of this function. In [22], T. C. Hales proves that orbital integrals are motivic "on average." Both of these papers pre-date the most general notion of a motivic function, and because of this, extra complications arise from having to explicitly track the parameters needed to define orbital integrals.

In order to be able to use the first approach to orbital integrals (as distributions), we define a notion of a "motivic distribution" (without this name, it first appears in this context in [11, §3]). We say that a distribution Φ is motivic if it is, in fact, a collection of distributions defined for each $F \in \mathcal{C}_{M'}$ for some $M' > 0$, and for any suitable family $(f_a)_{a \in S}$ of motivic test functions indexed by a parameter a varying over a definable set S (where by "suitable" we mean that $f_{F,a}$ is in the domain of Φ for all $F \in \mathcal{C}_M$ and $a \in S_F$), there exists a motivic function g on S and an $M \geq M'$ such that $\Phi(f_{F,a}) = g(a)$ for all $F \in \mathcal{C}_M$ and all $a \in S_F$.

The most general current results on orbital integrals are

(1) Let F be a local field. Let $\mathbf{G}$ be any connected reductive group over F, defined via its absolute root datum Ψ and a parameter $z \in Z$ (which is responsible for picking the F-form that is isomorphic to $\mathbf{G}$ over F), as discussed briefly in §5.1 above and in detail, in [8, §3.1]. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Assume that the characteristic of F is sufficiently large so that there exists a non-degenerate bilinear form $\langle,\rangle$ on $\mathfrak{g}(F)$ as in [1, §4.1] (the restriction on the characteristic of F here depends only on Ψ, as discussed in [8, §4.1]). Every co-adjoint orbit in $\mathfrak{g}(F)^*$ carries a volume form coming from the symplectic form defined by Kirillov (cf. [25, §17.3]), and the form $\langle,\rangle$ allows us to identify adjoint orbits with co-adjoint orbits. Let us denote the resulting volume form on the orbit of $X \in \mathfrak{g}(F)$ by ω_X. We prove that $(\omega_X)_{X \in \mathfrak{g}}$ is a family of definable volume forms indexed by $\mathfrak{g}$, see [32, §1.5.3] and [8, Proposition 4.3]. Thus, (cf. [8, Corollary 4.4]), *there exists a family of motivic measures on the orbits of all elements in the Lie algebra; in particular, with this choice of the normalization of the measures on adjoints orbits, all the orbital integrals form a family of motivic distributions.*

However, in many situations one needs to use a specific normalization of the invariant measures on the orbits in the group or the Lie algebra. The results for the other normalizations are listed below.

(2) Cluckers, Hales and Loeser prove in [9] that for $\mathbf{G}$ unramified over F, and with f fixed to be the characteristic function of $\mathfrak{g}(\mathcal{O}_F)$, $X \mapsto \mathcal{O}_X(f)$ is a motivic function on the set of strongly regular semisimple elements in $\mathfrak{g}(F)$, where the measures on the orbits are the measures used in the Fundamental Lemma. In fact, their method shows also that the orbital integrals of regular semisimple elements $X \in \mathfrak{g}(F)$ form a family of motivic distributions in the sense defined above.

(3) Still under the assumption that $\mathbf{G}$ is unramified, in [32, §B.6], we prove that if the centralizer $C_{\mathbf{G}}(\gamma)$ of a semisimple element $\gamma \in \mathbf{G}(F)$ splits over an unramified extension, our normalization of the measure coincides with the canonical measure on the orbit that one gets from the canonical measures (in the sense of [21]) on $C_{\mathbf{G}}(\gamma)(F)$ and $\mathbf{G}(F)$; if $C_{\mathbf{G}}(\gamma)$ is ramified, the motivic and canonical measures might differ by a constant not exceeding a fixed power of q. This result should easily generalize to the case when $\mathbf{G}$ is ramified as well (though one more constant from the measure on $\mathbf{G}(F)$ would appear).

We also note that even though we state the result for orbital integrals on the group, since this is what is required for the estimates that are the main goal of [32, Appendix B], in the course of the proof we essentially prove the analogous statement for the Lie algebra as well.

Note that unlike the results of [9] mentioned above, here no assumption that γ is regular is made.

(4) On the Lie algebra, one can consider the Fourier transform $\widehat{\mathcal{O}}_X(f) = \mathcal{O}_X(\widehat{f})$ of an orbital integral. By a theorem of Harish-Chandra in the characteristic zero case, this distribution is represented by a locally constant function supported on the set of regular elements $\mathfrak{g}(F)^{\mathrm{reg}}$. In positive characteristic, Huntsinger [3] proved that the *restriction of this distribution to the regular set* is represented by a locally constant function on $\mathfrak{g}(F)^{\mathrm{reg}}$. We denote this function in both cases by $\widehat{\mu}_X(Y)$, and think of it as a function on $\mathfrak{g}(F)$ that vanishes outside $\mathfrak{g}(F)^{\mathrm{reg}}$ and whose restriction to $\mathfrak{g}(F)^{\mathrm{reg}}$ is locally constant.

In [8, Theorem 5.8] we prove the following:

Theorem 13. *There exists $M > 0$, a motivic function h on $\mathfrak{g} \times \mathfrak{g}$, and a motivic constant c, such that $\widehat{\mu}_X(Y) = c_F^{-1} h_F(X, Y)$, for all $F \in \mathcal{C}_M$, where by a motivic constant we mean a motivic function on a one-point set.*

The fact that the restriction of $\widehat{\mu}_X(Y)$ to the regular set is motivic allows us to transfer Harish-Chandra's result to positive characteristic; namely, we prove in [8] that for *any* $X \in \mathfrak{g}(F)$, and any $f \in C_c^{\infty}(\mathbf{G}(F))$, one has $\mathcal{O}_X(\widehat{f}) = \int_{\mathfrak{g}(F)} \widehat{\mu}_X(Y) f(Y) dY$, provided the characteristic of F is large enough. This result is discussed below in §6.

5.4. **Classification of nilpotent orbits, and Shalika germs.** This section describes work that is still in progress.

It follows from the results of McNinch [27] and Jantzen [24] that the set of nilpotent elements $\mathcal{N}$ in the Lie algebra $\mathfrak{g}(F)$ is definable.

It is well-known that when the residue characteristic of the field F is large enough, there are finitely many nilpotent orbits in $\mathfrak{g}(F)$. Using the parametrization of nilpotent orbits for classical Lie algebras described explicitly by J.-L. Waldspurger in [34, §I.6], it is possible to prove that these orbits are parametrized by the residue-field points of a definable set. This approach is carried out in detail for odd orthogonal groups in [17]. Alternatively, one can use DeBacker's parametrization of nilpotent orbits that uses Bruhat-Tits theory, [16]. This approach is demonstrated for the Lie algebra of type G_2 (as well as for $\mathfrak{sl}_n$) in J. Diwadkar's thesis [18].

Using both parametrizations of nilpotent orbits, and the explicit matching between them due to M. Nevins, [29], L. Robson proves in [31] that for $\mathfrak{sp}_{2n}$, Shalika germs are motivic functions, up to a motivic constant. More precisely, the main result of [31] is:

Theorem 14. *Let* $\mathfrak{g} = \mathfrak{sp}_{2n}$. *Let* $\mathcal{N}$ *be the set of nilpotent elements in* $\mathfrak{g}$. *Then*

(1) *There exists a definable set* $\mathcal{E} \subset \mathrm{RF}^n$ *(that is, having only residue-field variables), and a definable function* $h : \mathcal{N} \to \mathcal{E}$, *such that for every* $d \in \mathcal{E}$, $h^{-1}(d)$ *is an adjoint orbit, and each orbit appears exactly once as the fibre of* h.

(2) *There exist a motivic constant* c, *a motivic function* Γ *on* $\mathcal{E} \times \mathfrak{g}^{\mathrm{reg}}$, *and a constant* $M > 0$, *such that for* $F \in \mathcal{C}_M$ *and every* $d \in \mathcal{E}_F$, *the function* $c_F^{-1} \Gamma_F(d, \cdot)$ *is a representative of the Shalika germ of the orbit* $h^{-1}(d)$ *on* $\mathfrak{g}(F)^{\mathrm{reg}}$, *in the sense that there exists a definable neighborhood of zero on which it coincides with the canonical Shalika germ of the orbit* $h^{-1}(d)$.

A refinement and generalization of this theorem to $\mathfrak{sl}_n$ appears in [19].

In a different context of groups over $\mathbb{C}((t))$, a motivic interpretation of the subregular germ is given in E. Lawes' thesis, [26].

5.5. **Transfer factors and other ingredients of the Fundamental Lemma.** In [9], the authors prove that transfer factors, weights, and other ingredients of the Fundamental Lemma are motivic functions. The same for the Fundamental Lemma of Jacquet-Rallis is proved in [37, Appendix A].

5.6. **Characters.** So far, the only results in motivic harmonic analysis on Harish-Chandra characters of representations have been restricted to the cases where the character can be related to a semisimple orbital integral. One of the fundamental problems is that the parametrization of supercuspidal representations via K-types (and the classification of the K-types due to J.-K. Yu and Ju-Lee Kim) uses multiplicative characters of certain subgroups of $\mathbf{G}(F)$, which in particular could be built from multiplicative characters of the field, and multiplicative characters are not presently incorporated into the motivic integration framework. Because of this, at present we do not have a way of talking about representations in a completely field-independent way. There are two partial results on characters of some representations in a suitable neighborhood of the identity, where Murnaghan-Kirillov theory holds.

(1) The main result of [20] can be expressed in modern terms: *the character of a uniform (i.e., coming from Deligne-Lusztig construction) depth-zero representation of* Sp_{2n} *or* SO_{2n+1} *is a motivic distribution (in the sense defined above) on the set of topologically unipotent elements.*

(2) For positive-depth so-called *toral, very supercuspidal* representations, it is shown in [11], that up to a constant, their Harish-Chandra characters are motivic exponential functions on *an explicitly described* neighborhood of the identity on which Murnaghan-Kirillov theory works, and an explicit parameter space (over the residue field) is constructed for these representations up to crude equivalence, where we identify two representations if their characters on this neighborhood of the identity are the same.

However, thanks to Harish-Chandra's local character expansion, extended to an expansion near an arbitrary tame semisimple element by J. Adler and J. Korman [2],

one can prove results about characters, using motivic integration, even without knowing that they are themselves motivic functions. We give an example of such an application in the next section.

We note that proving that the coefficients of Harish-Chandra's local character expansion are motivic is a question of the same order of difficulty as the same question about the characters themselves, near the identity (since the Fourier transforms of orbital integrals are now known to be motivic), and so it is presently far from known.

6. Local integrability results in positive characteristic

Following Harish-Chandra (and using the modification by R. Kottwitz [32, Appendix A] for non-regular elements), for a semisimple element $\gamma \in \mathbf{G}(F)$, we define the discriminant

$$D^G(\gamma) := \prod_{\substack{\alpha \in \Phi \\ \alpha(\gamma) \neq 1}} |1 - \alpha(\gamma)|, \quad \text{and} \quad D(\gamma) := \prod_{\alpha \in \Phi} |1 - \alpha(\gamma)|,$$

where Φ is the root system of $\mathbf{G}$.

On the Lie algebra, define $D(X) = \prod_{\alpha \in \Phi} |\alpha(X)|$. Following [25], we call a function h defined on $\mathfrak{g}(F)^{\mathrm{reg}}$ *nice* if it satisfies the following conditions:

- when extended by zero to all of $\mathfrak{g}(F)$, it is locally integrable, and
- the function $D(X)^{1/2} h(X)$ is locally bounded on $\mathfrak{g}(F)$.

Similarly, call a function on $\mathbf{G}(F)^{\mathrm{reg}}$ *nice* if it satisfies the same conditions on $\mathbf{G}(F)$, with $D(X)$ replaced by its group version $D(\gamma)$.

Combining the theorems of Harish-Chandra in characteristic zero with the Transfer Principles of §3 and the results of §5.3 above, we obtain the following theorem, which we state only in positive characteristic for emphasis, since we are using the characteristic zero version proved by Harish-Chandra in order to obtain it, and thus it says nothing new in the characteristic zero case.

Theorem 15. *([8, Theorem 2.1]) Given an absolute root datum Ψ, there exists a constant $M > 0$, such that for every $F \in \mathcal{B}_M$, every connected reductive group $\mathbf{G}$ defined over F with the absolute root datum Ψ, and every adjoint orbit $\mathcal{O}$ in $\mathfrak{g}(F)$, the function $\widehat{\mu}_{\mathcal{O}}$ is a nice function on $\mathfrak{g}(F)$.*

Our next result requires local character expansion in large positive characteristic; in characteristic zero it is due to Harish-Chandra [23, Theorem 16.2], and in *large positive characteristic* it is proved by DeBacker [15] near the identity, and by Adler-Korman [2] near a general tame semisimple element; we also need some weak information about the neighborhood of validity of this expansion, which follows from the results of DeBacker and Adler-Korman. These results require an additional hypothethis on the existence of the so-called mock exponential map $\mathbf{e}$, discussed in [2], which we will not quote here; we note only that for classical groups one can take $\mathbf{e}$ to be the Cayley transform.

Theorem 16. *([8, Theorem 2.2]) Given an absolute root datum Ψ, there exists a constant $M > 0$, such that for every $F \in \mathcal{B}_M$, every connected reductive group $\mathbf{G}$ defined over F with the absolute root datum Ψ, and every admissible representation π of $\mathbf{G}(F)$, the Harish-Chandra character θ_π is a nice function on $\mathbf{G}(F)$, provided that mock exponential map $\mathbf{e}$ exists for $\mathbf{G}(F)$.*

Finally, Theorem 15 also implies (thanks to results of DeBacker [15, Theorem 2.1.5, Corollary 3.4.6, and Remark 2.1.7]) that Fourier transforms of general invariant distributions on $\mathfrak{g}(F)$ with support bounded modulo conjugation are represented by nice functions in a neighborhood of the origin, cf. [8, Theorem 5.9].

7. Uniform bounds for orbital integrals

In order to state the next result, we need to introduce the notation for a family of test functions, namely, the generators of the spherical Hecke algebra. Assume that $\mathbf{G}$ is an unramified reductive group with root datum ξ. Let A be a maximal F-split torus; we have a Borel subgroup $B = TU$ with T an unramified maximal torus containing A. Let $K = \underline{G}(\mathcal{O}_F)$ be a hyperspecial maximal compact subgroup (where $\underline{G}$ is the smooth model associated with a hyperspecial point in an apartment of T by Bruhat-Tits theory). For $\lambda \in X_*(A)$, let τ_λ^G be the characteristic function of the double coset $K\lambda(\varpi)K$.

In [32, Appendix B], we prove, using Theorem 11,

Theorem 17. *Consider an unramified root datum ξ. Then there exist constants $M > 0$, a_ξ and b_ξ that depend only on ξ, such that for each non-Archimedean local field F with residue characteristic at least M, the following holds. Let $\mathbf{G}$ be a connected reductive algebraic group over F with the root datum ξ. Let A be a maximal F-split torus in $\mathbf{G}$, and let τ_λ^G be as above. Then for all $\lambda \in X_*(A)$ with $\|\lambda\| \leq \kappa$,*

$$|\mathcal{O}_\gamma(\tau_\lambda^G)| \leq q_F{}^{a_\xi + b_\xi \kappa} D^G(\gamma)^{-1/2}$$

for all semisimple elements $\gamma \in \mathbf{G}(F)$.

It is sometimes useful to reformulate theorems of this type to make an assertion about a group defined over a global field. This is possible because given a connected reductive group over a global field $\mathbf{F}$, there are finitely many possibilities for the root data of $\mathbf{G}_v = \mathbf{G} \otimes_{\mathbf{F}} \mathbf{F}_v$, as v runs over the set of finite places of $\mathbf{F}$, and $\mathbf{F}_v$ denotes the completion of $\mathbf{F}$ at v. Here we give an example of one such reformulation, which proved to be useful in [32] (we also refer to §7 of loc. cit. for a more detailed explanation of the setting).

Theorem 18. *Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbf{F}$, with a maximal split torus A. There exist constants $a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ that depend only on the global integral model of $\mathbf{G}$ such that for all but finitely many places v, for all $\lambda \in X_*(A_v)$ with $\|\lambda\| \leq \kappa$,*

$$|\mathcal{O}_\gamma(\tau_\lambda^G)| \leq q_v^{a_{\mathbf{G}} + b_{\mathbf{G}} \kappa} D^G(\gamma)^{-1/2}$$

for all semisimple elements $\gamma \in \mathbf{G}(\mathbf{F}_v)$, where q_v is the cardinality of the residue field of $\mathbf{F}_v$.

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