INTEGRABILITY OF OSCILLATORY FUNCTIONS ON LOCAL FIELDS: TRANSFER PRINCIPLES

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Abstract

For oscillatory functions on local fields coming from motivic exponential functions, we show that integrability over $\mathbb{Q}_p^n$ implies integrability over $\mathbb{F}_p((t))^n$ for large p, and vice versa. More generally, the integrability only depends on the isomorphism class of the residue field of the local field, once the characteristic of the residue field is large enough. This principle yields general local integrability results for Harish-Chandra characters in positive characteristic as we show in other work. Transfer principles for related conditions such as boundedness and local integrability are also obtained. The proofs rely on a thorough study of loci of integrability, to which we give a geometric meaning by relating them to zero loci of functions of a specific kind.

Contents

1. Introduction

Integrability conditions of oscillatory functions are often important but difficult to control. The idea of relating integrability conditions over $\mathbb{F}_p((t))^n$ to integrability over $\mathbb{Q}_p^n$ may sound tricky, the more so since integrability is not a first-order prop-

erty (in any seemingly natural language), and thus plays at a different level than, say, the classical Ax–Kochen principle [1, Theorem 6]. To relate integrability conditions over different local fields one must first relate oscillatory functions between the corresponding fields. To this end, we use the motivic exponential functions of [9], which specialize naturally to oscillatory functions on nonarchimedean local fields. Such motivic exponential functions play an increasingly important role in representation theory; for example, they played a crucial role in one of the methods for obtaining the fundamental lemma of the Langlands program and many of its variants in the characteristic zero case (see [6] and [28, Appendix]).

The main result of this paper is the transfer principle that relates L^1-integrability over $\mathbb{F}_p((t))^n$ to L^1-integrability over $\mathbb{Q}_p^n$ for specializations of motivic exponential functions and similarly for any pair of local fields with isomorphic residue fields. The first application is the proof of the local integrability of Harish-Chandra characters in positive (large) characteristic that we give in [5].

The proofs involve a precise understanding of what we call "loci of integrability": for a family of functions f_x with parameter x, the locus of integrability is the set of those parameters x for which the function f_x is integrable (see Definition 1.0.1). Integrability being a subtle and analytic condition, it is surprising that we can relate it back to geometry, that is, we relate loci of integrability to zero loci of functions of specific kinds. This geometric viewpoint on loci of integrability is key to proving our new transfer principles.

Let us explain basic versions of the transfer principles for a simple kind of motivic exponential function. Consider a formal expression F of the form

$$\sum_i |f_i| \cdot E(h_i),$$

where E is a formal symbol and where the f_i's and h_i's are infinitely many (first-order) definable functions in the language of valued fields with the same domain X and taking values in the valued field. For any nonarchimedean local field K and for any nontrivial additive character $\psi : K \to \mathbb{C}^\times$, the expression F specializes to the oscillatory function

$$F_{K,\psi} : X_K \to \mathbb{C} : x \mapsto \sum_i |f_{iK}(x)| \psi(h_{iK}(x)),$$

where $|\cdot|$ is the norm on K, X_K is the subset of K^n for some n obtained by interpreting the formula defining X in K, and the $f_{iK} : X_K \subset K^n \to K$ are the interpretations in K of the f_i's and similarly for the h_i's. There are many ways to endow X_K with a measure, a basic example being the restriction of the Haar measure on K^n to X_K in the case when X_K is open in K^n. We can now state a basic form of one of our main results.

TRANSFER PRINCIPLE FOR INTEGRABILITY (basic form)
As soon as the residue field k_K of K has sufficiently large characteristic, whether the statement

$$F_{K,\psi} \text{ is } L^1\text{-integrable over } X_K \text{ for each } \psi$$

holds or not depends only on (the isomorphism class of) k_K.

For applications one typically needs the family version of this transfer principle as given by Theorem 4.4.1. Variants with conditions like boundedness, local integrability, and local boundedness are also obtained (see Theorem 4.4.2). Thanks to the ubiquity of motivic exponential functions, the results apply to a large class of functions arising in representation theory, such as the functions representing the Fourier transforms of orbital integrals. In particular, our results lead to the first general proof of the local integrability of Harish-Chandra characters in large characteristics (see [5]), generalizing the results of, for example, [19]. While the work on the fundamental lemma in characteristic zero of [6] and [28, Appendix] combines insights about motivic exponential functions with results in positive characteristics of [21] and [28], the results of [5] combine the theorems of Harish-Chandra [17, Theorems 6.1, 6.5, 16.3] in characteristic zero with Theorem 4.4.1 of this paper. Waldspurger [26] has given an alternative approach for some results of [6], but it remains to be shown that techniques of, for example, [26] may also recover some results on integrability from [5].

Let us begin with the general definitions of loci of integrability, of boundedness, and of identical vanishing.

For arbitrary sets $A \subset X \times T$ and $x \in X$, write A_x for the set of $t \in T$ with $(x,t) \in A$. For $g : A \subset X \times T \to B$ a function and for $x \in X$, write $g(x,\cdot)$ for the function $A_x \to B$ sending t to $g(x,t)$.

Let T and X be arbitrary sets, and let $f : X \times T \to \mathbb{C}$ be a function.

Definition 1.0.1
Define the *locus of boundedness of f in X* as the set

$$\mathrm{Bdd}(f, X) := \big\{ x \in X \mid f(x,\cdot) \text{ is bounded on } T \big\}.$$

Define the *locus of identical vanishing of f in X* as the set

$$\mathrm{Iva}(f, X) := \big\{ x \in X \mid f(x,\cdot) \text{ is identically zero on } T \big\}.$$

If moreover T is equipped with a complete measure, we define the *locus of integrability of f in X* as the set

$$\mathrm{Int}(f, X) := \big\{ x \in X \mid f(x,\cdot) \text{ is measurable and integrable over } T \big\}.$$

We prove that loci of integrability for functions f_K coming from a motivic exponential function f on $X \times K^n$ are actually zero loci of functions coming from a motivic exponential function on X, and similarly for loci of boundedness and identical vanishing (see Theorem 4.4.4).

In fact, we develop all our results gradually: first in the case of summation over the integers in Section 2, then for p-adic integration in Section 3, and finally for motivic integration in Section 4. We also sharpen the results of [8] and [9] about stability under integration (see, e.g., Theorems 3.2.1 and 4.4.3), and we give general interpolation results of given functions by integrable functions (see, e.g., Theorems 4.3.1 and 4.3.3). It is the precise study of these loci that allows us to prove our new transfer principles. The biggest challenges that we had to address were of course related to oscillation, which appears from Section 3.2 onward. An important ingredient to control oscillation is the technical Proposition 3.2.5, which, in a certain sense, states that oscillation cannot interact too badly with definable conditions.

1.1. Conventions

For a function $f : A \to \mathbb{C}$, we write $Z(f)$ for the zero locus $\{a \in A \mid f(a) = 0\}$ of f, and similarly for an R-valued function $f : A \to R$ for any ring R.

For sets A_1, A_2, X and functions $f_i : A_i \to X$, a function $g : A_1 \to A_2$ is said to be over X when $f_2 \circ g = f_1$. Often, the f_i's will be coordinate projections to X.

Definition 1.0.1 will typically be applied to the counting measure on $\mathbb{Z}$, to the Haar measure on $\mathbb{Q}_p$ normalized such that $\mathbb{Z}_p$ has measure 1, and to product measures on Cartesian products of these sets. Our results also apply mutatis mutandis to measures on varieties V over $\mathcal{O}_K$ by working with affine charts and volume forms (see [8], [10]).

Recall that a complex-valued function is called bounded if and only if its range is contained in a compact subset of $\mathbb{C}$.

2. Summability over the integers

Summation over the integers and the results presented in this section are important for us since they lie behind p-adic integration: several of the p-adic results of Section 3, and even some of the motivic results of Section 4, will be reduced to the results of this section.

2.1. Presburger with base q

In this section, let $q > 1$ be a fixed real number.

By a Presburger set, one means a subset of $\mathbb{Z}^m$ for some $m \geq 0$ which can be described by a Boolean combination of sets of the following forms:

$$\{x \in \mathbb{Z}^m \mid f(x) \geq 0\},$$

$$\{x \in \mathbb{Z}^m \mid g(x) \equiv 0 \bmod n\},$$

where f and g are polynomials over $\mathbb{Z}$ of degree at most 1, and $n > 0$ is an integer. A Presburger function is a function between Presburger sets whose graph is also a Presburger set. A Presburger function is called linear if it is the restriction of an affine map $\mathbb{Q}^k \to \mathbb{Q}^\ell$. We write $\mathbb{N}$ for the set of nonnegative integers $\{z \in \mathbb{Z} \mid z \geq 0\}$.

Definition 2.1.1
Define the subring $\mathbb{A}_q \subset \mathbb{R}$ as

$$\mathbb{A}_q = \mathbb{Z}\Big[q, q^{-1}, \Big(\frac{1}{1-q^{-i}}\Big)_{i \in \mathbb{N}, 0 < i}\Big].$$

For S a Presburger set, let $\mathscr{P}_q(S)$ be the $\mathbb{A}_q$-algebra of $\mathbb{A}_q$-valued functions on S generated by
(1)   all Presburger functions $\alpha : S \to \mathbb{Z}$,
(2)   the functions $q^\beta : S \to \mathbb{A}_q : s \mapsto q^{\beta(s)}$ for all Presburger functions $\beta : S \to \mathbb{Z}$.

The functions in $\mathscr{P}_q(S)$ are called Presburger constructible functions on S (with base q). Note that a general function in $\mathscr{P}_q(S)$ is of the form

$$s \mapsto \sum_{i=1}^{N} a_i q^{\beta_i(s)} \prod_{j=1}^{M_i} \alpha_{ij}(s),$$

with the α_{ij}'s and β_i's Presburger functions $S \to \mathbb{Z}$, and the a_i's elements of $\mathbb{A}_q$. The constants $1/(1-q^{-i})$ are needed in $\mathbb{A}_q$ to make the framework closed under summation (see Theorem 2.1.6).

By the quantifier elimination results of [23], the image of a Presburger set under a Presburger function is again a Presburger set, as are finite intersections, finite unions, and complements of Presburger sets. The situation for zero loci of Presburger *constructible* functions is much more delicate. For finite unions and finite intersections there is no difficulty, as follows.

LEMMA 2.1.2
Let S be a Presburger set, and let the h_i's be in $\mathscr{P}_q(S)$ for $i = 1, \ldots, N$. Consider zero loci

$$Z(h_i) = \{s \in S \mid h_i(s) = 0\}.$$

Then there exist f and g in $\mathscr{P}_q(S)$ such that

$$Z(f) = \bigcap_{i=1}^{N} Z(h_i) \qquad \text{and} \qquad Z(g) = \bigcup_{i=1}^{N} Z(h_i).$$

Proof

One can just take the sum of the squares (resp., the product) of the h_i's.   □

Note that any Presburger subset A of S appears as the zero locus of a Presburger constructible function on S; indeed, $A = Z(1 - \chi_A)$ with χ_A the characteristic function of A in S. However, the converse is not true: there are Presburger constructible functions whose zero locus is not a Presburger set. Moreover, for $h \in \mathcal{P}_q(S)$, the complement of $Z(h)$ in S is not always equal to the zero locus of some function in $\mathcal{P}_q(S)$. It turns out that zero loci of Presburger constructible functions are closely related to loci of integrability, of boundedness, and of identical vanishing. Indeed, the zero loci of Presburger constructible functions are exactly the sets that arise as loci of integrability (against the counting measure) of Presburger constructible functions, and similarly for the loci of boundedness and of identical vanishing.

THEOREM 2.1.3 (Correspondences of loci)

Let f be in $\mathcal{P}_q(S \times \mathbb{Z}^m)$ for some Presburger set S and some $m \geq 0$. Then there exist h_1, h_2, and h_3 in $\mathcal{P}_q(S)$ such that

$$\text{Int}(f, S) = Z(h_1), \tag{2.1.1}$$

$$\text{Bdd}(f, S) = Z(h_2), \tag{2.1.2}$$

and

$$\text{Iva}(f, S) = Z(h_3), \tag{2.1.3}$$

where integrability in (2.1.1) is with respect to the counting measure on $\mathbb{Z}^m$, and where $Z(\cdot)$ stands for the zero locus.

Remark 2.1.4

Theorem 2.1.3 implies that the classes of sets which can appear as different kinds of loci for Presburger constructible functions are all equal, since for any given function h in $\mathcal{P}_q(S)$, there exists $f \in \mathcal{P}_q(S \times \mathbb{Z})$ such that

$$Z(h) = \text{Int}(f, S) = \text{Bdd}(f, S) = \text{Iva}(f, S).$$

Indeed, one can take $f(s, y) = h(s) \cdot y$. Hence the name "correspondences of loci" for Theorem 2.1.3.

One can interpolate Presburger constructible functions by Presburger constructible functions with maximal locus of integrability, as follows.

THEOREM 2.1.5 (Interpolation)
Let f be in $\mathcal{P}_q(S \times \mathbb{Z}^m)$ for some Presburger set S and some $m \geq 0$. Then there exists g in $\mathcal{P}_q(S \times \mathbb{Z}^m)$ with $\mathrm{Int}(g, S) = S$ and such that $f(s, y) = g(s, y)$ whenever s lies in $\mathrm{Int}(f, S)$.

The following result on the stability of $\mathcal{P}_q$ under summation generalizes [8, Theorem-Definition 4.5.1], which in turn goes back to [12, Lemma 3.2]. Theorem-Definition 4.5.1 of [8] is the special case of Theorem 2.1.6 for which $\mathrm{Int}(f, S) = S$.

THEOREM 2.1.6 (Integration)
Let f be in $\mathcal{P}_q(S \times \mathbb{Z}^m)$ for some Presburger set S and some $m \geq 0$. Then there exists a function $g \in \mathcal{P}_q(S)$ such that

$$g(s) = \sum_{y \in \mathbb{Z}^m} f(s, y)$$

whenever $s \in \mathrm{Int}(f, S)$.

The first auxiliary lemma can be obtained as a direct corollary of the o-minimality of the real number field with the exponential function, which follows from [27, Second Main Theorem, Corollary 5.4]. In addition, this lemma can also be proved in a more direct way, as communicated to us by Florent Martin.

LEMMA 2.1.7
Let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function of the form

$$h(x) = \sum_{i=1}^{r} c_i x^{a_i} b_i^x,$$

where the a_i's, b_i's, and c_i's are real numbers, the c_i's and b_i's are nonzero, and $r \geq 1$. Suppose that the pairs (a_i, b_i) are mutually different for different i's. Then the number of zeros of f is bounded by a constant only depending on r.

Proof
All functions h of the above form but with fixed r are members of a single definable family of functions with discrete zeros in the o-minimal structure of the real number field enriched with the exponential function. Now just note that discrete sets which appear as members of a family of sets in an o-minimal structure are finite and uniformly bounded in size (cf. [24]). □

LEMMA 2.1.8
Let $h : \mathbb{N}^m \to \mathbb{R}$ be a function of the form

$$h(x) = \sum_{i=1}^{r} c_i q^{b_{i1}x_1 + \cdots + b_{im}x_m} \prod_{j=1}^{m} x_j^{a_{ij}},$$

where the c_i's are nonzero real numbers, the a_{ij}'s and b_{ij}'s are integers, $a_{ij} \geq 0$, $m \geq 1$, and $r \geq 1$. Suppose that the tuples $(a_{i1}, \ldots, a_{im}, b_{i1}, \ldots, b_{im})$ are mutually different for different i's. Then h is not identically zero. Furthermore, h is summable over $\mathbb{N}^m$ if and only if $b_{ij} \leq -1$ for all i, j. Finally, h is bounded if and only if simultaneously all b_{ij}'s are less than or equal to 0 and for each i, j with $b_{ij} = 0$ one has $a_{ij} = 0$.

Proof
That h is not identically zero easily follows by induction on m and by Lemma 2.1.7 for the case $m = 1$. If all the b_{ij}'s are less than 0, then clearly h is summable. For the other direction, suppose that h is summable but some b_{ij} is greater than or equal to 0, say, $b_{11} \geq 0$. We may suppose that b_{11} is maximal among the b_{ij}'s. Put $I = \{i \mid b_{i1} = b_{11}\}$. We may suppose that a_{11} is maximal among the a_{i1}'s with $i \in I$. Put $J = \{i \in I \mid a_{i1} = a_{11}\}$. Then the function

$$x \mapsto q^{b_{11}x_1} x_1^{a_{11}} \sum_{i \in J} c_i q^{b_{i2}x_2 + \cdots + b_{im}x_m} \prod_{j=2}^{m} x_j^{a_{ij}}$$

must be identically zero on $\mathbb{N}^m$ by the summability of h, which is impossible by the first statement of the lemma. The statement about boundedness is obtained similarly. $\qquad \square$

The following result, [2, Theorem 3], will be the basis for the proofs of the results in this section.

THEOREM 2.1.9 (Parametric rectilinearization [2, Theorem 3])
Let S and $X \subset S \times \mathbb{Z}^m$ be Presburger sets. Then there exists a finite partition of X into Presburger sets such that, for each part A, there is a Presburger set $B \subset S \times \mathbb{Z}^m$ and a linear Presburger bijection $\rho : A \to B$ over S such that, for each $s \in S$, the set B_s is a set of the form $\Lambda_s \times \mathbb{N}^\ell$ for a finite subset $\Lambda_s \subset \mathbb{N}^{m-\ell}$ depending on s and for an integer $\ell \geq 0$ only depending on A.

Recall that B_s in Theorem 2.1.9 is the set $\{z \in \mathbb{Z}^m \mid (s, z) \in B\}$, and recall that for ρ to be over S means that ρ makes a commutative diagram with the projections from A and B to S (see Section 1.1). Some direct generalizations of Theorem 2.1.9 will be stated as Theorem 3.4.4 in a p-adic setting and as Theorem 4.5.4 in a uniform p-adic setting.

Proofs of Theorems 2.1.3 and 2.1.5

We first prove the existence of h_3 as in (2.1.3); that is, we first prove the result for $\mathrm{Iva}(f, S)$. Since the statement for $m = 1$ can be applied successively, it is enough to prove the case $m = 1$. By Theorem 2.1.9 and since Presburger functions are piecewise linear, there exists a finite partition of $S \times \mathbb{Z}$ and, for each part A, a Presburger bijection $\rho : A \to B$ over S such that either $B_x = \mathbb{N}$ or B_x is finite for each $x \in S$ and such that $f \circ \rho^{-1}$ is of the form

$$(x, t) \mapsto \sum_{i=1}^{r} c_i(x) t^{a_i} q^{b_i t}$$

for some integers a_i, b_i with $a_i \geq 0$ and some Presburger constructible functions c_i, where the pairs (a_i, b_i) are mutually different for different i's. Denote the image of A under the projection map $A \to S$ by S_A. By Lemma 2.1.7 there exists a constant $M \geq 0$ such that, for each fixed value of x, either the $c_i(x)$'s are all zero for $i = 1, \ldots, r$ or the function $t \mapsto \sum_{i=1}^{r} c_i(x) t^{a_i} q^{b_i t}$ has at most M zeros. Write $S_{A,1}$ for the set of $x \in S_A$ such that $|B_x| \leq M$, and let $S_{A,2}$ be $S_A \setminus S_{A,1}$. Note that $S_{A,1}$ and $S_{A,2}$ are Presburger sets. We take M Presburger functions $H_1, \ldots, H_M$ on $S_{A,1}$ such that the union of the graphs of the H_j's equals $B \cap (S_{A,1} \times \mathbb{Z})$. We write

$$Q_A := \left\{ x \in S_{A,1} \;\middle|\; \bigwedge_{j \in \{1, \ldots, M\}} \left(\sum_{i=1}^{r} c_i(x) H_j(x)^{a_i} q^{b_i H_j(x)} = 0 \right) \right\},$$

and

$$R_A := \left\{ x \in S_{A,2} \;\middle|\; \bigwedge_{i \in \{1, \ldots, r\}} \left(c_i(x) = 0 \right) \right\}.$$

By Lemma 2.1.2, each of the sets Q_A and R_A is the zero locus of a Presburger constructible function on S. Now one has

$$\mathrm{Iva}(f, S) = \bigcap_A \left(Q_A \cup R_A \cup (S \setminus S_A) \right).$$

By Lemma 2.1.2 and since the characteristic functions of the Presburger sets $S \setminus S_A$ are Presburger constructible, the existence of h_3 follows. This proves the existence of h_3 as in (2.1.3) for any given $f \in \mathscr{P}_q(S \times \mathbb{Z}^m)$.

We use this result to prove simultaneously Theorem 2.1.5 and the existence of h_1 and h_2. The statements clearly allow us to partition $S \times \mathbb{Z}^m$ into finitely many pieces A and to treat each one separately. (For the existence of h_1 and h_2, this uses Lemma 2.1.2.) We choose a partition such that all Presburger functions involved in f are Presburger linear, we refine this partition by using Theorem 2.1.9, and we consider

one resulting piece A. We can replace A by B and f by $f \circ \rho^{-1}$ with notation from Theorem 2.1.9, so that in the end, we get one Presburger set B on which we have

$$f(s, y) = \sum_{i=1}^{r} c_i(s) y^{a_i} q^{b_i \cdot y}, \tag{2.1.4}$$

where we use multiindex notation, $a_i, b_i \in \mathbb{Z}^m$ with $a_{ij} \geq 0$, the c_i's are Presburger constructible functions in $s \in S$, the tuples (a_i, b_i) are mutually different for different i's, and for each $s \in S$, one has $B_s = \Lambda_s \times \mathbb{N}^\ell$ for a fixed $\ell \geq 0$ and some finite set $\Lambda_s \subset \mathbb{N}^{m-\ell}$ depending on s. In fact, now we are already done by Lemma 2.1.8 and the existence of h_3 with $Z(h_3) = \mathrm{Iva}(g)$ for any given g. Indeed, let I be $\{i \mid b_{ij} \geq 0 \text{ for some } j = m - \ell + 1, \ldots, m\}$. Consider the function on B

$$h : (s, y) \mapsto \sum_{i \in I} c_i(s) y^{a_i} q^{b_i \cdot y}$$

for $s \in S$ and $y \in B_s$. Let $\widetilde{h}$ be the extension by zero of h to a function on $S \times \mathbb{Z}^m$. By Lemma 2.1.8, for $s \in S$, the family $\{f(s, y)\}_y$, where $y \in B_s$, is summable if and only if $\widetilde{h}(s, y) = 0$ for all $y \in \mathbb{Z}^m$. Since $\widetilde{h}$ is a Presburger constructible function on $S \times \mathbb{Z}^m$, (2.1.1) follows using h_1 with $Z(h_1) = \mathrm{Iva}(\widetilde{h})$. Taking $I' = \{i \mid (b_{ij} > 0 \text{ or } (b_{ij} = 0 \text{ and } a_{ij} > 0))\}$ for some $j = m - \ell + 1, \ldots, m\}$ instead of I in the above construction, one obtains the existence of h_2 for (2.1.2) in a similar way. Theorem 2.1.5 also follows, since we can define g piecewise for (s, y) in B by

$$g(s, y) = \sum_{i \in \{1, \ldots, r\} \setminus I} c_i(s) y^{a_i} q^{b_i \cdot y}. \qquad \square$$

Proof of Theorem 2.1.6
By the interpolation result Theorem 2.1.5, there exists g_0 in $\mathscr{P}_q(S \times \mathbb{Z}^m)$ with $\mathrm{Int}(g_0, S) = S$ and such that $f(s, y) = g_0(s, y)$ whenever s lies in $\mathrm{Int}(f, S)$. Now, by [8, Theorem-Definition 4.5.1], the function g that sends $s \in S$ to $\sum_{y \in \mathbb{Z}^m} g_0(s, y)$ lies in $\mathscr{P}_q(S)$. Clearly g is as required. $\qquad \square$

The above proof of Theorem 2.1.6 uses [8, Theorem-Definition 4.5.1]. For an alternative proof of Theorem 2.1.6 which does not rely on [8], one can proceed as follows. Use Theorem 2.1.9 to reduce to the case that g is a sum as in the right-hand side of (2.1.4), but with $m = 1$. If $y = y_1$ runs over $\mathbb{N}$, one knows that the b_i's from (2.1.4) are less than 0 by the proof of Theorem 2.1.3 and one uses explicit formulas for the summation of geometric power series and their derivatives. When $\Lambda_s \subset \mathbb{Z}$ is finite, with notation from (2.1.4), one may furthermore assume that Λ_s is of the form $\{z \in \mathbb{Z} \mid 0 \leq z \leq a(s)\}$, where a is a positively valued Presburger function in s and one uses geometric power series and their derivatives again to sum over Λ_s.

2.2. *Uniformity in the base q*

We show that the results of Section 2.1 hold uniformly in the base q. We will use this uniformity in the motivic setting. Write $\mathbb{R}_{>1}$ for $\{q \in \mathbb{R} \mid q > 1\}$.

Definition 2.2.1
As in [8], define the subring $\mathbb{A} \subset \mathbb{Q}(\mathbb{L})$ as

$$\mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1 - \mathbb{L}^{-i}}\Big)_{i \in \mathbb{N}, 0 < i}\Big],$$

where $\mathbb{L}$ is a formal symbol. Each $a \in \mathbb{A}$ is considered to be a function

$$a : \mathbb{R}_{>1} \to \mathbb{R} : q \mapsto a(q) \tag{2.2.1}$$

obtained by setting $\mathbb{L} = q$. For S a Presburger set, let $\mathcal{P}^u(S)$ be the $\mathbb{A}$-algebra of $\mathbb{R}$-valued functions on $S \times \mathbb{R}_{>1}$ generated by
(1) the functions $\alpha : S \times \mathbb{R}_{>1} \to \mathbb{R} : (s, q) \mapsto \alpha(s)$ for all Presburger functions $\alpha : S \to \mathbb{Z}$,
(2) the functions $q^\beta : S \times \mathbb{R}_{>1} \to \mathbb{R} : (s, q) \mapsto q^{\beta(s)}$ for all Presburger functions $\beta : S \to \mathbb{Z}$.
The functions in $\mathcal{P}^u(S)$ are called Presburger constructible functions on S with uniform base.

The ring $\mathbb{A}$ and a close variant of the rings $\mathcal{P}^u(S)$ also appear in [8]. The analogues of Theorems 2.1.3, 2.1.5, and 2.1.6 hold with almost the same proofs.

THEOREM 2.2.2 (Correspondences of loci)
Let S be a Presburger set, and let f be in $\mathcal{P}^u(S \times \mathbb{Z}^m)$ for some $m \geq 0$. Then there exist h_1, h_2, and h_3 in $\mathcal{P}^u(S)$ such that

$$\mathrm{Int}(f, S \times \mathbb{R}_{>1}) = Z(h_1),$$

$$\mathrm{Bdd}(f, S \times \mathbb{R}_{>1}) = Z(h_2),$$

and

$$\mathrm{Iva}(f, S \times \mathbb{R}_{>1}) = Z(h_3).$$

THEOREM 2.2.3 (Interpolation)
Let f be in $\mathcal{P}^u(S \times \mathbb{Z}^m)$ for some Presburger set S and some $m \geq 0$. Then there exists g in $\mathcal{P}^u(S \times \mathbb{Z}^m)$ such that $\mathrm{Int}(g, S \times \mathbb{R}_{>1}) = S \times \mathbb{R}_{>1}$ and such that $f(s, y, q) = g(s, y, q)$ whenever $(s, q) \in \mathrm{Int}(f, S \times \mathbb{R}_{>1})$ and $y \in \mathbb{Z}^m$.

THEOREM 2.2.4 (Integration)
Let f be in $\mathcal{P}^u(S \times \mathbb{Z}^m)$ for some Presburger set S and some $m \geq 0$. Then there exists a function $g \in \mathcal{P}^u(S)$ such that

$$g(s,q) = \sum_{y \in \mathbb{Z}^m} f(s,y,q)$$

whenever $(s,q) \in \mathrm{Int}(f, S \times \mathbb{R}_{>1})$.

Proofs of Theorems 2.2.2, 2.2.3, and 2.2.4
Since the Lemmas 2.1.7 and 2.1.8 are completely uniform in q, the proofs of Section 2.1 go through almost literally the same way.   □

3. Integrability over a fixed p-adic field

In this section, we study the loci of Definition 1.0.1 and obtain similar results as in Section 2, but now for functions on a finite-degree field extension of $\mathbb{Q}_p$. We do this first in a setting without oscillation, and then in a setting where the functions may oscillate due to the presence of additive characters. The key technical result to control the difficulties related to oscillation is provided by Proposition 3.2.5.

Let K be a fixed finite field extension of $\mathbb{Q}_p$ for a prime number p. Write q_K for the number of elements in the residue field k_K of K, and write $\mathcal{O}_K$ for the valuation ring of K with maximal ideal $\mathcal{M}_K$. Fix $\mathscr{L}_K$ to be either the semialgebraic language on K with coefficients from K, that is, Macintyre's language, or the subanalytic language on K (as in, e.g., [25] or [3]). Recall that Macintyre's language is the ring language $(\cdot, +, -, 0, 1)$ enriched with coefficients from K and, for each integer $n > 1$, a one-variable predicate for the set of nth powers in $K^\times$. The subanalytic language on K is Macintyre's language enriched with the field inverse $^{-1}$ on $K^\times$ extended by $0^{-1} = 0$ and, for each convergent power series $f : \mathcal{O}_K^n \to K$, a function symbol for the restricted analytic function

$$x \in K^n \mapsto \begin{cases} f(x) & \text{if } x \in \mathcal{O}_K^n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that one has quantifier elimination in $\mathscr{L}_K$ by Macintyre's result [20, Theorem 1] (see also [13]) and by [15] for the subanalytic case.

Write ϖ_K for a fixed uniformizer of K, and write $|\cdot|$ for the norm on K with $|\varpi_K| = q_K^{-1}$. Put the normalized Haar measure on K^n, denoted by $|dx|$ whenever x denotes a tuple of variables running over K^n and where the normalization is such that $\mathcal{O}_K^n$ has measure 1. For each integer $m > 0$ consider the map $\overline{\mathrm{ac}}_m : K \to \mathcal{O}_K/(\varpi_K^m)$ sending nonzero $x \in K$ to $\varpi_K^{-\mathrm{ord}\,x} \cdot x \mod (\varpi_K^m)$ and sending 0 to 0. We also write $\overline{\mathrm{ac}}$ for $\overline{\mathrm{ac}}_1$. To make the link with the motivic setting easier, we consider three sorted

structures for our fixed p-adic field K. To this end, we enrich the language $\mathcal{L}_K$ with the sorts $\mathbb{Z}$ for the value group and k_K for the residue field, together with the valuation map ord : $K^{\times} \to \mathbb{Z}$ and the angular component map $\overline{\mathrm{ac}}$. Let us denote this three-sorted language by $\mathcal{L}_K^3$. Let us for each $m > 1$ identify the map $\overline{\mathrm{ac}}_m : K \to \mathcal{O}_K/(\varpi_K^m)$ with a map $K \to k_K^m$, also denoted by $\overline{\mathrm{ac}}_m$, by using a bijection of finite sets $\mathcal{O}_K/(\varpi_K^m) \to k_K^m$.

Endow $K^n \times k_K^m \times \mathbb{Z}^r$ with the product topology of the valuation topology on K^n and the discrete topology on $k_K^m \times \mathbb{Z}^r$. In this section, definable will mean $\mathcal{L}_K^3$-definable.

3.1. Constructible functions

The ring of constructible functions $\mathcal{C}(X)$ on a definable set X is the $\mathbb{A}_{q_K}$-algebra of real-valued functions on X generated by functions of the form

(1) $f : X \to \mathbb{Z}$ whenever f is a definable function,

(2) $q_K^g : X \to \mathbb{A}_{q_K} : x \mapsto q_K^{g(x)}$ for definable functions $g : X \to \mathbb{Z}$.

The functions in $\mathcal{C}(X)$ are called constructible functions on X.

Now the analogues of Theorems 2.1.3, 2.1.5, and 2.1.6 hold in the p-adic setting. In fact, the interest in the rings of constructible functions lies in their stability under integration, which we generalize to the following result.

THEOREM 3.1.1 (Integration)
Let f be in $\mathcal{C}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exists $g \in \mathcal{C}(X)$ such that

$$g(x) = \int_{y \in K^m} f(x, y)|dy|$$

whenever $x \in \mathrm{Int}(f, X)$.

Remark 3.1.2
Under the extra condition that $\mathrm{Int}(f, X) = X$, Theorem 3.1.1 was known: in the subanalytic case this is [3, Theorem 4.2] and the semialgebraic case has the same proof as in [3], using the semialgebraic cell decomposition instead of the subanalytic cell decomposition. The first form of this kind of integration result (with some more conditions on f) goes back to Denef [14], where the functions of $\mathcal{C}(X)$ were introduced under a different name.

THEOREM 3.1.3 (Correspondences of loci)
Let f be in $\mathcal{C}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exist functions h_1, h_2, and h_3 in $\mathcal{C}(X)$ such that the zero loci of the h_i's equal, respectively,

$$\text{Int}(f, X), \qquad \text{Bdd}(f, X), \qquad and \qquad \text{Iva}(f, X),$$

for $i = 1$, 2, and 3, and with the normalized Haar measure on K^m.

Theorem 3.1.3 has the following corollary.

COROLLARY 3.1.4
Let f be in $\mathcal{C}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exist functions h_1 and h_2 in $\mathcal{C}(X)$ such that

$$\{x \in X \mid f(x, \cdot) \text{ is locally integrable on } K^m\} = Z(h_1) \qquad (3.1.1)$$

and

$$\{x \in X \mid f(x, \cdot) \text{ is locally bounded on } K^m\} = Z(h_2). \qquad (3.1.2)$$

Proof
Note that local integrability (resp., local boundedness) for a function r on K^n is equivalent to $\mathbf{1}_B \cdot r$ being integrable over K^n (resp., bounded on K^n) for each Cartesian product $B \subset K^n$ of balls in K, with characteristic function $\mathbf{1}_B$. Note that the family of all balls can be (possibly redundantly) realized as the members of a definable family (parameterized by, say, the radius and an element of the ball). Now the corollary follows from the three statements of Theorem 3.1.3, where the existence of h_3 is used to eliminate the variables that were used to parameterize the balls. $\qquad\square$

THEOREM 3.1.5 (Interpolation)
Let f be in $\mathcal{C}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exists g in $\mathcal{C}(X \times K^m)$ with $\text{Int}(g, X) = X$ and such that $f(x, y) = g(x, y)$ whenever x lies in $\text{Int}(f, X)$.

The above results will be proved using the cell decomposition Theorem 3.3.2 below and the analogous results of Section 2.1, but first we state the main *p*-adic results in the exponential setting, which will have completely different and more difficult proofs.

3.2. Constructible exponential functions
Fix an additive character $\psi_K : K \to \mathbb{C}^\times$ which is trivial on $\mathcal{M}_K$ but nontrivial on $\mathcal{O}_K$. (All characters are assumed to be unitary and continuous.) The ring of constructible exponential functions $\mathcal{C}^{\exp}(X)$ on a definable set X is the $\mathbb{A}_{q_K}$-algebra of complex-valued functions on X generated by functions of the form
(1) g with g in $\mathcal{C}(X)$;

(2)　　functions $\psi_K(f) : X \to \mathbb{C} : x \mapsto \psi_K(f(x))$ for any definable function $f :$
$X \to K$.

The functions in $\mathcal{C}^{\exp}(X)$ are called constructible exponential functions on X.

THEOREM 3.2.1 (Integration)
Let f be in $\mathcal{C}^{\exp}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exists $g \in \mathcal{C}^{\exp}(X)$ such that

$$g(x) = \int_{y \in K^m} f(x, y)|dy|$$

for all $x \in \mathrm{Int}(f, X)$.

In [9], Theorem 3.2.1 is proved under an extra restriction: f must be a finite sum of terms of the form $f_0 \psi_K(f_1)$ with $f_1 : X \times K^m \to K$ definable and $f_0 \in \mathcal{C}(X \times K^m)$ satisfying $\mathrm{Int}(f_0, X) = X$ (see also [18] for a similar result under a similar extra restriction).

We also find analogues of Theorems 3.1.3 and 3.1.5 in the exponential setting.

THEOREM 3.2.2 (Correspondences of loci)
Let f be in $\mathcal{C}^{\exp}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exist functions h_1, h_2, and h_3 in $\mathcal{C}^{\exp}(X)$ such that

$$\mathrm{Int}(f, X) = Z(h_1),$$

$$\mathrm{Bdd}(f, X) = Z(h_2),$$

and

$$\mathrm{Iva}(f, X) = Z(h_3).$$

THEOREM 3.2.3 (Interpolation)
Let f be in $\mathcal{C}^{\exp}(X \times K^m)$ for some definable set X. Then there exists g in $\mathcal{C}^{\exp}(X \times K^m)$ with $\mathrm{Int}(g, X) = X$ and such that $f(x, y) = g(x, y)$ whenever x lies in $\mathrm{Int}(f, X)$. Moreover, one can write any such g as a finite sum of terms of the form

$$f_0 \psi_K(f_1)$$

with $f_1 : X \times K^m \to K$ definable and $f_0 \in \mathcal{C}(X \times K^m)$ satisfying $\mathrm{Int}(f_0, X) = X$.

Theorem 3.2.2 implies the following corollary by the same reasoning as for Corollary 3.1.4.

COROLLARY 3.2.4

Let f be in $\mathcal{C}^{\exp}(X \times K^m)$ for some definable set X and some $m \geq 0$. Then there exist functions h_1 and h_2 in $\mathcal{C}^{\exp}(X)$ such that

$$\{x \in X \mid f(x, \cdot) \text{ is locally integrable on } K^m\} = Z(h_1) \qquad (3.2.1)$$

and

$$\{x \in X \mid f(x, \cdot) \text{ is locally bounded on } K^m\} = Z(h_2). \qquad (3.2.2)$$

The following key technical proposition excludes strange oscillatory behavior of exponential constructible functions. This will allow us to reduce to the techniques and results from the previous sections.

PROPOSITION 3.2.5

Let $m \geq 0$ and $s \geq 0$ be integers, let X and $U \subset X \times K^m$ be definable, and let $f_1, \ldots, f_s$ be in $\mathcal{C}^{\exp}(U)$. Write x for variables running over X, and write y for variables running over K^m. Then there exist an integer $d \geq 0$, a definable surjection $\varphi : U \to V \subset X \times \mathbb{Z}^t$ over X for some $t \geq 0$, definable functions $h_{\ell i} : U \to K$, and functions $G_{\ell i}$ in $\mathcal{C}^{\exp}(V)$ for $\ell = 1, \ldots, s$ such that the following conditions hold:
(a)   *for each ℓ with $1 \leq \ell \leq s$, one has*

$$f_\ell(x, y) = \sum_{i=1}^{N_\ell} G_{\ell i}\big(\varphi(x, y)\big)\psi_K\big(h_{\ell i}(x, y)\big),$$

   for some positive integer N_ℓ;
(b)   *if one sets, for $(x, r) \in V$,*

$$U_{x,r} := \big\{y \in U_x \mid \varphi(x, y) = (x, r)\big\}$$

   and

$$W_{x,r} := \big\{y \in U_{x,r} \mid \sup_{\ell,i}\big|G_{\ell i}(x, r)\big|_{\mathbb{C}} \leq \sup_\ell\big| f_\ell(x, y)\big|_{\mathbb{C}}\big\},$$

   where $|\cdot|_{\mathbb{C}}$ is the complex modulus, then

$$\mathrm{Vol}(U_{x,r}) \leq q_K^d \cdot \mathrm{Vol}(W_{x,r}) < +\infty,$$

   where the volume Vol is taken with respect to the Haar measure on K^m.

Roughly, the proposition for $s = 1$ says that if $|f_1|_{\mathbb{C}}$ is small, then f_1 is the sum of small terms of a very specific form. Indeed, for the functions $|G_{1i}(\varphi(x, \cdot)) \times \psi_K(h_{1i}(x, \cdot))|_{\mathbb{C}} = |G_{1i}(\varphi(x, \cdot))|_{\mathbb{C}}$ to be small, it suffices to know that they are small

on the sets $W_{x,r}$, since they are constant on each superset $U_{x,r}$. More precisely, if f_1 cannot be written as a sum of small terms as in Proposition 3.2.5(a), then $|f_1|_{\mathbb{C}}$ has to be large on a relatively large set, namely, on the set $W_{x,r}$.

For the proposition to make sense, the sets $U_{x,r}$ and $W_{x,r}$ have to be measurable, but this follows from the facts that each definable set is measurable, that functions in $\mathcal{C}^{\exp}(Z)$ are measurable for any definable Z, and that the space of measurable functions is closed under taking the complex modulus and the supremum. We prove this proposition in the next section. First, we need to set up some preliminaries.

3.3. Preliminaries for the p-adic proofs

We give a notion of p-adic cells which is adapted to the three sorts in $\mathcal{L}_K^3$ and which fits better with the motivic approach below than some previously used notions of p-adic cells (see especially the usage of ξ in the next definition).

Definition 3.3.1 (p-adic cells)
Let Y be a definable set. A 1-cell $A \subset Y \times K$ over Y is a (nonempty) set of the form

$$A = \{(y,t) \in Y' \times K \mid \alpha(y) \,\square_1\, \mathrm{ord}(t - c(y)) \,\square_2\, \beta(y),$$
$$\mathrm{ord}(t - c(y)) \in a + n\mathbb{Z}, \overline{\mathrm{ac}}_m(t - c(y)) = \xi(y)\}, \qquad (3.3.1)$$

with Y' a definable subset of Y, integers $a \geq 0$, $n > 0$, $m > 0$, $\alpha, \beta \colon Y' \to \mathbb{Z}$ and $\xi \colon Y' \to (\mathcal{O}_K/(\varpi_K^m))^\times$ definable, $c \colon Y' \to K$ definable, and $\square_i$ either $<$ or no condition, and such that A projects surjectively onto Y'. We call c the center, ξ the angular component, $a + n\mathbb{Z}$ the coset, α and β the boundaries, and Y' the base of A. A 0-cell $A \subset Y \times K$ over Y is a (nonempty) set of the form

$$A = \{(y,t) \in Y' \times K \mid t = c(y)\}, \qquad (3.3.2)$$

with Y' a definable subset of Y and $c \colon Y' \to K$ definable. In both cases we call A a cell over Y with center c.

Also our formulation of the cell decomposition result is somehow more relaxed than usual, due to having three sorts. For a slightly stronger cell decomposition result than Theorem 3.3.2, see, for example, [4, Theorem 3.3].

THEOREM 3.3.2 (*p-adic cell decomposition* [13, Theorem 1.8], [3, Theorem 2.3])
Let $X \subset Y \times K$ and $f_j \colon X \to \mathbb{Z}$ be definable for some definable set Y and $j = 1,\ldots,r$. Then there exists a finite partition of X into cells A_i (over Y) with center c_i such that for each occurring 1-cell A_i with coset $a_i + n_i\mathbb{Z}$ and base Y_i' one has

$$f_j(y,t) = h_{ij}(y) + a_{ij}\frac{\mathrm{ord}(t - c_i(y)) - a_i}{n_i}, \quad \text{for each } (y,t) \in A_i,$$

with integers a_{ij} and $h_{ij} : Y'_i \to \mathbb{Z}$ definable functions for $j = 1, \ldots, r$. Moreover, if also $g_j \in \mathcal{C}(X)$ are given for $j = 1, \ldots, r'$, then the cells A_i can be taken such that, for each $y \in y$, the function $g_i(y, \cdot)$ is constant on each ball contained in A_{iy}.

For $X \subset K$ open, a function $f : X \to K$ is called C^1 if f is differentiable at each point of X and the derivative $f' : X \to K$ of f is continuous. (This notion of C^1, although more naive than the ones in, e.g., [16], suffices for our purposes.) A ball in K is by definition a set of the form $\{t \in K \mid \operatorname{ord}(t - a) \geq z\}$ for some $a \in K$ and some $z \in \mathbb{Z}$. For X a subset of K, by a maximal ball contained in X we mean a ball $B \subset X$ which is maximal with respect to inclusion among all balls contained in X.

Definition 3.3.3 (Jacobian property)
Let $f : B_1 \to B_2$ be a function with $B_1, B_2 \subset K$. We say that f has the *Jacobian property* if the following conditions hold:
(a) f is a bijection from B_1 onto B_2 and B_1 and B_2 are balls in K;
(b) f is C^1 on B_1 with nonvanishing derivative f';
(c) $|f'|$ is constant on B_1;
(d)   for all $x, y \in B_1$ one has

$$\big| (x - y) \cdot f' \big| = \big| f(x) - f(y) \big|.$$

Definition 3.3.4 (1-Jacobian property)
Let $f : B_1 \to B_2$ be a function with $B_1, B_2 \subset K$. We say that f has the *1-Jacobian property* if f has the Jacobian property and moreover the following conditions hold:
(e) $\overline{\operatorname{ac}}(f')$ is constant on B_1;
(f)   for all $x, y \in B_1$ one has

$$\overline{\operatorname{ac}}(f') \cdot \overline{\operatorname{ac}}(x - y) = \overline{\operatorname{ac}}\big(f(x) - f(y) \big).$$

The following two results will be important for the proofs in the exponential setting.

PROPOSITION 3.3.5 ([7, Section 6])
Let Y and $X \subset Y \times K$ be definable sets, and let $F : X \to K$ be a definable function. Then there exists a finite partition of X into cells A_i over Y such that, for each i and each $y \in Y$, the restriction of $F(y, \cdot)$ to $A_{iy} := \{t \in K \mid (y, t) \in A_i\}$ is either constant or injective and such that in the latter case, for each ball $B \subset K$ such that $\{y\} \times B$ is contained in A_i, there is a ball $B^* \subset K$ such that $F(y, B) = B^*$ and such that the map

$$F_B : B \to B^* : t \mapsto F(y,t)$$

has the 1-*Jacobian property.*

LEMMA 3.3.6

Let $A \subset Y \times K$ and $h : A \to K$ be definable for some definable set Y. Suppose, for each $y \in Y$ and for each maximal ball B contained in A_y, that $h(y,\cdot)$ is constant modulo (ϖ_K) on B. Then there exists a finite partition of A into definable sets A_j and definable functions $h_j : Y \to K$ such that

$$\left| h(y,t) - h_j(y) \right| \le 1$$

holds for each $(y,t) \in A_j$ and for each j.

Remark 3.3.7

We state the lemma in its present form and prove it in such a way that makes it easy to adapt to the motivic case below.

Proof of Lemma 3.3.6

The proof is similar to the proof of [9, Theorem 2.2.2] and goes as follows. Up to a finite partition of A, we may suppose that, for each $y \in Y$, the function $h(y,\cdot)$ is injective (see, e.g., [4, Corollary 3.7]). Similarly we may suppose that h is as F in the conclusion of Proposition 3.3.5 already on the whole of A. We may moreover assume, for each two balls $\{y\} \times B_1$ and $\{y\} \times B_2$ contained in A, that the images $h(\{y\} \times B_1)$ and $h(\{y\} \times B_2)$ are balls with different radii, for example, by invoking Theorem 3.3.2. Now consider the graph of h in $A \times K$, and consider its image $W \subset Y \times K$ under the coordinate projection sending $(y,t,h(y,t))$ to $(y,h(y,t))$. Take a cell decomposition as in Theorem 3.3.2 of W into cells C_j. Write c_j for the center of C_j. Let us fix $m_j > 0$ such that the angular component of C_j (i.e., the map denoted by ξ in Definition 3.3.1) takes values in $\mathcal{O}_K/(\varpi_K^{m_j})$ in the case that C_j is a 1-cell and such that $m_j = 1$ if C_j is a 0-cell. Now it follows for all $(y,t) \in A$ with $h(y,t) \in C_j$ that

$$\left| h(y,t) - c_j(y) \right| \le q_K^{m_j - 1}.$$

Let A_j be the set consisting of $(y,t) \in A$ with $h(y,t) \in C_j$. If $m_j = 1$, we are done by taking $h_j = c_j$ on A_j. If $m_j > 1$, we can finish by further partitioning using the finiteness of the residue field. □

3.4. The p-adic proofs for constructible functions

For p-adic constructible functions (thus without additive character), the proofs reduce to the Presburger cases of Section 2.1 with $q = q_K$ via p-adic cell decomposition and the following definitions and results.

Definition 3.4.1

If $f_j \colon X \subset K^{m+1} \to \mathbb{Z}$ and the A_i's are as in Theorem 3.3.2, then call f_j *prepared* on the cells A_i. We call A_i a *full cell* and we call f_j *fully prepared* on the A_i if the base of A_i is itself a cell on which $h_{ij}(x)$—with notation from Theorem 3.3.2 for 1-cells and with $h_{ij} = f_j$ in the case of 0-cells—and the boundaries of A_i are prepared, and so on m times. It is also clear what we mean by a full cell $A \subset Y \times K^{m+1}$ *over* some definable set Y, in analogy to the notion of cells over Y of Definition 3.3.1. By the centers of a full cell, we mean a tuple of centers, consisting of the center of the cell A over Y, the center of the base A' of A, the center of the base of A', and so on.

Definition 3.4.2

Let $A \subset K^m$ be a full cell with center c_m in the last coordinate, up to center $c_1 \in K$ for the first coordinate. The skeleton of A is then the subset $S(A)$ of $(\mathbb{Z} \cup \{+\infty\})^m$, which is the image of A under the map

$$x \in A \mapsto \left(\operatorname{ord}(x_1 - c_1), \operatorname{ord}(x_2 - c_2(x_1)), \dots, \operatorname{ord}(x_m - c_m(x_1, \dots, x_{m-1}))\right),$$

where we have extended ord to a map $\operatorname{ord} \colon K \to \mathbb{Z} \cup \{+\infty\}$. Write s_A for the natural map $A \to S(A)$, which we call the skeleton map. Likewise, if $A \subset Y \times K^\ell$ is a full cell over Y for some definable set Y, it is clear what we mean by the skeleton $S_{/Y}(A)$ over Y and the skeleton map $s_{A/Y}$ of A over Y.

Remark 3.4.3

For $A \subset K^m$ a full cell, any fiber of the skeleton map s_A is a Cartesian product of singletons and balls, and hence, the volume of such a fiber $s_A^{-1}(r)$ equals 0 for each r or equals $q_K^{\alpha(r)}$ for a definable function $\alpha \colon A \to \mathbb{Z}$. Clearly the set of r such that $s_A^{-1}(r)$ has volume zero is a definable set.

Call a definable function $f \colon X \subset Y \times \mathbb{Z}^m \to Y \times \mathbb{Z}^\ell$ linear over Y if there is a definable function $a \colon Y \to \mathbb{Z}^\ell$ and an affine map $g \colon \mathbb{Q}^m \to \mathbb{Q}^\ell$ such that $f(y, z) = (y, g(z) + a(y))$ for all $(y, z) \in X$.

PROPOSITION 3.4.4 (Parametric rectilinearization for $\mathcal{L}_K^3$)

Let Y and $X \subset Y \times \mathbb{Z}^m$ be definable sets. Then there exists a finite partition of X into definable sets such that, for each part A, there is a set $B \subset Y \times \mathbb{Z}^m$ and a definable

bijection $\rho : A \to B$ *which is linear over* Y *such that, for each* $y \in Y$, *the set* B_y *is a set of the form* $\Lambda_y \times \mathbb{N}^\ell$ *for a finite subset* $\Lambda_y \subset \mathbb{N}^{m-\ell}$ *depending on* y *and for an integer* $\ell \geq 0$ *only depending on* A.

Proof

This follows from Theorem 2.1.9 as follows. If $Y \subset \mathbb{Z}^r$, then this is Theorem 2.1.9. If $Y \subset \mathbb{Z}^r \times k_K^n$, then the result follows from Theorem 2.1.9 and the orthogonality between the k_K-sort and the $\mathbb{Z}$-sort. (This orthogonality is the property that any definable subset of $\mathbb{Z}^r \times k_K^n$ equals a finite union of Cartesian products of definable subsets of $\mathbb{Z}^r$ and of k_K^n; this property follows from quantifier elimination of K-variables.) Now suppose that $Y \subset \mathbb{Z}^r \times K^\ell \times k_K^n$. By quantifier elimination of K-variables, there exist s and t, a definable function

$$\nu : Y \to k_K^t \times \mathbb{Z}^s,$$

and a definable set

$$X' \subset k_K^t \times \mathbb{Z}^{s+m}$$

such that X_y equals $X'_{\nu(y)}$ for every $y \in Y$. Applying the previous case to the set X' yields a partition of X' into parts A' with the desired properties. Let κ be the function

$$\kappa : X \to X' : (y, w) \mapsto (\nu(y), w).$$

Now the sets $\kappa^{-1}(A')$ form a partition of X with the desired properties. (As an alternative proof one may adapt the proof of [2, Theorem 3].) □

We now have the following variants of the Presburger Theorems 2.1.3 and 2.1.5.

COROLLARY 3.4.5

Let f *be in* $\mathcal{C}(X \times \mathbb{Z}^m)$ *for some definable set* X *and some* $m \geq 0$. *Then there exist* h_1, h_2, *and* h_3 *in* $\mathcal{C}(X)$ *such that*

$$\mathrm{Int}(f, X) = Z(h_1), \tag{3.4.1}$$

$$\mathrm{Bdd}(f, X) = Z(h_2), \tag{3.4.2}$$

and

$$\mathrm{Iva}(f, X) = Z(h_3), \tag{3.4.3}$$

where integrability in (3.4.1) is with respect to the counting measure on $\mathbb{Z}^m$.

COROLLARY 3.4.6 (Interpolation)
Let f be in $\mathcal{C}(X \times \mathbb{Z}^m)$ for some definable set X and some $m \geq 0$. Then there exists g in $\mathcal{C}(X \times \mathbb{Z}^m)$ with $\mathrm{Int}(g, X) = X$ and such that $f(x, y) = g(x, y)$ whenever x lies in $\mathrm{Int}(f, X)$.

Proofs of Corollaries 3.4.5 and 3.4.6
The proofs of Theorems 2.1.3 and 2.1.5 apply also in this setting, where one uses Theorem 3.4.4 instead of Theorem 2.1.9.   □

Proofs of Theorems 3.1.3 and 3.1.5
By an inductive application of the cell decomposition Theorem 3.3.2, partition $X \times K^m$ into finitely many full cells A_i over X such that, for each i, the restriction $f_{|A_i}$ factors through the skeleton map $s_{A_i/X}$ of A_i over X. Let us identify each skeleton with a definable set, for example, by replacing $\{+\infty\}$ by a disjoint copy of the singleton $\{0\}$. Let us write f_i for the map from the skeleton $S_{/X}(A_i)$ of A_i over X to $\mathbb{A}_{q_K}$ induced by $f_{|A_i}$. Then f_i lies in $\mathcal{P}_{q_K}(S_{/X}(A_i))$ for each i. The function M sending $z \in S_{/X}(A_i)$ to the volume of the fiber $(s_{A_i/X})^{-1}(z)$, taken inside K^m, either is identically zero on $S_{/X}(A_i)$ or is of the form $q_K^{\alpha_i(z)}$ on $S_{/X}(A_i)$ for some definable function α_i, since the fibers of $s_{A_i/X}$ are of a very simple form (see Remark 3.4.3). Hence, $\tilde{f}_i$ given by $z \mapsto f_i(z) \cdot M(z)$ lies in $\mathcal{C}(S_{/X}(A_i))$. For each i, let X_i be the image of A_i under the projection to X. Note that X_i also equals the image of $S_{/X}(A_i)$ under the projection to X. By Corollary 3.4.5, we now obtain for each i that there are h_{i1}, h_{i2}, and h_{i3} in $\mathcal{C}(X_i)$ such that

$$\mathrm{Int}(\tilde{f}_i, X_i) = Z(h_{i1}),$$

$$\mathrm{Bdd}(f_i, X_i) = Z(h_{i2}),$$

and

$$\mathrm{Iva}(f_i, X_i) = Z(h_{i3}).$$

Extend each of the h_{ij} by zero to a function $\tilde{h}_{ij}$ in $\mathcal{C}(X)$, thus defined on the whole of X. Now for $j = 1, 2,$ or 3, let h_j be the function

$$\sum_i \tilde{h}_{ij}^2.$$

Then the h_j's for $j = 1, 2, 3$ are as required.

     For the construction of g as desired for Theorem 3.1.5 one proceeds as follows. By Corollary 3.4.6 one finds for each i a function g_i in $\mathcal{C}(S_{/X}(A_i))$ with $\mathrm{Int}(g_i, X_i) = X_i$ and such that $\tilde{f}_i(x, s) = g_i(x, s)$ whenever x lies in $\mathrm{Int}(\tilde{f}_i, X_i)$.

Now we define the function g on $X \times K^m$ as follows. For $(x, y) \in A_i$, let $z = s_{A_i/X}(x, y)$, let $g(x, y) := f(x, y)$ if $M(z) = 0$, and let the quotient $g(x, y) := g_i(z)/M(z)$ when $M(z) \neq 0$. Note that g is a constructible function, by the simple form of M. The function g is as required. $\square$

Proof of Theorem 3.1.1
The theorem follows from Theorem 3.1.5 and [3, Theorem 4.2] in the same way that Theorem 2.1.6 follows from Theorem 2.1.5 and [8, Theorem-Definition 4.5.1]. Indeed, by the interpolation result Theorem 3.1.5, there exists g_0 in $\mathcal{C}(X \times K^m)$ with $\mathrm{Int}(g_0, X) = X$ and such that $f(x, y) = g_0(x, y)$ whenever x lies in $\mathrm{Int}(f, X)$. Now, by [3, Theorem 4.2] and by Remark 3.1.2 for the semialgebraic case, the function g which sends $x \in X$ to $\int_{K^m} g_0(x, y)|dy|$ lies in $\mathcal{C}(X)$. Clearly g is as required. $\square$

In fact, the proofs of Theorems 3.1.3 and 3.1.5 yield the following slightly more general variant.

COROLLARY 3.4.7
Let f be in $\mathcal{H}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$ for some definable set X, some $m \geq 0$, and some inclusion $\mathcal{C}(X) \subset \mathcal{H}(X)$ of $\mathbb{A}_q$-algebras of complex-valued functions on X. Then there exist functions h_1, h_2, h_3 in $\mathcal{H}(X)$ and g in $\mathcal{H}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$ such that the zero loci of the h_i's equal, respectively,

$$\mathrm{Int}(f, X), \qquad \mathrm{Bdd}(f, X), \qquad \text{and} \qquad \mathrm{Iva}(f, X)$$

and such that $\mathrm{Int}(g, X) = X$ and $f(x, y) = g(x, y)$ whenever x lies in $\mathrm{Int}(f, X)$. Moreover, any such g can be written as a finite sum of terms of the form

$$h_i \cdot f_i$$

with $h_i \in \mathcal{H}(X)$ and $f_i \in \mathcal{C}(X \times K^m)$ satisfying $\mathrm{Int}(f_i, X) = X$.

3.5. The p-adic proofs for constructible exponential functions
Consider a finite field $\mathbb{F}_q$ with a nontrivial additive character ψ. The following lemma and its corollary are classical exercises.

LEMMA 3.5.1
For any function $f : \mathbb{F}_q \to \mathbb{C}$ one has

$$\frac{1}{q}\|\hat{f}\|_{\sup} \leq \|f\|_{\sup} \leq \|\hat{f}\|_{\sup},$$

where $\| \cdot \|_{\sup}$ is the supremum norm and $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(y) = \sum_{x \in \mathbb{F}_q} f(x) \psi(-xy).$$

COROLLARY 3.5.2
Consider a function

$$f : \mathbb{F}_q \to \mathbb{C} : y \mapsto \sum_{j=1}^{s} c_j \psi(b_j y)$$

for some complex numbers c_j and some distinct $b_j \in \mathbb{F}_q$. Then there exists $y_0 \in \mathbb{F}_q$ with

$$\sup_{j=1}^{s} |c_j|_{\mathbb{C}} \leq |f(y_0)|_{\mathbb{C}}.$$

LEMMA 3.5.3
Let X be a definable set, and let f be in $\mathcal{C}^{\exp}(X)$. Then there exists a function g in $\mathcal{C}^{\exp}(X)$ such that, for each $x \in X$, $f(x)$ and $g(x)$ are conjugate complex numbers. In particular, $f(x)g(x)$ equals the square of the complex norm of $f(x)$ for each $x \in X$, and thus in particular $Z(f) = Z(fg)$.

Proof
The function g can be obtained from f by putting a minus sign in each of the arguments of the additive characters which occur in f. More precisely, write f as a finite sum of terms of the form $g_i \psi_K(h_i)$ for g_i in $\mathcal{C}(X)$ and definable functions $h_i : X \to K$, and define g as the corresponding sum with terms $g_i \psi_K(-h_i)$. Since the g_i's take real values, g is as desired. □

This lemma yields that any finite intersection or finite union of zero loci of functions in $\mathcal{C}^{\exp}(X)$ is again a zero locus of a function in $\mathcal{C}^{\exp}(X)$, analogous to Lemma 2.1.2.

COROLLARY 3.5.4
Let X be a definable set, and let the h_i's be in $\mathcal{C}^{\exp}(X)$ for $i = 1, \ldots, N$. Then there exist f and g in $\mathcal{C}^{\exp}(X)$ such that

$$Z(f) = \bigcap_{i=1}^{N} Z(h_i) \qquad \text{and} \qquad Z(g) = \bigcup_{i=1}^{N} Z(h_i).$$

The corresponding statement for the h_i's in $\mathcal{C}(X)$ also holds, yielding $f, g \in \mathcal{C}(X)$.

Proof

For f one can take $\sum_{i=1}^{N} h_i \overline{h_i}$, where $\overline{h_i}$ is the complex conjugate of h_i as given by Lemma 3.5.3. For g one simply takes the product of the h_i's. For the h_i's in $\mathcal{C}(X)$ one takes the sum of the squares of the h_i's for f and the product for g. □

As for Presburger constructible functions, the complement of $Z(h)$ in X for $h \in \mathcal{C}(X)$ is not always equal to the zero locus of some function in $\mathcal{C}(X)$, and similarly for h in $\mathcal{C}^{\exp}(X)$.

Proof of Proposition 3.2.5 for $m = 1$

The statement that we have to prove clearly allows us to work piecewise; if we have a finite partition of U into definable parts A, then it suffices to prove the proposition for f_ℓ restricted to each part A. We actually prove something slightly stronger than Proposition 3.2.5 for the case $m = 1$. That is, for a given definable function φ_0 : $U \to X \times \mathbb{Z}^{t_0}$ over X, we prove that in addition to the conclusions (a) and (b) of the proposition, we can require that also the following conditions (c) and (d) hold for each $x \in X$.

(c)   Each of the sets $U_{x,r}$ is either a singleton or equal to a maximal ball contained in U_x.

(d)   The function φ_0 factors through φ, that is, $\varphi_0 = \theta \circ \varphi$ for some definable function θ.

So, let a definable function $\varphi_0 : U \to X \times \mathbb{Z}^{t_0}$ over X be given. By the definition of $\mathcal{C}^{\exp}$, the f_ℓ's are finite sums of terms of the form $g \psi_K(h)$ for some $g \in \mathcal{C}(U)$ and some definable $h : U \to K$. Apply Theorem 3.3.2 to these functions $g \in \mathcal{C}(U)$ and to the last t_0 component functions of φ_0. By working piecewise, we may suppose that U is one of the so-obtained cells. If U is a 0-cell over X, there is nothing to prove. If U is a 1-cell over X, say, with center c, then let $\varphi_1 : U \to V_1 \subset X \times \mathbb{Z}$ be the definable surjective function sending (x, y) in U to $(x, \operatorname{ord}(y - c(x)))$. It follows that there are definable functions $h_{\ell i} : U \to K$ and functions $G_{\ell i}$ in $\mathcal{C}^{\exp}(V_1)$ such that for each ℓ one has

$$f_\ell(x, y) = \sum_{i=1}^{N_\ell} G_{\ell i}(\varphi_1(x, y)) \psi_K(h_{\ell i}(x, y)), \tag{3.5.1}$$

and that, for each x, the collection of sets $U_{x,r}^1 := \{y \in U_x \mid \varphi_1(x, y) = (x, r)\}$ equals the collection of maximal balls contained in U_x. Thus, the conditions (a), (c), and (d) already hold for φ_1. We now construct φ (and modify $G_{\ell i}$ and $h_{\ell i}$ accordingly) such that moreover (b) holds.

We will proceed by induction on $N := \sum_{\ell=1}^{s} (N_\ell - 1)$. Namely, assume that, for any finite family of functions $\{f_\ell\}$ on a definable set U (not necessarily the same

family and the same set as the given one) such that the functions f_ℓ have a presentation of the form (3.5.1) and satisfying the properties (a), (c), and (d), and with $\sum(N_\ell - 1) < N$, there exists a function φ such that the property (b) holds as well. Then we want to prove the same for any such family and presentation with $\sum(N_\ell - 1) = N$. The idea of the proof of the induction step is to increase the number of functions in the family without increasing the total number of terms in their presentations (3.5.1) and thus decrease $\sum(N_\ell - 1)$. To achieve this, one is still allowed to work piecewise, that is, to replace U with a subset that comes from using cell decomposition as above. Note that the constant d appearing in (b) will increase by at most 1 in each induction step, so that we actually obtain $d \le N$.

If $N = 0$, then all $N_\ell = 1$, and one is done, taking $\varphi = \varphi_1$ and $d = 0$. Indeed, if $N_\ell = 1$, then $|G_{\ell 1}(x, r)|_{\mathbb{C}}$ equals $|f_\ell(x, y)|_{\mathbb{C}}$, and thus, if $N = 0$, then $U_{x,r} = W_{x,r}$.

For general $N > 0$ we start by pulling the factor $\psi_K(h_{\ell 1}(x, y))$ out of (3.5.1), that is, we may assume that $h_{\ell 1}(x, y) = 0$ for all ℓ and all $(x, y) \in U$. By Proposition 3.3.5 we may moreover suppose that for each $(x, r) \in V_1$, each ℓ, and each i either $h_{\ell i}(x, \cdot)$ is constant on $U^1_{x,r}$ or $h_{\ell i}(x, \cdot)$ restricted to $U^1_{x,r}$ has the 1-Jacobian property. Hence, for each $(x, r) \in V_1$ there exist constants $b_{x,r,\ell,i} \in K$ such that, for all $y_1, y_2 \in U^1_{x,r}$ and all ℓ, i,

$$\mathrm{ord}\big(h_{\ell i}(x, y_1) - h_{\ell i}(x, y_2)\big) = \mathrm{ord}\big(b_{x,r,\ell,i} \cdot (y_1 - y_2)\big), \tag{3.5.2}$$

$$\overline{\mathrm{ac}}\big(h_{\ell i}(x, y_1) - h_{\ell i}(x, y_2)\big) = \overline{\mathrm{ac}}\big(b_{x,r,\ell,i} \cdot (y_1 - y_2)\big), \tag{3.5.3}$$

where $b_{x,r,\ell,1} = 0$ by a previous assumption, and where we write $\mathrm{ord} : K \to \mathbb{Z} \cup \{+\infty\}$. If, for all ℓ, i, x, r, the function $h_{\ell i}(x, \cdot)$ is constant modulo (ϖ_K) on $U^1_{x,r}$, then, up to a further finite partition of U, Lemma 3.3.6 applied to each of the $h_{\ell i}$'s brings us back to the case $N = 0$. We may thus in particular assume that, for each (x, r) in V_1, there exist ℓ, i with $b_{x,r,\ell,i} \ne 0$. Choose $\gamma_{x,r} \in K$ with

$$|\gamma_{x,r}| \cdot \max_{\ell,i} |b_{x,r,\ell,i}| = 1.$$

For each x, r, and ℓ, partition $\{1, \ldots, N_\ell\}$ into nonempty subsets $S_{\ell j}(x, r)$, $j \ge 1$, with the property that i_1, i_2 lie in the same part $S_{\ell j}(x, r)$ for some j if and only if

$$\mathrm{res}(\gamma_{x,r} b_{x,r,\ell,i_1}) = \mathrm{res}(\gamma_{x,r} b_{x,r,\ell,i_2}), \tag{3.5.4}$$

where $\mathrm{res} : \mathcal{O}_K \to k_K$ is the natural projection. By cutting U into finitely many pieces again, we may assume that the sets $S_{\ell j} := S_{\ell j}(x, r)$ do not depend on (x, r). Since $b_{x,r,\ell,1} = 0$, at least for one ℓ there are at least two different sets $S_{\ell,j}, S_{\ell,j'}$. Define for each ℓ, j and for $(x, y) \in U$

$$f_{\ell j}(x, y) := \sum_{i \in S_{\ell j}} G_{\ell i}\big(\varphi(x, y)\big) \psi_K\big(h_{\ell i}(x, y)\big),$$

and consider these functions $(f_{\ell j})_{\ell,j}$ as a single family. The total number of summands of the family $(f_{\ell j})_{\ell,j}$ is the same as for the functions f_ℓ, but there are more functions $f_{\ell j}$ than f_j, so we can apply induction on N to this family $(f_{\ell j})_{\ell,j}$, with the extra conditions (c) and (d) for φ_1 as part of the desired properties. Thus we find an integer $d \geq 0$, a definable surjection $\varphi : U \to V$ over X, definable functions $h_{\ell j i} : U \to K$, and functions $G_{\ell j i}$ with properties (a), (b), (c), and (d) for φ_1 and for this family.

Let us write $U_{x,r}$ for the sets defined by φ as in condition (b). Since $\varphi_1 = \theta \circ \varphi$ for some definable θ, one has $U_{x,r} \subset U^1_{x,r'}$ for each (x,r) and $(x,r') = \theta(x,r)$. By cutting U into pieces as before, we may assume that, for each x and r, not all $h_{\ell i}(x, \cdot)$'s are constant modulo (ϖ_K) on $U_{x,r}$, since, as before, this would bring us back to the case $N = 0$ for our original family $(f_\ell)_\ell$ via Lemma 3.3.6.

We will now show that the subset $M_{x,r}$ of $U_{x,r}$ consisting of those y satisfying both inequalities

$$\sup_{\ell,j,i} \big| G_{\ell j i}(x,r) \big|_{\mathbb{C}} \leq \sup_{\ell,j} \big| f_{\ell j}(x,y) \big|_{\mathbb{C}} \leq \sup_\ell \big| f_\ell(x,y) \big|_{\mathbb{C}} \qquad (3.5.5)$$

has big volume in the sense that

$$\mathrm{Vol}(U_{x,r}) \leq q_K^{d+1} \, \mathrm{Vol}(M_{x,r}). \qquad (3.5.6)$$

Once this is proved, we are done for our original family $(f_\ell)_\ell$ by replacing d with $d + 1$ while keeping the data of φ, $G_{\ell j i}$, and $h_{\ell j i}$.

Thus, to finish the proof, we fix x and r and it remains to show that $M_{x,r}$ as given by (3.5.5) has the property (3.5.6). Consider the partition of the ball $U_{x,r}$ into the balls B_ξ of the form $\xi + \gamma_{x,r} \mathcal{O}_K$. (The ball $U_{x,r}$ is indeed a union of such balls B_ξ by our choice of $\gamma_{x,r}$ since there exists a $h_{\ell i}(x, \cdot)$ that is nonconstant modulo (ϖ_K) on $U_{x,r}$.) First we will show that $|f_{\ell j}(x, \cdot)|_{\mathbb{C}}$ is constant on each such B_ξ. Second we will show that for each such B_ξ there is a subball $B'_\xi \subset B_\xi$ with $\mathrm{Vol}(B_\xi) = q_K \cdot \mathrm{Vol}(B'_\xi)$ and such that the second inequality of (3.5.5) holds for all $y \in B'_\xi$. These two facts together with the previous application of the induction hypothesis imply (3.5.6) and thus finish the proof for $m = 1$. Fix $B_\xi \subset U_{x,r}$, and write $y = \xi + \gamma_{x,r} y' \in B_\xi$ for $y' \in \mathcal{O}_K$. By (3.5.2), (3.5.3), and (3.5.4), for each ℓ and j there is a constant $c_{\ell j} \in \mathbb{C}$ such that

$$f_{\ell j}(x,y) = c_{\ell j} \psi_K(b'_{\ell j} y'),$$

where we can take $b'_{\ell j} = \gamma_{x,r} b_{x,r',\ell,i}$ for any $i \in S_{\ell j}$ where r' is such that $U_{x,r} \subset U^1_{x,r'}$. This shows that $|f_{\ell j}(x, \cdot)|_{\mathbb{C}}$ is constant on B_ξ. We now only have to construct B'_ξ. By renumbering, we can suppose that, on B_ξ, $|f_{1,1}|_{\mathbb{C}}$ is maximal among the $|f_{\ell j}|_{\mathbb{C}}$'s, so that the middle expression of (3.5.5) is equal to $|f_{1,1}|_{\mathbb{C}}$. In particular, it

suffices to choose B'_ξ such that $|f_{1,1}|_{\mathbb{C}} \le |f_1(x,y)|_{\mathbb{C}}$ for all $y \in B'_\xi$. Now let ψ be the additive character of $\mathbb{F}_{q_K}$ satisfying $\psi_K(y') = \psi(\text{res}(y'))$ for $y' \in \mathcal{O}_K$. By (3.5.4), we have $\text{res}(b'_{1j}) \ne \text{res}(b'_{1j'})$ for each $j \ne j'$, so we can apply Corollary 3.5.2 to

$$\tilde{f} : \mathbb{F}_{q_K} \to \mathbb{C} : \tilde{y} \mapsto \sum_j c_{1j} \psi\left(\text{res}(b'_{1j}) \cdot \tilde{y}\right)$$

and get a $\tilde{y}_0 \in \mathbb{F}_{q_K}$ with $|c_{1,1}|_{\mathbb{C}} \le |\tilde{f}(\tilde{y}_0)|_{\mathbb{C}}$. Set $B'_\xi := \{\xi + \gamma_{x,r} y' \mid y' \in \text{res}^{-1}(\tilde{y}_0)\}$. Since $f_1(x,y) = \tilde{f}(\text{res}(y'))$ and $|f_{1,1}|_{\mathbb{C}} = |c_{1,1}|_{\mathbb{C}}$, we are done. $\square$

Proof of Proposition 3.2.5 for $m > 1$

We proceed by induction on m. Denote $(y_1, \ldots, y_{m-1})$ by $\hat{y}$. Apply the $m = 1$ case by using $(x, \hat{y})$ as parameters and y_m as the only y-variable. This yields in particular an integer $d_1 > 0$, a surjection $\varphi_1 : U \to V_1 \subset X \times K^{m-1} \times \mathbb{Z}^{t_1}$, and an expression of each f_ℓ as a sum of terms of the form $G_1(\varphi_1(x,y)) \psi_K(h_1(x,y))$, where we omit the indices ℓ, i to simplify notation.

Now apply the induction hypothesis to the collection of functions G_1, this time by using $\hat{y}$ as the y-variables and the variables (x, r_1) as parameters, where r_1 is the variable running over $\mathbb{Z}^{t_1}$. This yields an integer d_2, a surjection $\varphi_2 : V_1 \to V_2 \subset X \times \mathbb{Z}^{t_1} \times \mathbb{Z}^{t_2}$, and an expression of each G_1 as a sum of terms of the form $G_2(\varphi_2(x, \hat{y}, r_1)) \psi_K(h_2(x, \hat{y}, r_1))$.

Now define φ as $\varphi_2 \circ \varphi_1$, and define $d = d_1 + d_2$. Then each f_ℓ is a sum of terms of the form $G_2(\varphi(x,y)) \psi_K(h_1(x,y) + h_2(\phi_1(x,y)))$, so (a) is satisfied and (b) also follows easily. $\square$

Proofs of Theorems 3.2.2 and 3.2.3

Let f be in $\mathcal{C}^{\exp}(X \times K^m)$ for some definable set X, and write f as $\sum_{i=1}^N G_i(\varphi(x, y)) \psi_K(h_i(x,y))$ as in Proposition 3.2.5 with $s = 1$, $f_1 = f$, and $U = X \times K^m$, so that in particular the $h_i : X \times K^m \to K$ and $\varphi : U \to V$ are definable functions and the $G_i(x,y)$'s lie in $\mathcal{C}^{\exp}(V)$. For each i let H_i be the function $G_i \circ \varphi$. Since K-valued functions can only depend piecewise trivially on $\mathbb{Z}$-variables, one has a natural isomorphism

$$\mathcal{C}^{\exp}(V) \cong \mathcal{C}^{\exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(V)$$

of $\mathcal{C}(X)$-algebras, and thus, the H_i's may be assumed to lie in $\mathcal{C}^{\exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$. To define these tensor products of $\mathbb{A}_q$-algebras we use the natural inclusions $\mathcal{C}(X) \subset \mathcal{C}(V)$, $\mathcal{C}(X) \subset \mathcal{C}^{\exp}(X)$, and $\mathcal{C}(X) \subset \mathcal{C}(X \times K^m)$, which are inclusions of algebras of $\mathbb{C}$-valued functions.

It is clear that, for any $x \in X$, if $x \in \text{Iva}(H_i, X)$ for all i, then $x \in \text{Iva}(f, X)$. Vice versa, if $f(x, \cdot)$ is identically zero, then $H_i(x, \cdot)$ is zero on each set $W_{x,r}$, and since it

is constant on each set $U_{x,r}$, $H_i(x,\cdot)$ is identically zero. Thus we just showed that

$$\text{Iva}(f, X) = \bigcap_i \text{Iva}(H_i, X). \tag{3.5.7}$$

A similar argument shows that

$$\text{Bdd}(f, X) = \bigcap_i \text{Bdd}(H_i, X) \quad \text{and} \quad \text{Int}(f, X) = \bigcap_i \text{Int}(H_i, X), \tag{3.5.8}$$

where in the case of $\text{Int}(f, X)$, we use the inequality between the volumes of $W_{x,r}$ and $U_{x,r}$ given by Proposition 3.2.5. By Corollary 3.4.7 with $\mathcal{H}(X) = \mathcal{C}^{\exp}(X)$, applied to each of the functions H_i, we find that each of the sets $\text{Int}(H_i, X)$, $\text{Bdd}(H_i, X)$, $\text{Iva}(H_i, X)$ is equal to a zero locus of a function in $\mathcal{C}^{\exp}(X)$. By Corollary 3.5.4 applied to the intersections from (3.5.7) and (3.5.8), the proof of Theorem 3.2.2 is finished.

Finally apply Corollary 3.4.7 with $\mathcal{H}(X) = \mathcal{C}^{\exp}(X)$ to each of the functions H_i to obtain g_i in $\mathcal{C}^{\exp}(X \times K^m)$ such that $\text{Int}(g_i, X) = X$ and $H_i(x, y) = g_i(x, y)$ whenever x lies in $\text{Int}(H_i, X)$. Now take $g = \sum_i g_i \psi_K(h_i)$ as required for Theorem 3.2.3. $\qquad\square$

For the proof of Theorem 3.2.1 we will apply [9, Theorem 8.6.1(1)], which has stringent integrability conditions. These conditions can be satisfied by the last part of the statement of Theorem 3.2.3.

Proof of Theorem 3.2.1
Let g be given by Theorem 3.2.3. Since, by the same theorem, g is a finite sum of terms of the form

$$f_0 \psi_K(f_1)$$

with $f_1 : X \times K^m \to K$ definable and $f_0 \in \mathcal{C}(X \times K^m)$ satisfying $\text{Int}(f_0, X) = X$, the function g falls under the scope of [9, Theorem 8.6.1(1)], which yields the desired conclusion. $\qquad\square$

4. Transfer principles for integrability and boundedness
In this section we use motivic functions from [8] and [9] to study loci of integrability and other loci uniformly in all local nonarchimedean fields whose residue fields have large characteristic (including $\mathbb{F}_q((t))$). We give uniform analogues of the results of Section 3. This finally leads to the main results of the paper: the transfer principles for integrability and boundedness. Again, we first prove the results without oscillation using the constructible motivic functions from [8] and subsequently in a setting with

oscillation using the motivic exponential functions from [9]. We start by recalling the necessary definitions.

4.1. Notation

Let $\mathcal{O}$ be a ring of integers of a number field. We will use the first-order Denef–Pas language with coefficients in $\mathcal{O}[[t]]$, and denote it by $\mathscr{L}_{\mathrm{DP}}$. By definable we will from now on mean $\mathscr{L}_{\mathrm{DP}}$-definable, without using other coefficients than those from $\mathcal{O}[[t]]$. Recall that $\mathscr{L}_{\mathrm{DP}}$ has three sorts: the valued field, the residue field, and the value group. The language $\mathscr{L}_{\mathrm{DP}}$ has as symbols the usual logical symbols, the language of rings $(+, -, \cdot, 0, 1)$ with coefficients from $\mathcal{O}[[t]]$ for the valued field, another copy of the language of rings for the residue field, the Presburger language $(+, -, \leq, \{\cdot \equiv \cdot \bmod n\}_{n>1}, 0, 1)$ for the value group, the symbol ord for the valuation map on the nonzero elements of the valued field, and the symbol $\overline{\mathrm{ac}}$ for an angular component map. All structures for $\mathscr{L}_{\mathrm{DP}}$ that we will consider are triples $(L, k_L, \mathbb{Z})$ with L a complete discretely valued field, $\mathcal{O}_L$ its valuation ring with residue field k_L, and the value group identified with $\mathbb{Z}$, together with the information of how the symbols of $\mathscr{L}_{\mathrm{DP}}$ are interpreted in this triple. To fix the meaning of the symbols of $\mathscr{L}_{\mathrm{DP}}$ one fixes a ring homomorphism $\lambda_{\mathcal{O},L} : \mathcal{O}[[t]] \to \mathcal{O}_L$ respecting 1 and sending t to a uniformizer ϖ of $\mathcal{O}_L$. If one fixes such $\lambda_{\mathcal{O},L}$, then all the symbols of $\mathscr{L}_{\mathrm{DP}}$ have a unique meaning where we require that $\overline{\mathrm{ac}} : L \to k_L$ is the unique multiplicative map which extends the projection $\mathcal{O}_L^\times \to k_L^\times$ and sends ϖ to 1; it is given by

$$\overline{\mathrm{ac}} : L \to k_L : \begin{cases} x\varpi^{-\operatorname{ord} x} \bmod (\varpi) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(The other symbols have their natural meanings.) Note that, by the completeness of L, a ring homomorphism $\mathcal{O} \to \mathcal{O}_L$ and the choice of the uniformizer ϖ of $\mathcal{O}_L$ determine a ring homomorphism $\mathcal{O}[[t]] \to \mathcal{O}_L$ sending t to ϖ.

Let $\mathcal{A}_{\mathcal{O}}$ be the collection of nonarchimedean local fields K of characteristic zero with a ring homomorphism $\mathcal{O} \to K$ and a uniformizer ϖ_K of $\mathcal{O}_K$. Let $\mathcal{B}_{\mathcal{O}}$ be the collection of all local fields K of positive characteristic with a ring homomorphism $\mathcal{O} \to K$ and a uniformizer ϖ_K of $\mathcal{O}_K$. Let $\mathcal{C}_{\mathcal{O}}$ be the union of $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{B}_{\mathcal{O}}$. For an integer $M > 0$, denote by $\mathcal{A}_{\mathcal{O},M}$, $\mathcal{B}_{\mathcal{O},M}$ (resp., $\mathcal{C}_{\mathcal{O},M}$) those fields in $\mathcal{A}_{\mathcal{O}}$, $\mathcal{B}_{\mathcal{O}}$ (resp., $\mathcal{C}_{\mathcal{O}}$) that have residue characteristic larger than M. For K in $\mathcal{C}_{\mathcal{O}}$, write $\mathcal{M}_K$ for the maximal ideal of $\mathcal{O}_K$, write k_K for the residue field, and write q_K for the number of elements of k_K. For $x \in \mathcal{O}_K$, denote by $\overline{x} \in k_K$ the reduction of x modulo (ϖ_K).

For $K \in \mathcal{C}_{\mathcal{O}}$, write $\mathcal{D}_K$ for the collection of additive characters $\psi : K \to \mathbb{C}^\times$ which are trivial on the maximal ideal $\mathcal{M}_K$ and which coincide on $\mathcal{O}_K$ with the character sending $x \in \mathcal{O}_K$ to

$$\exp\left(\frac{2\pi i}{p}\operatorname{Tr}_{k_K}(\bar{x})\right),$$

where Tr_{k_K} is the trace of k_K over its prime subfield and p is the characteristic of k_K. Note that there is no restriction in only considering additive characters lying in $\mathcal{D}_K$, since, in our setup, all other additive characters on K appear naturally by using a parameter over the valued field.

For any $K \in \mathcal{C}_{\mathcal{O}}$, the measure we put on $K^n \times k_K^m \times \mathbb{Z}^r$ is the product measure of the Haar measure on K^n normalized so that $\mathcal{O}_K^n$ has measure 1 with the discrete measure (the counting measure) on $k_K^m \times \mathbb{Z}^r$. Likewise, we endow $K^n \times k_K^m \times \mathbb{Z}^r$ with the product topology of the valuation topology on K^n with the discrete topology on $k_K^m \times \mathbb{Z}^r$.

4.2. The motivic setting
We recall the terminology and notation from [8] and [9].

4.2.1. Definable subassignments
For any field k of characteristic zero, we consider the Laurent series field $k((t))$ over k with the uniformizer t and the corresponding angular component map and discrete valuation.

Any $\mathcal{L}_{\mathrm{DP}}$-formula φ in m free-valued field variables, n free residue field variables, and r free-value group variables and any field k of characteristic zero which contains our fixed ring of integers $\mathcal{O}$ as a subring give rise to a subset of

$$k((t))^m \times k^n \times \mathbb{Z}^r$$

consisting of the points satisfying φ, which can be written symbolically as follows:

$$\left\{(x, y, z) \in k((t))^m \times k^n \times \mathbb{Z}^r \mid \varphi(x, y, z)\right\};$$

this subset is denoted by $\varphi_{k((t))}$.

By a definable subassignment we mean the map X which sends k to $X(k) := \varphi_{k((t))}$ for some $\mathcal{L}_{\mathrm{DP}}$-formula φ, where k runs over characteristic zero fields which contain $\mathcal{O}$ as a subring. Denote by h the definable subassignment which sends k to the singleton $\{0\}$, also written as $k((t))^0 \times k^0 \times \mathbb{Z}^0$.

For any definable subassignment X and for nonnegative integers m, n, r, write $X[m, n, r]$ for the definable subassignment sending k to

$$X(k) \times k((t))^m \times k^n \times \mathbb{Z}^r.$$

For example, $h[m, n, r]$ sends k to $k((t))^m \times k^n \times \mathbb{Z}^r$. We will also write $X \times \mathbb{Z}^r$ for $X[0, 0, r]$.

A point on a definable subassignment X consists of a pair (x, k) with k a characteristic zero field having $\mathcal{O}$ as a subring and with x an element of $X(k)$. We write $|X|$ for the collection of all points that lie on X. (We leave it to the reader to choose whether to consider $|X|$ as an actual class or to work in a fixed large universe.)

The usual set-theoretic operations make sense for definable subassignments. If $X(k) \subset Y(k)$ for each k, then we also call X a definable subassignment of Y.

By a definable morphism $f : X \to Y$ between definable subassignments X and Y we mean a definable subassignment $G \subset X \times Y$ such that $G(k)$ is the graph of a function from $X(k)$ to $Y(k)$ for each k and we call G the graph of f. We write f_k for the function from $X(k)$ to $Y(k)$ with the graph $G(k)$.

Write Def for the category of definable subassignments with definable morphisms as morphisms.

4.2.2. Definable subassignments and local fields

We have seen that behind a definable subassignment X lies an $\mathcal{L}_{\mathrm{DP}}$-formula φ which describes the sets $X(k)$. Clearly such a formula φ corresponding to X is not unique. However, if we fix such a φ for a definable subassignment X of $h[m, n, r]$, which we call fixing a representative of X, then for each $K \in \mathcal{C}_{\mathcal{O}}$, we can consider the subset φ_K of $K^m \times k_K^n \times \mathbb{Z}^r$ consisting of the points satisfying φ. Indeed, all the symbols of $\mathcal{L}_{\mathrm{DP}}$ can be interpreted in the three sorts $K, k_K, \mathbb{Z}$, where elements of $\mathcal{O}[[t]]$ are interpreted in K via the ring homomorphism $\mathcal{O}[[t]] \to K$ coming from the ring homomorphism $\mathcal{O} \to K$ and sending t to the uniformizer ϖ_K.

For all motivic objects in this paper, we will make a link with objects (usually sets and functions) on local fields. In particular, given a definable subassignment X, we will often implicitly fix a representative φ and write X_K instead of φ_K for K in $\mathcal{C}_{\mathcal{O},M}$ with sufficiently large M. Although X_K depends on the choice of φ, we have the following phenomenon:

> For any two representatives φ and φ' of a definable subassignment X, there exists $M > 0$ such that $\varphi_K = \varphi'_K$ for all K in $\mathcal{C}_{\mathcal{O},M}$.

Remark 4.2.3

The operation of taking representatives in this context is similar to the notion of taking a model over $\mathbb{Z}$ of a variety defined over $\mathbb{Q}$ in the context of algebraic geometry, as one typically does for counting the number of $\mathbb{F}_q$-rational points.

Similarly, any definable morphism $f : X \to Y$ between definable subassignments gives rise, up to fixing a formula γ corresponding to the graph of f, to a function

$$f_K : X_K \to Y_K$$

whose graph is γ_K for any K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large. (If the characteristic of the residue field of K is small, then γ_K might not define the graph of a function and one may define f_K as being the zero function in this case, by convention.)

4.2.4. *Constructible motivic functions*

Recall that $h[0,0,1]$ can be identified with $\mathbb{Z}$, since $h[0,0,1](k) = \mathbb{Z}$ for all k. Let X be in Def, that is, let X be a definable subassignment. A definable morphism $\alpha : X \to h[0,0,1]$ gives rise to a function $|X| \to \mathbb{Z}$ (also denoted by α) sending a point (x,k) on X to $\alpha_k(x)$. Likewise, such α gives rise to the function $\mathbb{L}^{\alpha}$ from $|X|$ to $\mathbb{A}$ which sends a point (x,k) on X to $\mathbb{L}^{\alpha_k(x)}$ and where $\mathbb{A}$ is as in Section 2.2.

Following [8], we define the ring $\mathcal{P}(X)$ of constructible Presburger functions on X as the subring of the ring of functions $|X| \to \mathbb{A}$ generated by

(1)   all constant functions into $\mathbb{A}$,
(2)   all functions $\alpha : |X| \to \mathbb{Z}$ with $\alpha : X \to h[0,0,1]$ a definable morphism,
(3)   all functions of the form $\mathbb{L}^{\beta}$ with $\beta : X \to h[0,0,1]$ a definable morphism.

Note that, although $|X|$ is not a set, $\mathcal{P}(X)$ can be regarded as a set since it does not have too many generators.

For Y a definable subassignment of X, write $\mathbf{1}_Y$ for the characteristic function of Y, sending a point (x,k) on X to 1 if it lies on Y and to zero otherwise.

Define the group $\mathcal{Q}(X)$ as the quotient of the free abelian group over symbols $[Y]$ with Y a definable subassignment of $X[0,m,0]$ for some $m \geq 0$, by the following scissor relations:

$$[Y] = [Y'] \tag{4.2.1}$$

if there exists a definable isomorphism $Y \to Y'$ which commutes with the projections $Y \to X$ and $Y' \to X$;

$$[Y_1 \cup Y_2] + [Y_1 \cap Y_2] = [Y_1] + [Y_2] \tag{4.2.2}$$

for Y_1 and Y_2 definable subassignments of a common $X[0,m,0]$ for some m.

We will still write $[Y]$ for the class of $[Y]$ in $\mathcal{Q}(X)$ for $Y \subset X[0,m,0]$. Note that, in [8] and [9], the notation $K_0(\mathrm{RDef}_X)$ is used instead of $\mathcal{Q}(X)$. Denote by $\mathcal{P}^0(X)$ the subring of $\mathcal{P}(X)$ generated by the characteristic functions $\mathbf{1}_Y$ for all definable subassignments Y of X and by the constant function $\mathbb{L}$. Using the canonical ring morphism $\mathcal{P}^0(X) \to \mathcal{Q}(X)$ and sending $\mathbf{1}_Y$ to $[Y]$ and $\mathbb{L}$ to the class of $X[0,1,0]$, we define the ring $\mathcal{C}(X)$ as

$$\mathcal{P}(X) \otimes_{\mathcal{P}^0(X)} \mathcal{Q}(X).$$

Elements of $\mathcal{C}(X)$ are called constructible motivic functions on X.

Let F be a function in $\mathcal{C}(X[m,0,0])$ for some $m \geq 0$. Using notation from [8, Section 13.2], we say that F is motivically X-integrable if and only if its class in $C^m(X[m,0,0] \to X)$ lies in $I_X C(X[m,0,0] \to X)$, where $X[m,0,0] \to X$ is the projection. We do not need the notion of motivic integrability for the transfer principles, and we refer to [8] for a more detailed definition. Let us just give an intuitive explanation of motivic X-integrability. The condition for F to be motivically X-integrable is a strong uniform form of the condition that, for all K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, $F_K(x,\cdot)$ is integrable over K^m with respect to the Haar measure for each $x \in X_K$. This motivic condition is defined, via cell decomposition techniques, in terms of the X-integrability of functions G in $\mathcal{P}(X \times \mathbb{Z}^m)$, and reduces to summability over $\mathbb{Z}^m$, as follows. A function G in $\mathcal{P}(X \times \mathbb{Z}^m)$ is considered X-integrable if and only if, for each $(x,k) \in |X|$, the family

$$\big(G_k(x,z)(q)\big)_{z \in \mathbb{Z}^m}$$

is summable (in the usual sense) for each real $q > 1$, where we use evaluation at $\mathbb{L} = q$ as in (2.2.1). In fact, the existence of g as in (c) of Theorem 4.3.3 gives a precise relation between integrability over local fields with large residue field characteristic and motivic integrability.

4.2.5. Constructible motivic functions and local fields

Each f in $\mathcal{P}(X)$, with X a definable subassignment, can be written as a finite sum of terms of the form $a\mathbb{L}^\beta \prod_{i=1}^\ell \alpha_i$ with $a \in \mathbb{A}$ and the α_i's and β's definable morphisms from X to $h[0,0,1] = \mathbb{Z}$. Let us take representatives α_i' and β' of the α_i's and β's, that is, the $\mathcal{L}_{\mathrm{DP}}$-formulas describing the graphs. We have seen in Section 4.2.2 that, for K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, α_{iK}' and β_K' are the graphs of functions from X_K to $\mathbb{Z}$ and we have denoted these functions by α_{iK} and β_K. Now we extend this notation to elements f of $\mathcal{P}(X)$, where we write f_K for the function sending $x \in X_K$ to

$$\sum_j a_j(q_K) q_K^{\beta_{jK}(x)} \prod_{i=1}^{\ell_j} \alpha_{ijK}(x)$$

whenever f equals

$$\sum_j a_j \mathbb{L}^{\beta_j} \prod_{i=1}^{\ell_j} \alpha_{ij},$$

where $a_j \in \mathbb{A}$, $a_j(q_K)$ is the evaluation of a_j at $\mathbb{L} = q_K$ as in (2.2.1), and the α_{ij}'s and β_j's are definable morphisms from X to $\mathbb{Z}$. In a similar sense to that of Sec-

tion 4.2.2, the function $f_K : X_K \to \mathbb{Q}$ is independent of the choice of the representatives for the α_{ij}'s and β_i's whenever K is in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large.

Likewise, since each g in $\mathcal{Q}(X)$ can be written as $[Y] - [Z]$ for some definable subassignments $Y \subset X[0,n,0]$ and $Z \subset X[0,n',0]$, by taking representatives, one can consider Y_K and Z_K for K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, and we denote by g_K the function on X_K sending $x \in X_K$ to

$$\#Y_x - \#Z_x,$$

where Y_x is the (finite) set $\{r \in k_K^n \mid (x,r) \in Y_K\}$ of size $\#Y_x$ and likewise for Z_x.

Since for $f \in \mathcal{P}^0(X)$ and f' its image in $\mathcal{Q}(X)$ one has $f_K = f'_K$ for all K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, one can define, for F in $\mathcal{C}(X)$ and for K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, the function F_K as

$$F_K : X_K \to \mathbb{Q} : x \mapsto \sum_i a_{iK}(x) b_{iK}(x)$$

whenever $F = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{P}(X)$ and $b_i \in \mathcal{Q}(X)$. In our usual sense, this is independent of the choice of representatives when K is in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large.

4.2.6. Motivic exponential functions

Let X be in Def. We consider the category $Q_X^{\exp}$ whose objects are the triples (Y, ξ, g) with Y a definable subassignment of $X[0,n,0]$ for some $n \geq 0$ and $\xi : Y \to h[0,1,0]$ and $g : Y \to h[1,0,0]$ definable morphisms. A morphism $(Y', \xi', g') \to (Y, \xi, g)$ in $Q_X^{\exp}$ is a definable morphism $h : Y' \to Y$ which makes a commutative diagram with the projections to X and such that $\xi' = \xi \circ h$ and $g' = g \circ h$.

To the category $Q_X^{\exp}$ one assigns a ring $\mathcal{Q}^{\exp}(X)$ defined as follows. As an abelian group it is the quotient of the free abelian group over the symbols $[Y, \xi, g]$ with (Y, ξ, g) in $Q_X^{\exp}$ by the following four relations:

$$[Y, \xi, g] = [Y', \xi', g'] \tag{4.2.3}$$

for (Y, ξ, g) isomorphic to (Y', ξ', g'),

$$[Y \cup Y', \xi, g] + [Y \cap Y', \xi_{|Y \cap Y'}, g_{|Y \cap Y'}]$$
$$= [Y, \xi_{|Y}, g_{|Y}] + [Y', \xi_{|Y'}, g_{|Y'}] \tag{4.2.4}$$

for Y and Y' definable subassignments of some common $X[0,n,0]$ for some $n \geq 0$ and ξ, g defined on $Y \cup Y'$,

$$[Y, \xi, g + g'] = [Y, \xi + \bar{g}, g'] \tag{4.2.5}$$

for $g : Y \to h[1,0,0]$ a definable morphism with $\mathrm{ord}(g(y)) \geq 0$ for all y in Y and $\bar{g}$ the reduction of g modulo the maximal ideal, and

$$[Y[0,1,0], \xi + p, g] = 0 \tag{4.2.6}$$

when $p : Y[0,1,0] \to h[0,1,0]$ is the projection and when g and ξ factorize through the projection $Y[0,1,0] \to Y$.

LEMMA 4.2.7 ([9, Lemma 3.1.1])
We may endow $\mathcal{Q}^{\exp}(X)$ with a ring structure by setting

$$[Y, \xi, g] \cdot [Y', \xi', g'] = [Y \otimes_X Y', \xi \circ p_Y + \xi' \circ p_{Y'}, g \circ p_Y + g' \circ p_{Y'}],$$

where $Y \otimes_X Y'$ is the fiber product of Y and Y', p_Y is the projection to Y, and $p_{Y'}$ is the projection to Y'.

By [9, Lemma 3.1.3] there is a natural injection of rings $\mathcal{Q}(X) \to \mathcal{Q}^{\exp}(X)$ sending $[Y]$ to $[Y,0,0]$. Hence, we may define the ring $\mathcal{C}^{\exp}(X)$ of motivic exponential functions by

$$\mathcal{C}^{\exp}(X) := \mathcal{C}(X) \otimes_{\mathcal{Q}(X)} \mathcal{Q}^{\exp}(X). \tag{4.2.7}$$

Remark 4.2.8
Note that, in [9], $\mathcal{Q}(X)$ is denoted by $K_0(\mathrm{RDef}_X)$, $\mathcal{Q}^{\exp}(X)$ is denoted by $K_0(\mathrm{RDef}_X^{\exp})$, and $\mathcal{C}^{\exp}(X)$ is denoted by $\mathcal{C}(X)^{\exp}$.

Let F be a motivic exponential function in $\mathcal{C}^{\exp}(X[m,0,0])$ for some $m \geq 0$. Using notation from [9], we say that F is motivically X-integrable if and only if its class in $C^m(X[m,0,0] \to X)^{\exp}$ lies in $I_X C(X[m,0,0] \to X)^{\exp}$, where $X[m,0,0] \to X$ is the projection. Again the notion of X-integrability essentially boils down to the condition of summability for countable families, similarly to as explained at the end of Section 4.2.4. Theorem 4.3.3(c) gives a precise and new relation between integrability over local fields with large residue field characteristic and motivic integrability.

4.2.9. Motivic exponential functions and local fields

In this section we explain, following [9], how to find actual functions $f_{K,\psi} : X_K \to \mathbb{C}$ for $f \in \mathcal{C}^{\exp}(X)$, K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, $\psi \in \mathcal{D}_K$, and X a definable subassignment. For f in $\mathcal{C}(X)$ this is already explained above in Section 4.2.5. Take $G = [Y, \xi, g]$ in $\mathcal{Q}^{\exp}(X)$, with $Y \subset X[0,n,0]$, take representatives of Y, ξ, and g, and let K be in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, so that ξ_K and g_K are functions from Y_K to k_K (resp., to K). Then we define $G_{K,\psi}$ as the function sending $x \in X_K$ to the exponential sum

$$\sum_{r \in Y_x} \psi\big(\xi_K(x,r) + g_K(x,r)\big),$$

which is well defined since ψ is trivial on $\mathcal{M}_K$ and since $\xi_K(x,r)$ can be considered as an element of $\mathcal{O}_K$ mod $\mathcal{M}_K$. Finally, for $f \in \mathcal{C}^{\exp}(X)$, K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large, and $\psi \in \mathcal{D}_K$, we define $f_{K,\psi}$ by

$$f_{K,\psi} : X_K \to \mathbb{C} : x \mapsto \sum_i a_{iK}(x) b_{iK,\psi}(x)$$

whenever $f = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{C}(X)$ and $b_i \in \mathcal{Q}^{\exp}(X)$.

We recapitulate how $f_{K,\psi}$ is independent of the choice of representatives for K in $\mathcal{C}_{\mathcal{O},M}$ with M sufficiently large. For any two different collections C_1 and C_2 of representatives of the $\mathcal{L}_{DP}$-formulas that go into the description of f, there exists M' such that, for all K in $\mathcal{C}_{\mathcal{O},M'}$ and all $\psi \in \mathcal{D}_K$, one has that $f_{K,\psi}$ is independent of the choice between C_1 and C_2.

4.3. The constructible setting

We find motivic analogues of our thematic results, namely, the analogues of the p-adic Theorems 3.1.1, 3.1.3, and 3.1.5.

THEOREM 4.3.1 (Integration)
Let f be in $\mathcal{C}(X[m,0,0])$ for some $m \geq 0$ and some definable subassignment X. Then there exists g in $\mathcal{C}(X)$ such that, for all K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic,

$$g_K(x) = \int_{y \in K^m} f_K(x,y)$$

whenever $x \in \mathrm{Int}(f_K, X_K)$.

The special case of the above theorem when f is motivically X-integrable follows from [8] and [9].

THEOREM 4.3.2 (Correspondences of loci)
Let f be in $\mathcal{C}(X[m,0,0])$ for some definable subassignment X and some $m \geq 0$. Then there exists $h_1, h_2, h_3 \in \mathcal{C}(X)$ such that, for all K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic, the zero locus of h_{iK} in X_K equals $\mathrm{Int}(f_K, X_K)$, $\mathrm{Bdd}(f_K, X_K)$, or $\mathrm{Iva}(f_K, X_K)$ when i is 1, 2, or 3, respectively.

THEOREM 4.3.3 (Interpolation)
Let f be in $\mathcal{C}(X[m,0,0])$ for some $m \geq 0$ and some definable subassignment X.

Then there exists g in $\mathcal{C}(X[m,0,0])$ such that the following hold for K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic:

(a) $f_K(x,y) = g_K(x,y)$ *whenever* $x \in \mathrm{Int}(f_K, X_K)$ *and for all* $y \in K^m$,

(b) $\mathrm{Int}(g_K, X_K) = X_K$,

(c)   *g is motivically X-integrable.*

4.4. The exponential setting and transfer principles

The following two theorems constitute the new general transfer principles of this paper for integrability, local integrability, boundedness, and local boundedness.

THEOREM 4.4.1 (Transfer principle for integrability)
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some $m \geq 0$ and some definable subassignment X. Then, for all $K \in \mathcal{C}_{\mathcal{O},M}$ for some large M, the truth of each of the following statements depends only on (the isomorphism class of) the residue field of K:

(a)   *For all $x \in X_K$ and for all $\psi \in \mathcal{D}_K$, the function $f_{K,\psi}(x,\cdot)$ is integrable over K^m; that is, $\mathrm{Int}(X_K, f_{K,\psi}) = X_K$ for all $\psi \in \mathcal{D}_K$.*

(b)   *For all $x \in X_K$ and for all $\psi \in \mathcal{D}_K$, the function $f_{K,\psi}(x,\cdot)$ is locally integrable on K^m.*

THEOREM 4.4.2 (Transfer principle for boundedness)
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some $m \geq 0$ and some definable subassignment X. Then, for all $K \in \mathcal{C}_{\mathcal{O},M}$ for some large M, the truth of each of the following statements depends only on (the isomorphism class of) the residue field of K:

(a)   *For all $x \in X_K$ and for all $\psi \in \mathcal{D}_K$, the function $f_{K,\psi}(x,\cdot)$ is bounded on K^m.*

(b)   *For all $x \in X_K$ and for all $\psi \in \mathcal{D}_K$, the function $f_{K,\psi}(x,\cdot)$ is locally bounded on K^m.*

The transfer principles will follow from motivic analogues of our thematic results, which we now state in our final, exponential setting.

THEOREM 4.4.3 (Integration)
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some $m \geq 0$ and some definable subassignment X. Then there exists g in $\mathcal{C}^{\exp}(X)$ such that the following holds for all K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic and for all $\psi \in \mathcal{D}_K$:

$$g_{K,\psi}(x) = \int_{y \in K^m} f_{K,\psi}(x,y),$$

whenever $x \in \mathrm{Int}(f_{K,\psi}, X_K)$.

THEOREM 4.4.4 (Correspondences of loci)
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some definable subassignment X and some $m \geq 0$. Then there exist $h_1, h_2, h_3 \in \mathcal{C}^{\exp}(X)$ such that, for all K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic and for each $\psi \in \mathcal{D}_K$, the zero locus of $h_{iK,\psi}$ in X_K equals $\mathrm{Int}(X_K, f_{K,\psi})$, $\mathrm{Bdd}(X_K, f_{K,\psi})$, or $\mathrm{Iva}(X_K, f_{K,\psi})$ for $i = 1, 2,$ or 3, respectively.

Theorem 4.4.4 implies the following corollary by the same reasoning as that for Corollary 3.2.4. The analogue of Corollary 3.1.4 (i.e., the statement without the exponentials) in the motivic context holds similarly but is left to the reader.

COROLLARY 4.4.5
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some definable subassignment X and some $m \geq 0$. Then there exist functions h_1 and h_2 in $\mathcal{C}^{\exp}(X)$ such that, for all K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic and for each $\psi \in \mathcal{D}_K$, the zero locus of $h_{1K,\psi}$ in X_K equals

$$\{x \in X_K \mid f_{K,\psi}(x, \cdot) \text{ is locally integrable on } K^m\}$$

and the zero locus of $h_{2K,\psi}$ in X_K equals

$$\{x \in X_K \mid f_{K,\psi}(x, \cdot) \text{ is locally bounded on } K^m\}.$$

THEOREM 4.4.6 (Interpolation)
Let f be in $\mathcal{C}^{\exp}(X[m,0,0])$ for some definable subassignment X and some $m \geq 0$. Then there exist g in $\mathcal{C}^{\exp}(X[m,0,0])$ and $M > 0$ such that for all K in $\mathcal{C}_{\mathcal{O},M}$ and all $\psi \in \mathcal{D}_K$ one has

(a) $f_{K,\psi}(x, y) = g_{K,\psi}(x, y)$ *whenever x lies in $\mathrm{Int}(X_K, f_{K,\psi})$,*
(b) $\mathrm{Int}(X_K, g_{K,\psi}) = X_K$,
(c)   *g is motivically X-integrable.*

Remark 4.4.7
By standard techniques of motivic integration, all the above results (Theorems 4.4.1–4.4.6 and similar results in the previous sections) imply the corresponding results where $X[m,0,0]$ is replaced by an arbitrary subassignment $U \subset X[m,0,0]$.

4.5. Proofs of the motivic results
We begin with some preliminaries.

Definition 4.5.1
Consider a definable subassignment X. A residual parameterization of X is by defi-

nition a definable isomorphism over X of the form

$$\sigma : X \to X_{\mathrm{par}} \subset X[0, m, 0]$$

for some $m \geq 0$. For $F : X \to Y$ a definable morphism, write F_{par} for the corresponding definable morphism $F \circ \sigma^{-1} : X_{\mathrm{par}} \to Y$. Likewise, given $f \in \mathcal{C}^{\exp}(X)$, write f_{par} for the natural corresponding function in $\mathcal{C}^{\exp}(X_{\mathrm{par}})$, and so on.

Note that parameterizing is a way of working piecewise in a uniform way (creating, for each K, at most $\#k_K^s$ pieces for a parameterization σ which introduces s new residue field variables, the pieces being the fibers of the coordinate projection to k_K^s). In [9], for $\sigma : X \to X_{\mathrm{par}}$ and $f \in \mathcal{C}^{\exp}(X)$, one denotes f_{par} by $(\sigma^{-1})^*(f)$, which is the compositional pullback of f along σ^{-1} and which becomes the actual composition $f_K \circ \sigma_K^{-1}$ when specializing for K in $\mathcal{C}_{\mathcal{O}, M}$. Up to residual parameterization (i.e., up to replacing X by X_{par} for a well-chosen residual parameterization), all results of Section 3.3 go through in a uniform way, as we now explain.

Definition 4.5.2 (Uniform cells)
Consider $A \subset \Lambda[1, 0, 0]$ for some definable subassignment Λ. Then A is called a uniform 1-cell (resp., a uniform 0-cell) over Λ if there exists $M > 0$ such that for all K in $\mathcal{A}_{\mathcal{O}, M}$ one has that A_K is a p-adic 1-cell (resp., a p-adic 0-cell) over Λ_K.

The next theorem follows from Denef–Pas cell decomposition [22] and the results of [7, Section 6].

THEOREM 4.5.3 (Uniform version of the cell decomposition Theorem 3.3.2 and the Jacobian property Proposition 3.3.5)
Consider $X \subset \Lambda[1, 0, 0]$ with Λ and X definable subassignments, let $f_j : X \to h[0, 0, 1]$ and $F : X \to h[1, 0, 0]$ be definable morphisms, and let g_j be in $\mathcal{C}(X)$. Then there exists a residual parameterization $\sigma : X \to X_{\mathrm{par}} \subset X[0, m, 0]$ for some $m \geq 0$ such that the following holds. There exist $M > 0$ and a finite partition of X_{par} into definable subassignments A such that, for all K in $\mathcal{A}_{\mathcal{O}, M}$, one has that the sets A_K and the restrictions of the functions $f_{j, \mathrm{par}, K}$ and $g_{j, \mathrm{par}, K}$ (resp., $F_{\mathrm{par}, K}$) to A_K are as in Theorem 3.3.2 (resp., Proposition 3.3.5), with $Y = \Lambda[0, m, 0]_K$.

From now on we will work and prove results for all K in $\mathcal{C}_{\mathcal{O}, M}$ for some M, instead of only in $\mathcal{A}_{\mathcal{O}, M}$, which will be allowed by the uniform nature of the above results and by the classical Ax–Kochen principle [1, Theorem 6] for first-order statements in the language $\mathcal{L}_{\mathrm{DP}}$. This variant of the classical Ax–Kochen principle follows directly from the quantifier elimination result of [22]; we will use this variant, for

first-order statements in the language $\mathcal{L}_{DP}$, as an ingredient in our proofs. For a recent geometric treatment of classical Ax–Kochen principles, see [11].

We give uniform variants of some Presburger results.

PROPOSITION 4.5.4 (Uniform rectilinearization)
Let Y and $X \subset Y \times \mathbb{Z}^m$ be definable subassignments. Then there exist finitely many definable subassignments $A_i \subset Y \times \mathbb{Z}^m$ and $B_i \subset Y \times \mathbb{Z}^m$ and definable isomorphisms $\rho_i : A_i \to B_i$ over Y such that the following holds for large enough M and each K in $\mathcal{C}_{\mathcal{O},M}$: The sets $A_{i,K}$ are disjoint and their union equals X_K; for every i, the function $\rho_{i,K}$ is linear over Y_K; and for each $y \in Y_K$, the set $B_{i,K,y}$ is a set of the form $\Lambda_y \times \mathbb{N}^{\ell_i}$ for a finite subset $\Lambda_y \subset \mathbb{N}^{m-\ell_i}$ depending on y, with an integer $\ell_i \geq 0$ only depending on i.

Proof
The proof goes exactly the same way as the one for Proposition 3.4.4, using the quantifier elimination result for valued field variables of [22] instead of the one for fixed K. (As an alternative proof one may again adapt the proof of [2, Theorem 3].) □

The following uniform variants of Corollaries 3.4.5 and 3.4.6 will be used to prove Theorems 4.3.2 and 4.3.3.

COROLLARY 4.5.5
Let f be in $\mathcal{C}(X \times \mathbb{Z}^m)$ for some definable subassignment X and some $m \geq 0$. Then there exist h_1, h_2, and h_3 in $\mathcal{C}(X)$ such that for large enough M and each K in $\mathcal{C}_{\mathcal{O},M}$

$$\text{Int}(f_K, X_K) = Z(h_{1,K}), \tag{4.5.1}$$

$$\text{Bdd}(f_K, X_K) = Z(h_{2,K}), \tag{4.5.2}$$

and

$$\text{Iva}(f_K, X_K) = Z(h_{3,K}). \tag{4.5.3}$$

COROLLARY 4.5.6
Let f be in $\mathcal{C}(X \times \mathbb{Z}^m)$ for some definable subassignment X and some $m \geq 0$. Then there exists g in $\mathcal{C}(X \times \mathbb{Z}^m)$ with $\text{Int}(g, X) = X$ and such that for large enough M and each K in $\mathcal{C}_{\mathcal{O},M}$ one has $f_K(x, y) = g_K(x, y)$ whenever x lies in $\text{Int}(f_K, X_K)$. Moreover, one can take g motivically X-integrable.

Proofs of Corollaries 4.5.5 and 4.5.6
The same proofs as those of Corollaries 3.4.5 and 3.4.6 apply, where one uses Theorem 4.5.4 instead of Theorem 3.4.4. We give some extra details about how one can ensure that g can be taken motivically X-integrable. If $X \subset h[0, n, r]$ for some n and some r, then the proofs of Corollaries 3.4.5 and 3.4.6 directly adapt by the definition of $\mathcal{C}(h[0, n, r])$ and the orthogonality between the value group and residue field in $\mathcal{L}_{\text{DP}}$. (This orthogonality is the property that any definable subassignment of $h[0, n, r]]$ equals a finite union of Cartesian products of definable subsets of $\mathbb{Z}^r$ and of $h[0, n, 0]$, which follows from quantifier elimination of valued field variables in the language $\mathcal{L}_{\text{DP}}$ (see [22]).) The general case follows from the observation, which also follows from quantifier elimination of valued field variables in $\mathcal{L}_{\text{DP}}$, that there exists a definable morphism

$$\Delta : X[0, 0, m] \to h[0, n, r + m]$$

over $\mathbb{Z}^m$ and a motivic constructible function H in $\mathcal{C}(h[0, n, r + m])$ such that the compositional pullback $\Delta^*(H)$ (with notation from [8]) equals f. The just obtained special case that $X \subset h[0, n, r]$, applied to H, yields a motivically $h[0, n, r]$-integrable function g_H in $\mathcal{C}(h[0, n, r + m])$. This allows us to put $g = \Delta^*(g_H)$ as required by Corollary 4.5.6. □

The uniform analogue of Lemma 3.3.6 goes as follows.

LEMMA 4.5.7
Let $A^0 \subset Y^0[1, 0, 0]$ be a definable subassignment, and let $h^0 : A^0 \to h[1, 0, 0]$ be a definable morphism for some definable subassignment Y^0. Let $M_0 \geq 0$ be given. Suppose that, for each $K \in \mathcal{C}_{\mathcal{O}, M_0}$, for each $y \in Y_K^0$, and for each maximal ball B contained in $A_{K,y}^0$, $h_K^0(y, \cdot)$ is constant modulo (ϖ_K) on B. Then there exists a residual parameterization $\sigma : A^0 \to A_{\text{par}}^0 \subset A^0[0, s, 0]$ for some $s \geq 0$ such that, if we write A for A_{par}^0, Y for $Y^0[0, s, 0]$, and h for $h^0 \circ \sigma^{-1}$, the following holds. There exist $M \geq M_0$, a finite partition of A into definable subassignments A_j, and definable morphisms $h_j : Y \to h[1, 0, 0]$ such that for each K in $\mathcal{C}_{\mathcal{O}, M}$

$$\left| h_K(y, t) - h_{j,K}(y) \right| \leq 1$$

holds for each $(y, t) \in A_{j,K}$ and for each j.

Proof
This follows from [9, Theorem 2.2.2] or, alternatively, from the same proof as that of Lemma 3.3.6, where automatically all the m_j's are equal to 1 at the end of that proof. □

Up to using a residual parameterization, the uniform version of Proposition 3.2.5 also holds, as follows.

PROPOSITION 4.5.8

Let X^0 and $U^0 \subset X^0[m,0,0]$ be definable subassignments, and let $f_1^0, \ldots, f_s^0$ be in $\mathcal{C}^{\exp}(U^0)$. There exists a residual parameterization $\sigma : U^0 \to U^0_{par} \subset U^0[0,s,0]$ for some $s \geq 0$ such that, if we write X for $X^0[0,s,0]$, U for U^0_{par}, and f_ℓ for $f_\ell^0 \circ \sigma^{-1}$, then the following holds. There exist an integer $d \geq 0$, definable morphisms $h_{\ell i} : U \to h[1,0,0]$, a definable surjection $\varphi : U \to V \subset X[0,0,t]$ for some $t \geq 0$, and functions $G_{\ell i}$ in $\mathcal{C}^{\exp}(V)$ such that, for each K in $\mathcal{C}_\mathcal{O}$ with large enough residue field characteristic and for each ψ in $\mathcal{D}_K$, Propositions 3.2.5(a) and 3.2.5(b) hold for d, $f_{\ell K, \psi}$, U_K, V_K, φ_K, $h_{\ell i K}$, and $G_{\ell i K, \psi}$.

Proof

The proof of Proposition 3.2.5 works uniformly in K in $\mathcal{A}_{\mathcal{O},M}$ for large enough M, where one uses Lemma 4.5.7 and relation (4.2.5) instead of Lemma 3.3.6, and Theorem 4.5.3 instead of both Theorem 3.3.2 and Proposition 3.3.5. The only difference in the uniform proof is that we have to use residual parameterizations. Each time we need a parameterization, for the remainder of the proof, we replace all objects (sets and functions) by the corresponding parameterized objects, similarly to as in the statement of the proposition. This happens when we apply Theorem 4.5.3, Theorem 4.5.7, Lemma 4.5.7, or the induction hypothesis. □

LEMMA 4.5.9

Let X be a definable subassignment, and let f be in $\mathcal{C}^{\exp}(X)$. Then there exists a function g in $\mathcal{C}^{\exp}(X)$ such that, for each K in $\mathcal{C}_{\mathcal{O},M}$, each ψ in $\mathcal{D}_K$, and each $x \in X_K$, $f_{K,\psi}(x)$ and $g_{K,\psi}(x)$ are conjugate complex numbers.

Proof

For each K in $\mathcal{C}_{\mathcal{O},M}$ for large enough M the function g can be obtained from f by putting a minus sign in each of the arguments of the additive characters which occur in f. More precisely, write f as a finite sum of terms of the form $g_i \otimes [Y_i, \xi_i, h_i]$ for g_i in $\mathcal{C}(X)$, Y_i a definable subassignment of $X[0,n_i,0]$ for some n_i, and definable functions ξ_i, h_i, on Y_i, and define g as the corresponding sum with terms $g_i \otimes [Y_i, -\xi_i, -h_i]$. Since the g_{iK}'s take real values, g is as desired. □

The analogue of Corollary 3.5.4 also holds.

COROLLARY 4.5.10

Let X be a definable subassignment, and let the h_i's be in $\mathcal{C}^{\exp}(X)$ for $i = 1, \ldots, N$. Then there exist f and g in $\mathcal{C}^{\exp}(X)$ and $M > 0$ such that, for each K in $\mathcal{C}_{\mathcal{O},M}$ and each ψ in $\mathcal{D}_K$,

$$Z(f_{K,\psi}) = \bigcap_{i=1}^{N} Z(h_{i,K,\psi}) \quad \text{and} \quad Z(g_{K,\psi}) = \bigcup_{i=1}^{N} Z(h_{i,K,\psi}).$$

The corresponding statement for the h_i's in $\mathcal{C}(X)$ also holds, yielding $f, g \in \mathcal{C}(X)$.

Proof

For f one can take $\sum_{i=1}^{N} h_i \overline{h_i}$, where $\overline{h_i}$ is the complex conjugate of h_i as given by Lemma 3.5.3. For g one simply takes the product of the h_i's. $\qquad\square$

The final ingredients are the following basic forms of the transfer principle and the specialization principle of [9]. The transfer principle enables one to change the characteristic of K for equalities between the specializations of exponential motivic functions. The specialization principle of [9] states that taking motivic integrals commutes with taking specializations to local fields and taking the classical integral over the local field.

PROPOSITION 4.5.11 ([9, Proposition 9.2.1])

Let φ be in $\mathcal{C}^{\exp}(X)$ for some definable subassignment X. Then, there exists an integer M such that, for all K_1, K_2 in $\mathcal{C}_{\mathcal{O},M}$ with isomorphic residue fields k_{K_1}, k_{K_2}, the following holds:

$$\varphi_{K_1,\psi_{K_1}} = 0 \quad \text{for all } \psi_{K_1} \in \mathcal{D}_{K_1}$$

if and only if

$$\varphi_{K_2,\psi_{K_2}} = 0 \quad \text{for all } \psi_{K_2} \in \mathcal{D}_{K_2}.$$

PROPOSITION 4.5.12 ([9, Theorem 9.1.4])

Let φ be in $\mathcal{C}^{\exp}(X \subset \Lambda[m, 0, 0])$ for some definable subassignments X, Λ and some $m \geq 0$. Suppose that φ is motivically Λ-integrable; namely, with notation from [9], the class $\overline{\varphi}$ of φ in $C^m(X \to \Lambda)^{\exp}$ lies in $\mathrm{IC}(X \to \Lambda)^{\exp}$. Let $\theta \in \mathcal{C}^{\exp}(\Lambda)$ be the motivic integral of $\overline{\varphi}$ relative to the projection $X \to \Lambda$; namely, $\theta = \mu_\Lambda(\overline{\varphi})$ with notation from [9, (8.7.10)]. Then, there exists an integer M such that for all K in $\mathcal{C}_{\mathcal{O},M}$ and all ψ in $\mathcal{D}_K$ the following holds for each $\lambda \in \Lambda_K$. The function $w \mapsto \varphi_{K,\psi}(\lambda, w)$ is integrable over $X_{K,\lambda}$ against the Haar measure on K^m and

$$\int_{w \in X_{K,\lambda}} \varphi_{K,\psi}(\lambda, w)|dw| = \theta_{K,\psi}(\lambda),$$

with $|dw|$ the normalized Haar measure on K^m.

With all this at hand, we are ready to complete the proofs of our main results.

Proof of Theorem 4.3.2
Let X and $f \in \mathcal{C}(X[m, 0, 0])$ be given as in the theorem. The proof goes in two steps. First we show that there is a suitable residual parameterization σ of $X[m, 0, 0]$ such that the theorem holds for the function f_{par} in $\mathcal{C}(X[m, 0, 0]_{\mathrm{par}})$, where, as in Definition 4.5.1, $X[m, 0, 0]_{\mathrm{par}}$ and f_{par} are obtained from $X[m, 0, 0]$ and f using σ. In the second step, we show that the result for $X[m, 0, 0]_{\mathrm{par}}$ and f_{par} implies the result for $X[m, 0, 0]$ and f itself.
Let us now treat the first step. We follow the steps of the proof of Theorem 3.1.3. By an inductive application of the cell decomposition Theorem 4.5.3, we find a residual parameterization

$$\sigma : X[m, 0, 0] \to X[m, 0, 0]_{\mathrm{par}} \subset X[m, s, 0]$$

of $X[m, 0, 0]$ for some s, a finite partition of $X[m, 0, 0]_{\mathrm{par}}$ into definable subassignments A_i, and an $M > 0$ such that for each $K \in \mathcal{C}_{\mathcal{O}, M}$ the following holds, where we write X' for $X[0, s, 0]$. The nonempty sets among the $A_{i,K}$'s are full cells over X'_K, which together form a finite partition of $(X[m, 0, 0]_{\mathrm{par}})_K$, and for nonempty $A_{i,K}$'s the restriction $(f_{\mathrm{par},K})|_{A_{i,K}}$ factors through the skeleton map $s_{A_{i,K}/X'_K}$ of $A_{i,K}$ over X'_K. As in the proof of Theorem 3.1.3, we identify each skeleton with a definable set, for example, by replacing $\{+\infty\}$ by a disjoint copy of the singleton $\{0\}$. Let us write $f_{i,K}$ for the map from the skeleton $S_{/X'_K}(A_{i,K})$ of $A_{i,K}$ over X'_K to $\mathbb{A}_{q_K}$ induced by $f_{\mathrm{par},K|A_{i,K}}$. Then the function $f_{i,K}$ lies in $\mathcal{P}_{q_K}(S_{/X'_K}(A_{i,K}))$ for each i. The function $M_{0,K}$ sending $z \in S_{/X'_K}(A_{i,K})$ to the volume of the fiber $(s_{A_{i,K}/X'_K})^{-1}(z)$, taken inside K^m, lies in $\mathcal{C}(S_{/X'_K}(A_{i,K}))$ and clearly is the specialization of a motivic constructible function which is the product of a characteristic function of a definable subassignment and a motivic constructible function of the form $\mathbb{L}^\alpha$ for some definable morphism α (see Remark 3.4.3). Hence, $z \mapsto f_{i,K}(z) \cdot M_{0,K}(z)$ also lies in $\mathcal{C}(S_{/X'_K}(A_{i,K}))$ and is the specialization of a motivic constructible function $\tilde{f}_i$. For each i, let X'_i be the image of the subassignment A_i under the projection to the subassignment X'. By Corollary 4.5.5, we now obtain for each i that there are h_{i1}, h_{i2}, and h_{i3} in $\mathcal{C}(X'_i)$ such that

$$\mathrm{Int}(\tilde{f}_{i,K}, X'_{i,K}) = Z(h_{i1,K}),$$

$$\mathrm{Bdd}(\tilde{f}_{i,K}, X'_{i,K}) = Z(h_{i2,K}),$$

and

$$\mathrm{Iva}(\tilde{f}_{i,K}, X'_{i,K}) = Z(h_{i3,K}).$$

Extend each of the h_{ij}'s by zero to a function $\tilde{h}_{ij}$ in $\mathcal{C}(X')$. Now for $j = 1, 2$, or 3, let h_j be the function

$$\sum_i \tilde{h}_{ij}^2.$$

Then the $h_j \in \mathcal{C}(X')$ for $j = 1, 2, 3$ are as required by the theorem for f_{par} and $X[m, 0, 0]_{\mathrm{par}}$ instead of for f and $X[m, 0, 0]$. This finishes the first step.

In the second and final step we show how to get rid of the residue field parameters that were introduced with σ. Let us denote by $h_{j,\mathrm{par}}$ the functions obtained from step one, which are as desired by the theorem but for f_{par} and $X[m, 0, 0]_{\mathrm{par}}$ instead of for f and $X[m, 0, 0]$, with residual parameterization σ. Let us write $v : X' = X[0, s, 0] \to X$ for the natural projection. Observe that v is a coordinate projection which only omits some residue field variables. Now we define h_j as the motivic integral of $h_{j,\mathrm{par}}^2$ relative to $v : X' \to X$, namely, $\mu_X(h_{j,\mathrm{par}}^2)$ with notation from [8, Section 14.2] (which coincides with notation from [9, (8.7.10)]), and which in this case coincides with $v_!(h_{j,\mathrm{par}}^2)$ of [8, Section 5.6]). By Proposition 4.5.12, $h_{j,K}(x)$ for $x \in X_K$ equals the sum of the terms $h_{j,\mathrm{par},K}^2(x, \xi)$, where ξ runs over the fiber $v_K^{-1}(x) \subset X'_K$, which is a finite set of size at most q_K^s. Since a sum of squares of real numbers is nonzero if and only if each of the real numbers is nonzero, the h_i's are as desired. This completes the proof of Theorem 4.3.2. $\square$

Proof of Theorem 4.3.3
Let X and $f \in \mathcal{C}(C[m, 0, 0])$ be given as in the theorem. The proof goes again in two steps, similar to the proof of Theorem 4.3.2. First we show that there is a suitable residual parameterization σ of $X[m, 0, 0]$ such that the theorem holds for the function f_{par} in $\mathcal{C}(X[m, 0, 0]_{\mathrm{par}})$. In the second step, we again show that the result for $X[m, 0, 0]_{\mathrm{par}}$ and f_{par} implies the result for $X[m, 0, 0]$ and f itself. For the first step, one repeats the first step of the proof of Theorem 4.3.2 up to the introduction of the X'_i's; we use the same notation as in the proof of Theorem 4.3.2. One then proceeds as follows. By Corollary 4.5.6, one finds for each i a function g_i in $\mathcal{C}(S_{/X}(A_i))$ such that g_i is motivically X'_i-integrable and, for all K in $\mathcal{C}_{\mathcal{O},M}$ for a large M, $\mathrm{Int}(g_{i,K}, X'_{i,K}) = X'_i$ and $\tilde{f}_{i,K}(x, s) = g_{i,K}(x, s)$ whenever x lies in $\mathrm{Int}(\tilde{f}_{i,K}, X'_{i,K})$. Now for g one takes the function in $\mathcal{C}(X[m, 0, 0]_{\mathrm{par}})$ which equals f_{par} on the definable subassignment where $M_0 = 0$ and which equals g_i/M_0 elsewhere. Then this g is as desired by Theorem 4.3.3 for f_{par} and $X[m, 0, 0]_{\mathrm{par}}$ instead of for f and $X[m, 0, 0]$.

(See Remark 3.4.3 to see that dividing by M_0 in this way is harmless.) This finishes the first step.

For the second step, let us denote by g_{par} the function obtained from step one, which is as desired by the theorem but for f_{par} and $X[m, 0, 0]_{par}$ instead of for f and $X[m, 0, 0]$, and with residual parameterization σ. Since σ is a definable isomorphism which is moreover a coordinate projection which omits only residue field variables, we can define g as $\sigma^*(g_{par})$, being nothing else than the pullback of g_{par} in the notation of [8]. For K in $\mathcal{C}_{\mathcal{O}, M}$ for large M, g_K equals the composition $g_{par, K} \circ \sigma_K$, by the definition of pullback. Hence, the integrability conditions for each K in $\mathcal{C}_{\mathcal{O}, M}$ for large M are preserved. By the construction of the motivic integrability conditions, motivic integrability is also preserved when passing from g_{par} to g in the form desired by the theorem. Recall that $f_{par, K}$ equals $f_K \circ \sigma_K^{-1}$ by the definition of f_{par}. Hence, the equality between the $g_{par, K}$'s and the $f_{par, K}$'s on the integrable locus of $f_{par, K}$ yields the desired equality between the g_K's and the f_K's on the integrable locus of f_K. Hence, g is as desired by the theorem for f. □

Similarly to the p-adic case, the proofs of Theorems 4.3.2 and 4.3.3 yield the following slightly more general variant.

COROLLARY 4.5.13
Let f be in $\mathcal{C}^{exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X[m, 0, 0])$ for some definable subassignment X and some $m \geq 0$. Then there exist $M \geq 0$, h_1, h_2, h_3 in $\mathcal{C}^{exp}(X)$, and g in $\mathcal{C}^{exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X[m, 0, 0])$ such that, for each K in $\mathcal{C}_{\mathcal{O}, M}$ and each ψ in $\mathcal{D}_K$, the zero loci of the $h_{i, K, \psi}$'s equal, respectively,

$$\text{Int}(f_{K, \psi}, X_K), \qquad \text{Bdd}(f_{K, \psi}, X_K), \qquad and \qquad \text{Iva}(f_{K, \psi}, X_K)$$

and such that $\text{Int}(g_{K, \psi}, X_K) = X_K$ and $f_{K, \psi}(x, y) = g_{K, \psi}(x, y)$ whenever x lies in $\text{Int}(f_{K, \psi}, X_K)$. Moreover, any such g can be written as a finite sum of terms of the form

$$h_i \cdot f_i$$

with $h_i \in \mathcal{C}^{exp}(X)$ and $f_i \in \mathcal{C}(X[m, 0, 0])$ satisfying $\text{Int}(f_{i, K, \psi}, X_K) = X_K$ and such that the f_i's are motivically X-integrable.

Proofs of Theorems 4.4.4 and 4.4.6
The proof consists of the two usual steps, as in the proofs of Theorems 4.3.2 and 4.3.3. In the first step, we prove the result up to a residual parameterization σ. For that step, let us change the notation: denote the given function $f \in \mathcal{C}^{exp}(X[m, 0, 0])$ by f_0, and let us write X^0 for X and U^0 for $X^0[m, 0, 0]$. By Proposition 4.5.8, there

exists a residual parameterization $\sigma : U^0 \to U^0_{\text{par}} \subset X^0[0,s,0]$ such that if we write X for $X^0[0,s,0]$, U for U^0_{par}, and f for $f^0 \circ \sigma^{-1}$, then the following holds. There exist an integer $d \geq 0$, definable morphisms $h_i : U \to h[1,0,0]$, a definable surjection $\varphi : U \to V \subset X[0,0,t]$ for some $t \geq 0$, and functions G_i in $\mathcal{C}^{\exp}(V)$ such that, for each K in $\mathcal{C}_{\mathcal{O}}$ with large enough residue field characteristic and for each ψ in $\mathcal{D}_K$, Propositions 3.2.5(a) and 3.2.5(b) hold for d, $f_{K,\psi}$, U_K, V_K, φ_K, h_{iK}, and $G_{iK,\psi}$. For each i let H_i be the function $G_i \circ \varphi$. By [9, (3.3.1) and Proposition 3.3.2], one has a natural isomorphism

$$\mathcal{C}^{\exp}(V) \cong \mathcal{C}^{\exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(V)$$

of $\mathcal{C}(X)$-algebras, and thus, the H_i's may be considered to lie in $\mathcal{C}^{\exp}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X[m,0,0])$. For these tensor products of $\mathcal{C}(X)$-algebras we use the natural inclusions of $\mathcal{C}(X)$-algebras $\mathcal{C}(X) \subset \mathcal{C}(V), \mathcal{C}(X) \subset \mathcal{C}^{\exp}(X)$, and $\mathcal{C}(X) \subset \mathcal{C}(X[m,0,0])$. From Proposition 3.2.5(b) for the mentioned data, it follows that, for K in $\mathcal{C}_{\mathcal{O},M}$, any ψ in $\mathcal{D}_K$, and any $x \in X_K$, if $x \in \text{Iva}(H_{i,K,\psi}, X_K)$ for all i, then $x \in \text{Iva}(f_K, X_K)$. Vice versa, if $f_K(x,\cdot)$ is identically zero, then $H_{i,K,\psi}(x,\cdot)$ is zero on each set $W_{K,x,r}$, and since the function $H_{i,K,\psi}(x,\cdot)$ is constant on each set $U_{K,x,r}$, it is identically zero. Thus we just showed that

$$\text{Iva}(f_{K,\psi}, X_K) = \bigcap_i \text{Iva}(H_{i,K,\psi}, X_K).$$

A similar argument shows that

$$\text{Bdd}(f_{K,\psi}, X_K) = \bigcap_i \text{Bdd}(H_{i,K,\psi}, X_K),$$

$$\text{Int}(f_{K,\psi}, X_K) = \bigcap_i \text{Int}(H_{i,K,\psi}, X_K),$$

where in the case of $\text{Int}(f_{K,\psi}, X_K)$, we use the inequality between the volumes of $W_{K,x,r}$ and $U_{K,x,r}$ from Proposition 3.2.5(b). Now apply Corollary 4.5.13 to each of the functions H_i to find h_{ij} in $\mathcal{C}^{\exp}(X)$ for $j = 1,2,3$ and g_i in $\mathcal{C}^{\exp}(X \times K^m)$. By Corollary 4.5.10 for the finite intersections displayed above, Theorem 4.4.4 holds for f_{par}. For g as required by Theorem 4.4.6 for f_{par} one takes the obvious function in $\mathcal{C}^{\exp}(X[m,0,0])$, which specializes to $\sum_i g_{i,K} \psi(h_{i,K})$ for K in $\mathcal{C}_{\mathcal{O},M}$ and ψ in $\mathcal{D}_K$, namely, the motivic exponential function $\sum_i g_i \cdot [X,0,h_i]$. This finishes the first step of the proofs.

In the second step, where we remove the residual parameterization, we proceed differently for the two theorems. For Theorem 4.4.4, let $h_{j,\text{par}}$ for $j = 1,2,3$ be obtained using the first step, using a residual parameterization σ. These $h_{j,\text{par}}$'s live in $\mathcal{C}^{\exp}(X[0,s,0])$ where the s residue field variables were introduced with σ. We define

the final h_j by $\mu_X(h_j\overline{h}_j)$, where $\overline{h}_j$ is given by Lemma 4.5.9 and where μ_X is as in [9, (8.7.10)] for the coordinate projection from $X[0, s, 0]$ to X. (This is integration on the fibers, which in our case means integration over the s residue variables.) Similarly to the argument ending the proof of Theorem 4.3.2, these h_j's are as desired by Theorem 4.4.4 for f. For Theorem 4.4.6, let g_{par} be obtained using the first step with residual parameterization σ. One defines g as $\sigma^*(g_{\mathrm{par}})$ and concludes as in the proof of Theorem 4.3.3.   □

Proofs of Theorems 4.3.1 and 4.4.3
The specialization principle of Theorem 4.5.12 yields that taking motivic integrals, for integrable functions, commutes with specialization for K and ψ for all $K \in \mathcal{C}_{\mathcal{O},M}$ and ψ in $\mathcal{D}_K$, for large enough M. Now the results follow from Theorems 4.3.3 and 4.4.6, in exactly the same way as Theorems 3.1.1 and 3.2.1 follow from Theorems 3.1.5 and 3.2.3.   □

Finally, we can give the proofs of our new transfer principles, which follows from the work we have done.

Proofs of Theorems 4.4.1 and 4.4.2
For the first statement of Theorem 4.4.1 (resp., of Theorem 4.4.2), take h_1 (resp., h_2) as given by Theorem 4.4.4. For the second statement of Theorem 4.4.1 (resp., of Theorem 4.4.2), take h_1 (resp., h_2) as given by Corollary 4.4.5. In all cases the proof is finished by applying Proposition 4.5.11 to h_1 (resp., to h_2).   □

Note that the basic form of the transfer principle for integrability as stated in the introduction follows from Theorem 4.4.1. Indeed, in that basic form, the functions $F_{K,\psi}$ for K in $\mathcal{C}_{\mathcal{O}}$ are a special case of functions that come from a motivic exponential function since $\mathcal{L}_{\mathrm{DP}}$ is richer than the language of valued fields that is used in the introduction. Moreover, the family of additive characters $x \mapsto \psi(yx)$ with nonzero parameter y in the valued field and for $\psi \in \mathcal{D}_K$ clearly allows one to deduce the basic form (which does not impose conditions on the conductor) from the results we have established. First use Theorem 4.4.4, for the family with nonzero parameter y in the character as we just described, to find h_1 in $\mathcal{C}^{\exp}(h[1, 0, 0])$, extended by zero on zero. Use Theorem 4.4.4 for this h_1 to find a new function h_3 in $\mathcal{C}^{\exp}(h[0, 0, 0])$. Finally apply Proposition 4.5.11 to h_3 to recover the form of the transfer principle as stated in the introduction.

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